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A QUALITATIVE ANALYSIS OF A SECOND-ORDER ANISOTROPIC PHASE-FIELD TRANSITION SYSTEM ENDOWED WITH A GENERAL CLASS OF NONLINEAR DYNAMIC BOUNDARY CONDITIONS

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Abstract. The paper is concerned with the study of a nonlinear second-order anisotropic phase-field transition system of Caginalp type, subject to nonlinear and in-homogeneous dynamic boundary conditions (in both unknown functions). Under certain hypothesis on the input data: $f_1(t,x)$, $f_2(t,x)$, $w_1(t,x)$, $w_2(t,x)$, $u_0(x)$, $\alpha_0(x)$, $\varphi_0(x)$ and $\xi_0(x)$, we prove the well-posedness (the existence, a priori estimates, regularity and uniqueness) of a classical solution in the Sobolev space $W_p^{1,2}(Q)\times W_p^{1,2}(\Sigma)$, $W_\nu^{1,2}(Q)\times W_p^{1,2}(\Sigma)$. Here we extend the previous results concerned with nonlinearity of cubic type, allowing to the present mathematical model to be more capable for describing the complexity of a wide class of real physical phenomena (moving interface problems, image processing, the phase changes at the boundary $\partial\Omega$, etc.).

1. **Introduction.** In a compact domain $\Omega \subset I\!R^n$, $n \leq 3$, with a C^2 boundary $\partial\Omega = \Gamma$ and $[0,T]$ a generic time interval, we consider the following nonlinear second-order system of coupled PDEs with respect to the unknown functions $u(t,x)$ and $\varphi(t,x)$ (**hereafter** u,φ):

$$\begin{cases}\dfrac{\partial}{\partial t}u - \mathrm{div}\Big(K\big(t,x,u,\nabla u\big)\nabla u(t,x)\Big) = -\dfrac{\ell}{2}\dfrac{\partial}{\partial t}\varphi(t,x) + f_1(t,x) & \text{in } Q\\ \dfrac{\partial}{\partial t}\varphi - \mathrm{div}\Big(\Psi\big(t,x,\varphi,\nabla\varphi\big)\nabla\varphi(t,x)\Big) = p_1\big[\varphi-\varphi^3\big] + p_2 u + f_2(t,x) & \text{in } Q,\end{cases} \tag{1}$$

subject to the general class of nonlinear and in-homogeneous dynamic boundary conditions in both unknown functions $u(t,x)$ and $\varphi(t,x)$, i.e.

$$\begin{cases}\dfrac{\partial}{\partial \mathbf{n}}u + \dfrac{\partial}{\partial t}u - \Delta_\Gamma u + hu + g_1(t,x,u) = w_1(t,x) & \text{on } \Sigma\\ \dfrac{\partial}{\partial \mathbf{n}}\varphi + \dfrac{\partial}{\partial t}\varphi - \Delta_\Gamma\varphi + c_0\varphi + g_2(t,x,\varphi) = w_2(t,x) & \text{on } \Sigma,\end{cases} \tag{2}$$

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and with the initial conditions

$$u(0,x) = u_0(x), \qquad \varphi(0,x) = \varphi_0(x) \ \text{ in } \ \Omega, \tag{3}$$

where $Q = (0,T] \times \Omega$, $\Sigma = (0,T] \times \partial\Omega$ and:

- $t \in (0,T]$, $x = (x_1, \ldots, x_n) \in \Omega$;
- ℓ, p_1, p_2, h and c_0 are positive parameters;
- $\dfrac{\partial}{\partial t} v(t,x)$ is the partial derivative of $v(t,x)$ with respect to t;
- $u(t,x)$ represents the *reduced temperature distribution* in Q and denote by $\nabla u(t,x) = u_x(t,x)$ ($\nabla u = u_x$ **in short**) the gradient of $u(t,x)$ in x, that is
$$\nabla u(t,x) = \left(\frac{\partial}{\partial x_1} u(t,x), \frac{\partial}{\partial x_2} u(t,x), \ldots, \frac{\partial}{\partial x_n} u(t,x) \right).$$
We set $\dfrac{\partial}{\partial x_i} u = u_{x_i}$, $i = 1, 2, \ldots, n$, and thus $u_x = \big(u_{x_1}, u_{x_2}, \ldots, u_{x_n}\big)$;
- $\varphi(t,x)$ is the *phase function* (the order parameter), used to distinguish between the states (phases) of material which occupies the region Ω at every time $t \in [0,T]$; similarly as above, we denote by $\nabla\varphi(t,x) = \varphi_x(t,x)$ ($\nabla\varphi = \varphi_x$ **in short**) the gradient of $\varphi(t,x)$ in x, that is
$$\nabla \varphi(t,x) = \left(\frac{\partial}{\partial x_1} \varphi(t,x), \frac{\partial}{\partial x_2} \varphi(t,x), \ldots, \frac{\partial}{\partial x_n} \varphi(t,x) \right).$$
We set $\dfrac{\partial}{\partial x_i} \varphi = \varphi_{x_i}$, $i = 1, 2, \ldots, n$, and so $\varphi_x = \big(\varphi_{x_1}, \varphi_{x_2}, \ldots, \varphi_{x_n}\big)$;
- $K\big(t,x,u,u_x\big)$ - is a positive and bounded nonlinear real function with bounded derivatives, having the role of *controlling the speed of the diffusion process* in $(1)_1$;
- $\Psi(t,x,\varphi,\varphi_x)$ - is a positive and bounded nonlinear real function with bounded derivatives, having the role of *controlling the speed of the diffusion process* in $(1)_2$;
- $f_1(t,x) \in L^p(Q)$, $f_2(t,x) \in L^q(Q)$ are given functions (see Remark 2 below), where p and q satisfy
$$q \geq p \geq 2; \tag{4}$$
- $w_1(t,x), w_2(t,x) \in W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\Sigma)$ are given functions (see Remark 2 below);
- $u_0 \in W_\infty^{2-\frac{2}{p}}(\Omega)$, with
$$\frac{\partial}{\partial \mathbf{n}} u_0 - \Delta_\Gamma u_0 + h u_0 + g_1(0,x,u_0) = w_1(0,x),$$
and $\varphi_0 \in W_\infty^{2-\frac{2}{q}}(\Omega)$, with
$$\frac{\partial}{\partial \mathbf{n}} \varphi_0 - \Delta_\Gamma \varphi_0 + c_0 \varphi_0 + g_2(0,x,\varphi_0) = w_2(0,x).$$
- $\mathbf{n}$=$\mathbf{n}$(x) is the outward unit normal vector to Ω at a point $x \in \partial\Omega$. $\frac{\partial}{\partial \mathbf{n}}$ denotes differentiation along $\mathbf{n}$;
- Δ_Γ is the Laplace-Beltrami operator;

- Regarding the nonlinearities $g_k : \Sigma \times \mathbb{R} \to \mathbb{R}$, $k = 1, 2$, we will assume the following hypotheses:
  - are Carathéodory functions, i.e., $g_k(\cdot, \cdot, z) : \Sigma \to \mathbb{R}$ is measurable, $\forall z \in \mathbb{R}$;
  - $g_k(t, x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous, $\forall (t, x) \in \Sigma$, with $g_k(\cdot, \cdot, 0) \in L^\infty(\Sigma)$ (see [7, Definition 2.106, p. 42]);
  - (G_1): $\Big(g_k(t, x, z_1) - g_k(t, x, z_2)\Big)(z_1 - z_2) \geq b_3(z_1 - z_2)^2$, $\forall (t, x) \in \Sigma$, $z_1, z_2 \in \mathbb{R}$, for a constant $b_3 > 0$;
  - (G_2): there is a function $\bar{G} : \Sigma \times \mathbb{R}^2 \to \mathbb{R}$ verifying the relations

$$\Big(g_k(t, x, z_1) - g_k(t, x, z_2)\Big)^2 \leq \bar{G}(t, x, z_1, z_2)(z_1 - z_2)^2,$$

$$\bar{G}(t, x, z_1, z_2) \leq b_4(1 + |z_1|^{2(r'-1)} + |z_2|^{2(r'-1)}), \ \forall (t, x) \in \Sigma, z_1, z_2 \in \mathbb{R},$$

for a constant $b_4 > 0$ and $r' \geq 1$ such that (see (26) below)

$$r' \leq \frac{n+2}{n+2-2p} \quad if \ \ \frac{1}{p} - \frac{2}{n+2} > 0; \tag{5}$$

  - (G_3): $g_k(t, x, z)z \geq b_5 z^2$, $\forall (t, x) \in \Sigma$, $z \in \mathbb{R}$, with $b_5 > 0$.

Lemma 1.1. *Assumption* (G_2) *implies that* g_k, $k = 1, 2$, *fulfils the polynomial growth condition*

$$|g_k(t, x, z)| \leq b_6(1 + |z|^{r'}), \qquad \forall (t, x) \in \Sigma, \quad z \in \mathbb{R} \tag{6}$$

where b_6 *is a positive constant.*

Proof. Indeed, setting $z_1 = z$ and $z_2 = 0$ in (G_2), we get $(k = 1, 2)$

$$|g_k(t, x, z)| \leq |g_k(t, x, 0)| + \bar{G}(t, x, z, 0)^{\frac{1}{2}}|z|$$

$$\leq |g_k(t, x, 0)| + b_4^{\frac{1}{2}}(1 + |z|^{2(r'-1)})^{\frac{1}{2}}|z|, \qquad \forall z \in \mathbb{R}.$$

Since $g_k(t, x, 0) \in L^\infty(\Sigma)$, $k = 1, 2$, estimate (6) follows. □

Remark 1. For the sake of simplicity, in what follows we will take g_k, $k = 1, 2$, independent of time and space variables, i.e. $g_k(t, x, z) = g_k(z)$, $k = 1, 2$, since the major difficulty in the study of the nonlinear second-order problem (1)-(3) consists, among other, in treating the nonlinearity g_k, $k = 1, 2$, with respect to z.

Remark 2. Besides classical meanings, like the density of heat sources or sinks of heat, the pairs of given functions $\{f_1, f_2\}$ and $\{w_1, w_2\}$ in (1)-(2), can be also interpreted as *distributed* and *boundary control*, respectively, which opens a wide field of applicability for the nonlinear parabolic system (1)-(3), such as optimal control.

For reader's convenience, we will write problem (1) in the equivalent form

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) - \dfrac{\partial}{\partial u_{x_j}}\Big[K(t,x,u,u_x)u_{x_i}\Big]u_{x_jx_i} \\ \qquad = A_1(t,x,u,u_x) - \dfrac{\ell}{2}\dfrac{\partial}{\partial t}\varphi(t,x) + f_1(t,x) & \text{in } Q \\ \dfrac{\partial}{\partial t}\varphi(t,x) - \dfrac{\partial}{\partial \varphi_{x_j}}\Big[\Psi(t,x,\varphi,\varphi_x]\varphi_{x_i}\Big]\varphi_{x_jx_i} \\ \qquad = A_2(t,x,\varphi,\varphi_x) + p_1\big[\varphi(t,x) - \varphi^3(t,x)\big] + p_2 u(t,x) + f_2(t,x) & \text{in } Q, \end{cases} \tag{7}$$

with

$$A_1(t,x,u,u_x) = \frac{\partial}{\partial u}\Big[K(t,x,u,u_x)u_{x_i}\Big]u_{x_i} + \frac{\partial}{\partial x_i}\Big[K(t,x,u,u_x)u_{x_i}\Big], \tag{8}$$

$$A_2(t,x,\varphi,\varphi_x) = \frac{\partial}{\partial \varphi}\Big[\Psi(t,x,\varphi,\varphi_x)\varphi_{x_i}\Big]\varphi_{x_i} + \frac{\partial}{\partial x_i}\Big[\Psi(t,x,\varphi,\varphi_x)\varphi_{x_i}\Big] \tag{9}$$

and ($v = u$ or $v = \varphi$)

$$v_{x_i} = \frac{\partial}{\partial x_i}v(t,x), \quad v_{x_jx_i} = \frac{\partial^2}{\partial x_j \partial x_i}v(t,x), \quad i,j = 1,...,n.$$

It is easy to recognize $(7)_1$ and $(2)_1$ as being a quasi-linear one of type (2.4) in [14, p. 3 and p. 11] (see also [18, p. 229], [9, (1.7), p. 97]), with

$$\begin{aligned} a^1_{ij}(t,x,u,u_x) &= \frac{\partial}{\partial u_{x_j}}\Big[K(t,x,u,u_x)u_{x_i}\Big], \quad i = 1,...,n, \\ a^1(t,x,u,u_x,\varphi) &= -A_1(t,x,u,u_x) + \frac{\ell}{2}\frac{\partial}{\partial t}\varphi - f_1(t,x). \end{aligned} \tag{10}$$

Similarly, for $(7)_2$ and $(2)_2$ we have

$$\begin{aligned} a^2_{ij}(t,x,\varphi,\varphi_x) &= \frac{\partial}{\partial \varphi_{x_j}}\Big[\Psi(t,x,\varphi,\varphi_x)\varphi_{x_i}\Big], \quad i = 1,...,n, \\ a^2(t,x,u,\varphi,\varphi_x) &= -A_2(t,x,\varphi,\varphi_x) - p_1\big[\varphi - \varphi^3\big] - p_2 u(t,x) - f_2(t,x). \end{aligned} \tag{11}$$

In addition, unless otherwise stated, we assume that equations (1) [or (7)] are *uniformly parabolic*, which means fulfilment of the conditions

$$\begin{aligned} \nu_1(|u|)\zeta^2 &\le \frac{\partial}{\partial z_j}K(t,x,u,z)\zeta_i\zeta_j \le \nu_2(|u|)\zeta^2 \\ \nu_1(|\varphi|)\zeta^2 &\le \frac{\partial}{\partial z_j}\Psi(t,x,\varphi,z)\zeta_i\zeta_j \le \nu_2(|\varphi|)\zeta^2, \end{aligned} \tag{12}$$

for arbitrary u, φ, z and $\zeta = (\zeta_1,...,\zeta_n)$ an arbitrary real vector, where $\nu_1(s)$ and $\nu_2(s)$ are positive (nonincreasing and nondecreasing, respectively) continuous functions of $s \ge 0$.

In the present paper we study the solvability of the nonlinear second-order boundary value problems of the form (1)-(3) (or (7) plus (2)-(3)) in the class

$W_p^{1,2}(Q) \times W_p^{1,2}(\Sigma)$, $W_\nu^{1,2}(Q) \times W_p^{1,2}(\Sigma)$. The new mathematical formulation expressed by (1)-(3) is characterized by the presence of some new physical parameters: p_1, p_2, $K\big(t,x,u(t,x),u_x(t,x)\big)$, $\Psi(t,x,\varphi(t,x),\varphi_x(t,x)))$, the principal part being in *divergence form* and by considering the classical *regular potential* (reaction term) (see [3], [4], [6], [11], [14], [15]-[20], [25]-[27]). The most important novelty in our paper concerns the inhomogeneous dynamic boundary conditions of nonlinear type, not treated until now (in this new context, that is: the principal is in *divergence form* in both unknown functions $u(t,x)$ and $\varphi(t,x)$) in the mathematical literature. Thus, significant aspects of the delicate physical features are expected to be reflected more accurately. In this regard, as applications of problem (1)-(3), we indicate the *moving interface problems*, e.g. *phase separation and transition* (see [2]-[6], [8]-[11], [15]-[26], [28], [29]), *anisotropy effects* (see [2], [9], [10], [13], [18], [26]), *image denoising* and *segmentation* (see [1], [11], [26] and references therein) etc. In addition, the general hypotheses formulated on g_k, $k = 1, 2$, also allows to take in the dynamic boundary conditions a nonlinearity with a larger growth exponent $r' \leq (n+2)/(n+2-2p)$ if $n+2 > 2p$ (see (5)), for each unknown functions u and φ. It extends the already studied types of boundary conditions and therefore makes the new formulation of model (1)-(3) to be more able to describe a wide variety of industrial applications of two-phase systems, in particular, the interactions with the walls in confined systems (i.e. the *phase changes* at the boundary of Ω).

Different formulations of the nonlinear phase-field transition system (1)-(3) with different nonlinearities, as well as different physical parameters and boundary conditions, can be found in the works: Benincasa, Favini and Moroşanu [3], Boldrini, Caretta and Fernández-Cara [4], Cârjă, Miranville and Moroşanu [6], Cavaterra, Gal, Grasselli and Miranville [8], Conti, Gatti and Miranville [10], Gatti and Miranville [13], Miranville and Moroşanu [15]-[18], Moroşanu [19]-[22], Moroşanu and Croitoru [23], Moroşanu and Motreanu [24], Moroşanu and Pavăl [25], [26], Vaz and all [29]. In the present work we have limited the nonlinear reaction therm in $(1)_2$ to depend only of φ because, as we already have mentioned, the major difficulty in treating the parabolic nonlinear problem (1) lies just in such sort of nonlinearities. Examples of nonlinearities depending on t, x and φ can be found in Moroşanu and Motreanu [24].

The rest of the paper is organized as follows. In Section 2, we first present the technical method involved in treating the boundary conditions of dynamic type. Next, we recall the notations and the methods to be used in the proof of the main result, Theorem 2.2, formulated at the end of the section. The well-posedness of solutions to a nonlinear reaction-diffusion equation, supplied with a general class of nonlinear and non-homogeneous dynamic boundary conditions, is discussed in Section 3 (Theorem 3.2). The Section 4 provides the proof of the main result Theorem 2.2.

2. **Preliminaries and main result.**

In order to approach the parabolic nonlinear system (1)-(3), we will use the same idea as in Cârjă, Miranville and Moroşanu [6], Choban and Moroşanu [9], Miranville and Moroşanu [15], Moroşanu [20]. In this regards, let $\alpha = u$ and $\xi = \varphi$ be a further variables such that $\alpha(0,x) = u_0$, $\xi(0,x) = \varphi_0$ on Γ, while for the remaining data in (1)-(3) we will keep the same meanings formulated at the beginning. Corresponding, the boundary conditions in

(2) will be approached in the sequel by

$$\begin{cases} u = \alpha, \qquad \varphi = \xi & \text{on } \Sigma \\ \dfrac{\partial}{\partial \mathbf{n}} u + \dfrac{\partial}{\partial t}\alpha - \Delta_\Gamma \alpha + h\alpha + g_1(\alpha) = w_1(t,x) & \text{on } \Sigma \\ \dfrac{\partial}{\partial \mathbf{n}} \varphi + \dfrac{\partial}{\partial t}\xi - \Delta_\Gamma \xi + c_0\xi + g_2(\xi) = w_2(t,x) & \text{on } \Sigma \\ \alpha(0,x) = \alpha_0(x), \qquad \psi(0,x) = \psi_0(x) & x \in \Gamma, \end{cases} \tag{13}$$

where $\alpha_0, \xi_0 \in W_\infty^{2-\frac{2}{p}}(\Gamma)$.

Accordingly, the problem (7), (2)-(3) can be rewritten suitably as follows

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) - \dfrac{\partial}{\partial u_{x_j}} \left(K(t,x,u,u_x) u_{x_i}\right) u_{x_j x_i} & \\ \qquad = A_1(t,x,u,u_{x_i}) - \dfrac{\ell}{2}\dfrac{\partial}{\partial t}\varphi + f_1(t,x) & \text{in } Q \\ u(t,x) = \alpha(t,x) & \text{on } \Sigma \\ \dfrac{\partial}{\partial \mathbf{n}} u + \dfrac{\partial}{\partial t}\alpha - \Delta_\Gamma \alpha + h\alpha + g_1(\alpha) = w_1(t,x) & \text{on } \Sigma \\ u(0,x) = u_0(x) & \text{on } \Omega \\ \alpha(0,x) = \alpha_0(x) & x \in \Gamma, \end{cases} \tag{14}$$

$$\begin{cases} \dfrac{\partial}{\partial t} \varphi(t,x) - \dfrac{\partial}{\partial \varphi_{x_j}} \Big(\Psi(t,x,\varphi,\varphi_x)\varphi_{x_i}\Big) \varphi_{x_j x_i} & \\ \qquad = A_2(t,x,\varphi,\varphi_{x_i}) + p_1\left[\varphi - \varphi^3\right] + p_2 u(t,x) + f_2(t,x) & \text{in } Q \\ \varphi(t,x) = \xi(t,x) & \text{on } \Sigma \\ \dfrac{\partial}{\partial \mathbf{n}} \varphi + \dfrac{\partial}{\partial t}\xi - \Delta_\Gamma \xi + c_0\xi + g_2(\xi) = w_2(t,x) & \text{on } \Sigma \\ \varphi(0,x) = \varphi_0(x) & \text{on } \Omega \\ \xi(0,x) = \xi_0(x) & x \in \Gamma. \end{cases} \tag{15}$$

Definition 2.1. Any solution $(u, \alpha, \varphi, \xi)$ of the nonlinear second-order boundary value problem (14)-(15) is called the `classical solution` if it is continuous in Q, have continuous derivatives u_t, u_x, u_{xx}, φ_t, φ_x, φ_{xx} in Q and α_t, α_x, α_{xx}, ξ_t, ξ_x, ξ_{xx} on Σ, satisfy the equation $(14)_1$-$(15)_1$ at all points $(t,x) \in Q$ as well as the conditions $(14)_{2-3}$-$(15)_{2-3}$ and $(14)_{4-5}$-$(15)_{4-5}$ for $(t,x) \in \Sigma$ and for $t = 0$, respectively.

Our main results regarding the existence, uniqueness and regularity of solutions to problem (14)-(15) (practically, well-posedness of the solutions to the nonlinear second-order boundary value problem (1)-(3) or (7), (2)-(3)) is the following

Theorem 2.2. *Suppose* $\big\{(u,\alpha),\ (\varphi,\xi)\big\} \in \big[C^{1,2}(Q)\times C^{1,2}(\Sigma)\big]^2$ *is a classical solution of problem* (14)-(15) *and for positive numbers:*

$$M,\ M_0,\ M_1,\ M_2,\ M_3,\ M_4, \quad \textit{and} \quad N,\ N_0,\ N_1,\ N_2,\ N_3,\ N_4,$$

one has

$\mathbf{I}_1$. $|u(t,x)| < M$ *for any* $(t,x)\in Q$ *and for any* t,x,z, *the map* $K(t,x,u,z)$ *is continuous, differentiable with respect to* $x,\ u,\ z$, *its* x*-derivatives,* u*-derivatives and* z*-derivatives are measurable bounded, satisfies* $(12)_1$, *and*

$$\begin{gathered} 0 < K_m \le K(t,x,u,u_x) < K_M, \qquad for \quad (t,x)\in Q, \\ \sum_{i=1}^{n}\left[|K(t,x,u,z)u_{x_i}| + \left|\frac{\partial}{\partial u}(K(t,x,u,z)u_{x_i})\right|\right](1+|z|) \\ +\sum_{i,j=1}^{n}\left|\frac{\partial}{\partial x_j}(K(t,x,u,z)u_{x_i})\right| \le M_0(1+|z|)^2. \end{gathered} \tag{16}$$

$\mathbf{I}_2$. *For every* $\varepsilon>0$, *the functions* $u(t,x)$ *and* $K(t,x,u,u_x)$ *satisfy the relations*

$$\|u\|_{L^s(Q)} \le M_1, \qquad \|K(t,x,u,u_x)u_{x_i}\|_{L^r(Q)} < M_2, \quad i=1,...,n,$$

where

$$r=\begin{cases}\max\{p,4\} & p\ne 4\\ 4+\varepsilon & p=4,\end{cases} \qquad s=\begin{cases}\max\{p,2\} & p\ne 2\\ 2+\varepsilon & p=2.\end{cases}$$

$\mathbf{I}_3$. *The hypotheses* (G_1)-(G_3) $(k=1)$ *are fulfilled.*

$\mathbf{J}_1$. $|\varphi(t,x)| < N$ *for any* $(t,x)\in Q$ *and for any* t,x,z, *the map* $\Psi(t,x,\varphi,z)$ *is continuous, differentiable with respect to* $x,\ \varphi,\ z$, *its* x*-derivatives,* φ*-derivatives and* z*-derivatives are measurable bounded, satisfies* $(12)_2$, *and*

$$\begin{gathered} 0 < \Psi_m \le \Psi(t,x,\varphi,\varphi_x) < \Psi_M, \qquad for \quad (t,x)\in Q, \\ \sum_{i=1}^{n}\left[|\Psi(t,x,\varphi,z)\varphi_{x_i}| + \left|\frac{\partial}{\partial \varphi}(\Psi(t,x,\varphi,z)\varphi_{x_i})\right|\right](1+|z|) \\ +\sum_{i,j=1}^{n}\left|\frac{\partial}{\partial x_j}(\Psi(t,x,\varphi,z)\varphi_{x_i})\right| \le N_0(1+|z|)^2. \end{gathered} \tag{17}$$

$\mathbf{J}_2$. *For every* $\varepsilon>0$, *the functions* $\varphi(t,x)$ *and* $\Psi(t,x,\varphi,\varphi_x)$ *satisfy the relations*

$$\|\varphi\|_{L^s(Q)} \le N_1, \qquad \|\Psi(t,x,\varphi,\varphi_x)\varphi_{x_i}\|_{L^r(Q)} < N_2, \quad i=1,...,n,$$

where the quantities r *and* s *were defined in* $\mathbf{I}_2$.

$\mathbf{J}_3$. *The hypotheses* (G_1)-(G_3) $(k=2)$ *are fulfilled.*

Then, there exists a unique solution $u\in W^{1,2}_p(Q)$, $\varphi\in W^{1,2}_\nu(Q)$ $(\nu=min\{q,\mu\})$,

$\alpha, \xi \in W_p^{1,2}(\Sigma)$ *to* (14)-(15), $p, q \neq \frac{3}{2}$, *and satisfies*

$$\|u\|_{W_p^{1,2}(Q)} + \|\varphi\|_{W_\nu^{1,2}(Q)} + \|\alpha\|_{W_p^{1,2}(\Sigma)} + \|\xi\|_{W_p^{1,2}(\Sigma)}$$
$$\leq C \Bigg[1 + \|u_0\|_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|\varphi_0\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{q}}(\Omega)} + \|\alpha_0\|_{W_\infty^{2-\frac{2}{p}}(\Gamma)} + \|\xi_0\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{p}}(\Gamma)} \tag{18}$$
$$+\|f_1\|_{L^{p'}(Q)} + \|f_2\|_{L^q(Q)} + \|w_1\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)} + \|w_2\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\Bigg],$$

where the constant $C > 0$ *depends on* $|\Omega|$ (*the measure of* Ω), T, n, p, q, b_3, b_4, b_5, b_6 *and physical parameters, but is independent of* u, φ, α, ξ, f_1, f_2, w_1 *and* w_2.

If $(u^1, \alpha^1, \varphi^1, \xi^1)$, $(u^2, \alpha^2, \varphi^2, \xi^2)$ *are two solutions to* (14)-(15) *corresponding to* $(u_0^1, \alpha_0^1, \varphi_0^1, \xi_0^1)$, $(u_0^2, \alpha_0^2, \varphi_0^2, \xi_0^2) \in W_\infty^{2-\frac{2}{p}}(\Omega) \times W_\infty^{2-\frac{2}{p}}(\Gamma) \times W_\infty^{2-\frac{2}{q}}(\Omega) \times W_\infty^{2-\frac{2}{p}}(\Gamma)$, (f_1^a, f_2^a), $(f_1^b, f_2^b) \in L^p(Q) \times L^q(Q)$, w_1^a, w_2^a, $w_1^b, w_2^b \in W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)$, *respectively, such that*

$$\begin{cases} \|u^1\|_{W_p^{1,2}(Q)},\ \|u^2\|_{W_p^{1,2}(Q)} \leq M_3, & \|\alpha^1\|_{W_p^{1,2}(\Sigma)},\ \|\alpha^2\|_{W_p^{1,2}(\Sigma)} \leq M_4, \\ \|\varphi^1\|_{W_p^{1,2}(Q)},\ \|\varphi^2\|_{W_\nu^{1,2}(Q)} \leq N_3, & \|\xi^1\|_{W_p^{1,2}(\Sigma)},\ \|\xi^2\|_{W_p^{1,2}(\Sigma)} \leq N_4, \end{cases} \tag{19}$$

then the following estimate holds

$$\max_{(t,x)\in Q} |u^1 - u^2| + \max_{(t,x)\in\Sigma} |\alpha^1 - \alpha^2| + \max_{(t,x)\in Q} |\varphi^1 - \varphi^2| + \max_{(t,x)\in\Sigma} |\xi^1 - \xi^2|$$
$$\leq C_1 e^{CT} \max\Bigg\{ \max_{(t,x)\in\Omega} |u_0^1 - u_0^2|,\ \max_{(t,x)\in\Gamma} |\alpha_0^1 - \alpha_0^2|,$$
$$\max_{(t,x)\in\Omega} |\varphi_0^1 - \varphi_0^2|,\ \max_{(t,x)\in\Gamma} |\xi_0^1 - \xi_0^2|, \tag{20}$$
$$\max_{(t,x)\in Q} |f_1^a - f_1^b|,\ \max_{(t,x)\in Q} |f_2^a - f_2^b|,$$
$$\max_{(t,x)\in\Sigma} |w_1^a - w_1^b|,\ \max_{(t,x)\in\Sigma} |w_2^a - w_2^b| \Bigg\},$$

where the positive constants $C_1 > 0$, $C > 0$, *are independent of* $\Big\{u^1, \alpha^1, \varphi^1, \xi^1, f_1^a, w_1^a, u_0^1, \alpha_0^1, \varphi_0^1, \xi_0^1\Big\}$ *and* $\Big\{u^2, \alpha^2, \varphi^2, \xi^2, f_2^a,\ w_2^a, u_0^2, \alpha_0^2, \varphi_0^2, \xi_0^2\Big\}$. *In particular, the uniqueness of solution to problem* (14)-(15) *holds.*

Basic tools in our approach are:

- the Leray-Schauder degree theory (see [17, p. 221] and references therein);
- Green's first identity

$$\begin{cases} -\int_\Omega y \operatorname{div} z\, dx = \int_\Omega \nabla y \cdot z\, dx - \int_{\partial\Omega} y \frac{\partial}{\partial \mathbf{n}} z\, d\gamma, \\ -\int_\Omega y \Delta z\, dx = \int_\Omega \nabla y \cdot \nabla z\, dx - \int_{\partial\Omega} y \frac{\partial}{\partial \mathbf{n}} z\, d\gamma, \end{cases} \tag{21}$$

for any scalar-valued function y and z - a continuously differentiable vector field in n dimensional space;

- the Lions and Peetre embedding Theorem (see [17, p. 14]) to ensure the existence of a continuous embedding $W_p^{1,2}(Q) \subset L^{\mu_1}(Q)$, where the real μ_1 is given by (see (4)):

$$\mu_1 = \begin{cases} \text{any positive number } \geq 3p & \text{if } \dfrac{1}{p} - \dfrac{2}{n+2} \leq 0, \\ \\ \dfrac{p\,(n+2)}{n+2-2p} & \text{if } \dfrac{1}{p} - \dfrac{2}{n+2} > 0, \end{cases} \tag{22}$$

and, for $m \in \{1, 2, \cdots\}$ and $1 \leq p \leq \infty$, $W_p^{m,2m}(Q)$ denotes the Sobolev space on Q:

$$W_p^{m,2m}(Q) = \left\{ y \in L^p(Q) : \frac{\partial^r}{\partial t^r} \frac{\partial^q}{\partial x^q}\, y \in L^p(Q), \text{ for } 2r + q \leq 2m \right\}, \tag{23}$$

i.e., the spaces of functions whose t-derivatives and x-derivatives up to the order m and $2m$, respectively, belong to $L^p(Q)$ (see [17, p. 13-14]).

- Also, we shall use the set $C^{1,2}(\bar{Q})$ ($C^{1,2}(Q)$) of all continuous functions in $\bar{Q}$ (in Q) having continuous derivatives u_t, u_x, u_{xx} in $\bar{Q}$ (in Q), as well as the Sobolev spaces $W_p^{\ell}(\Omega)$, $W_p^{\ell,\ell/2}(\Sigma)$ with non integral ℓ for the initial and boundary conditions, respectively (see [17, p. 14]).
- As far as the techniques used in the paper are concerned, it should be noted that we derive the *a priori* estimates in $L^p(Q)$ and $L^p(\Sigma)$.

In the following we will denote by C several positive constants, being understood that the extra dependencies will be set out on occurrence.

3. **Well-posedness of solutions to the nonlinear second-order reaction-diffusion equation (15) in the class $W_p^{1,2}(Q) \times W_p^{1,2}(\Sigma)$.** We consider the following nonlinear second-order reaction-diffusion problem:

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi(t,x) - \dfrac{\partial}{\partial \Phi_{x_j}}\Big(\Psi(t,x,\Phi,\Phi_x)\Phi_{x_i}\Big)\Phi_{x_j x_i} \\ \qquad = A_2(t,x,\Phi,\Phi_{x_i}) + p_1\left[\Phi(t,x) - \Phi^3(t,x)\right] + \hat{f}_2(t,x) & \text{in } Q, \\ \Phi(t,x) = \xi(t,x) & \text{on } \Sigma \\ \dfrac{\partial}{\partial \mathbf{n}}\Phi + \dfrac{\partial}{\partial t}\xi - \Delta_\Gamma \xi + c_0 \xi + g_2(\xi) = w_2(t,x) & \text{on } \Sigma \\ \Phi(0,x) = \varphi_0(x) & \text{on } \Omega \\ \xi(0,x) = \xi_0(x) & x \in \Gamma, \end{cases} \tag{24}$$

where $A_2(t,x,\Phi,\Phi_{x_i}) = \dfrac{\partial}{\partial \Phi}\Big[\Psi(t,x,\Phi,\Phi_x)\Phi_{x_i}\Big]\Phi_{x_i} + \dfrac{\partial}{\partial x_i}\Big[\Psi(t,x,\Phi,\Phi_x)\Phi_{x_i}\Big]$, $\hat{f}_2 \in L^p(Q)$, $\varphi_0 \in W_\infty^{2-\frac{2}{q}}(\Omega)$ and $\varphi_0 = \xi_0$ on Γ .

The equation (24) was introduced by Allen-Cahn (see [17] and reference therein) to describe the motion of anti-phase boundaries in crystalline solids. Recently, the Allen-Cahn equation has been widely applied to many complex moving interface problems, like: the *mixture of two incompressible fluids*, the *nucleation of solids*, *vesicle membranes*, etc (see Calatroni and Colli [5], Vaz and Boldrini [29] and references therein). For additional details relative to a extensive class of problems of type (24) (transport phenomena, reaction-diffusion equation, for instance), as well as different types for the nonlinear term $F(\Phi)$ and boundary conditions, we direct the reader to the books by Miranville and Moroşanu [17], Moroşanu [19].

Definition 3.1. Any solution $\big(\Phi,\xi\big)$ of the nonlinear second-order boundary value problem (24) is called the `classical solution` if it is continuous in Q, have continuous derivatives Φ_t, Φ_x, Φ_{xx} in Q and ξ_t, ξ_x, ξ_{xx} on Σ, satisfy the equation $(24)_1$ at all points $(t,x)\in Q$ as well as the conditions $(24)_{2-3}$ and $(24)_{4-5}$ for $(t,x)\in\Sigma$ and for $t=0$, respectively.

The main result of this section establishes the solvability of the problem (24), characterized by

- the presence of some new physical parameters $(p_1, c_0, \Psi(t,x,\Phi,\Phi_x))$;
- the principal part in *divergence form*;
- considering the cubic nonlinearity $\Phi-\Phi^3$ (a classical *regular potential*), satisfying the condition H_0 in [24]:

  $H_0: \quad \big[\Phi(t,x)-\Phi^3(t,x)\big]|\Phi(t,x)|^{3p-4}\Phi(t,x)\le 1+|\Phi(t,x)|^{3p-1}-|\Phi(t,x)|^{3p};$
- the nonlinear in-homogeneous dynamic boundary conditions.

Precisely, in Theorem 3.2 below we prove the existence, a priori estimates and regularity for the solution of problem (24) in the class $W_p^{1,2}(Q)\times W_p^{1,2}(\Sigma)$ (see (23) for $m=1$).

Theorem 3.2. *For any classic solution* $\big(\Phi(t,x),\xi(t,x)\big)\in C^{1,2}(Q)\times C^{1,2}(\Sigma)$ *of* (24)*, suppose there are* $N, N_0, N_1, N_2\in(0,\infty)$ *such that the hypotheses* $\mathbf{J}_1$ - $\mathbf{J}_3$ *are satisfied.*

Then, $\forall\hat{f}_2\in L^p(Q)$, $\varphi_0(x)\in W_\infty^{2-\frac{2}{q}}(\Omega)$, $\xi_0(x)\in W_\infty^{2-\frac{2}{p}}(\Gamma)$, $w_2\in W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)$, $q\ne\frac{3}{2}$, *there exists a solution* $\Phi\in W_p^{1,2}(Q)$, $\xi\in W_p^{1,2}(\Sigma)$ *to* (24) *and satisfies*

$$\begin{aligned}\|\Phi\|_{W_p^{1,2}(Q)}+\|\xi\|_{W_p^{1,2}(\Sigma)} \quad &\le C\ \Bigg[1+\|\varphi_0\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{q}}(\Omega)}+\|\xi_0\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{p}}(\Gamma)} \\ &\qquad +\|\hat{f}_2\|_{L^p(Q)}+\|w_2\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\Bigg],\end{aligned}\tag{25}$$

where the constant $C>0$ *depends on physical parameters but is independent of* φ, ξ, $\hat{f}_2$ *and* w_2*.*

In the particular case of the linear reaction term, the results like those established by Theorem 3.2 has been proved in Choban and Moroşanu [9].

Proof. In order to prove the Theorem 3.2, we use the Leray-Schauder principle (see [17, p. 221] and references therein). In this respect, let p' chosen as follows

$$\mu_1\ge p'=\begin{cases}\text{any positive number}\ge pr' & if\ \ \dfrac{1}{p}-\dfrac{2}{n+2}\le 0,\\[2ex] \text{any number in}\left[pr',\dfrac{p(n+2)}{n+2-2p}\right] & if\ \ \dfrac{1}{p}-\dfrac{2}{n+2}>0.\end{cases}\tag{26}$$

Notice that (26) makes sense due to (5).

Consider the Banach space

$$B^H = W_p^{0,1}(Q) \cap L^{3p}(Q) \times L^{p'}(\Sigma),$$

endowed with the norm $\|\cdot\|_{B^H}$, expresed by

$$\|(v,\bar{v})\|_{B^H} = \|v\|_{L^p(Q)} + \|v_x\|_{L^p(Q)} + \|\bar{v}\|_{L^{p'}(\Sigma)},$$

and a nonlinear operator $H : B^H \times [0,1] \to B^H$ defined by

$$(\Phi,\xi) = \Big(\Phi(v,\bar{v},\lambda), \xi(v,\bar{v},\lambda)\Big) = H(v,\bar{v},\lambda) \quad \forall (v,\bar{v}) \in B^H,\ \forall \lambda \in [0,1], \tag{27}$$

where $\big(\Phi(v,\bar{v},\lambda), \xi(v,\bar{v},\lambda)\big)$ is the unique solution to the following linear second-order boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi(t,x) - \left[\lambda \dfrac{\partial}{\partial v_{x_j}}(\Psi(t,x,v,v_x)v_{x_i}) - (1-\lambda)\delta_i^j\right]\Phi_{x_i x_j} & \\ \qquad\qquad = \lambda\Big[A_2(t,x,v,v_{x_i}) + p_1\left[v(t,x) - v^3(t,x)\right] + \hat{f}_2(t,x)\Big] & \text{in } Q \\ \Phi(t,x) = \xi(t,x) & \text{on } \Sigma \\ \Phi(0,x) = \lambda\varphi_0(x) & \text{on } \Omega \\ \dfrac{\partial}{\partial \mathbf{n}}\Phi + \dfrac{\partial}{\partial t}\xi - \Delta_\Gamma \xi + c_0\xi = \lambda\big[-g_2(\bar{v}) + w_2(t,x)\big] & \text{on } \Sigma \\ \xi(0,x) = \lambda\xi_0(x) & x \in \Gamma. \end{cases} \tag{28}$$

For beginning, we shall prove the following technical lemma

Lemma 3.3. *We assume hypotheses* $\mathbf{J}_1$ *and* $\mathbf{J}_2$ *to be valid,* $\forall v \in W_p^{1,2}(Q) \subset W_p^{0,1}(Q) \cap L^{3p}(Q)$*. Then*

$$A_2(t,x,v,v_{x_i}) + p_1[v - v^3] + \hat{f}_2(t,x) \in L^p(Q). \tag{29}$$

Proof. Indeed, since $v \in W_p^{1,2}(Q) \subset L^{\mu_1}(Q) \subset L^{3p}(Q)$ (see (22)), then $\|v\|_{L^{3p}(Q)} \le Konst$ and thus

$$\|v^3\|_{L^p(Q)} = \left(\int_Q |v^3|^p dxdt\right)^{\frac{1}{p}} = \left[\left(\int_Q |v|^{3p} dxdt\right)^{\frac{1}{3p}}\right]^{3p\frac{1}{p}} = \|v\|^3_{L^{3p}(Q)} \le (Konst)^3,$$

i.e., the nonlinear term in (29) belongs to $L^p(Q)$, $\forall v \in W_p^{1,2}(Q) \subset W_p^{0,1}(Q) \cap L^{3p}(Q)$ (see also Miranville and Moroşanu [18]).

Next, we prove that $A_2(t,x,v,v_{x_i})) \in L^p(Q)$, $\forall v \in W_p^{1,2}(Q) \subset W_p^{0,1}(Q) \cap L^{3p}(Q)$. Making use of (9), we get $(v_{x_i} = \frac{\partial}{\partial x_i}v(t,x))$

$$\begin{aligned} A_2(t,x,v,v_{x_i}) &= \tfrac{\partial}{\partial x_i}\left[\Psi(t,x,v,v_x)v_{x_i}\right] + \tfrac{\partial}{\partial v}\left[\Psi(t,x,v,v_x)v_{x_i}\right]v_{x_i} \\ &= \tfrac{\partial}{\partial x_i}\left[\Psi(t,x,v,v_x)\right]v_{x_i} + \Psi(t,x,v,v_x)\tfrac{\partial}{\partial x_i}\Big(\tfrac{\partial}{\partial x_i}v(t,x)\Big) + \tfrac{\partial}{\partial v}\left[\Psi(t,x,v,v_x)v_{x_i}\right]v_{x_i} \\ &= \Big\{\tfrac{\partial}{\partial x_i}\Psi(t,x,v,v_x) + \tfrac{\partial}{\partial \varphi}\left[\Psi(t,x,v,v_x)v_{x_i}\right] + \textstyle\sum_{j=1}^n \tfrac{\partial}{\partial v_{x_j}}\Psi(t,x,v,v_x)v^2_{x_j x_i}\Big\}v_{x_i} \\ &\quad + \Psi(t,x,v,v_x)v^2_{x_i x_i} + \tfrac{\partial}{\partial v}\left[\Psi(t,x,v,v_x)\right](v_{x_i})^2. \end{aligned}$$

Denote

$T_1 = \frac{\partial}{\partial x_i}\left[\Psi(t,x,v,v_x)\right] v_{x_i}$,

$T_2 = \frac{\partial}{\partial v}\left[\Psi(t,x,v,v_x)\right] (v_{x_i})^2$,

$T_3 = \Psi(t,x,v,v_x) v^2_{x_i x_i}$,

$G_j = v_{x_i} \frac{\partial}{\partial v_{x_j}} \Psi(t,x,v,v_x) v^2_{x_j x_i}, j = 1, 2, \cdots, n.$

According to the hypothesis, we have:

i. $\frac{\partial}{\partial x_i}\Psi(t,x,v,v_x)$ is measurable bounded and $v_{x_i} \in L^p(Q)$;

ii. $\frac{\partial}{\partial v}\Psi(t,x,v,v_x)$ is measurable bounded and $(v_{x_i})^2 \in L^p(Q)$;

iii. $\Psi(t,x,v,v_x)$ is measurable bounded (see $(17)_1$) and $v^2_{x_i x_i}$ is continuous;

iv. $\frac{\partial}{\partial v_{x_j}}\Psi(t,x,v,v_x)$ is measurable bounded, v_{x_i} and $v^2_{x_j x_i}$ are continuous.

Using classical measure theory, from i.-iv. it results that T_1, T_2, T_3, $G_j, j = 1, 2, \cdots, n$ are in $L^p(Q)$ and thus $A_2(t,x,v,v_{x_i}) \in L^p(Q)$.

Finally, we recall that $\hat{f}_2(t,x) \in L^p(Q)$ and, owing to the above, we easy derive that the statement expressed by (29) is true. □

3.1. **The proof of Theorem 3.2 (continued).** Let us show that the nonlinear operator H defined by (27) satisfies the following two properties **P1** and **P2**, that is:

P1. H is well-defined.

P2. H is continuous and compact.

P1. H **is well-defined** if the problem (28) has a unique solution. Making use of Lemma 3.3, from the right-hand side of (28) it follows that $\forall v \in W_p^{0,1}(Q) \cap L^{3p}(Q)$, then $A_2(t,x,v,v_x) + p_1\left[v - v^3\right] + \hat{f}_2(t,x) \in L^p(Q)$. On the other hand, according to (6), we have that $g_2(t,x,\bar{v}) \in L^{\frac{p'}{r'}}(\Sigma)$ whenever $\bar{v} \in L^{p'}(\Sigma)$. Moreover, (5) implies $g_2(t,x,\bar{v}) \in L^p(\Sigma)$. Applying **Lemma 7.4** in Choban and Moroşanu [9, p. 114] with

$f_3 = \lambda\left[A_2(t,x,v,v_{x_i}) - p_1\left[v(t,x) - v^3(t,x)\right] + \hat{f}_2(t,x)\right] \in L^p(Q)$ and

$g_3 = \lambda\left[-g_2(\bar{v}) + w_2(t,x)\right] \in L^p(\Sigma)$,

the solution (Φ, ξ) to problem (28) exists and is unique. Furthermore, $\forall (v,\bar{v}) \in B^H$, $\forall\ \lambda \in [0,1]$,

$$(\Phi,\xi) = \Big(\Phi(v,\bar{v},\lambda), \xi(v,\bar{v},\lambda)\Big) \in W_p^{1,2}(Q) \times W_p^{1,2}(\Sigma). \tag{30}$$

Since $\mu_1 = \dfrac{p\,(n+2)}{n+2-2p} \geq p$ if $\dfrac{1}{p} - \dfrac{2}{n+2} > 0$ (see (22)), we can take $\mu_1 > p$ in all cases required by (22) and (26). Consequently, we have the continuous embeddings (see [17, p. 14])

$$\begin{cases} W_p^{1,2}(Q) \subset W_p^{0,1}(Q) \cap L^{3p}(Q) \subset L^p(Q) \\ W_p^{1,2}(\Sigma) \subset L^{p'}(\Sigma) \subset L^p(\Sigma), \end{cases} \tag{31}$$

which means that $H(v,\bar v,\lambda)=(\Phi,\xi)\in B^H$ for all $(v,\bar v)\in B^H$ and $\forall\ \lambda\in[0,1]$.

P2. Let us now show that H **is continuous and compact.** In this respect, we consider $v^n\to v$ in $W_p^{0,1}(Q)\cap L^{3p}(Q)$, $\bar v^n\to\bar v$ in $L^{p'}(\Sigma)$ and $\lambda^n\to\lambda$ in $[0,1]$. Using the notation

$$(\Phi^{n,\lambda_n},\xi^{n,\lambda_n})=H(v^n,\bar v^n,\lambda^n),\ (\Phi^{n,\lambda},\xi^{n,\lambda})=H(v^n,\bar v^n,\lambda),\ (\Phi^{\lambda},\xi^{\lambda})=H(v,\bar v,\lambda),$$

and considering the difference $H(v^n,\bar v^n,\lambda^n)-H(v^n,\bar v^n,\lambda)$, we obtain from (27) and (28)

$$\begin{cases}\dfrac{\partial}{\partial t}(\Phi^{n,\lambda_n}-\Phi^{n,\lambda})\\ \quad-\left[\lambda\dfrac{\partial}{\partial v^n_{x_j}}(\Psi(t,x,v^n,v^n_x)v^n_{x_i})+(1-\lambda)\delta_i^j\right](\Phi^{n,\lambda_n}_{x_ix_j}-\Phi^{n,\lambda}_{x_ix_j})\\ \quad=(\lambda_n-\lambda)\Big\{\Big[\dfrac{\partial}{\partial v^n_{x_j}}\Big(\Psi(t,x,v^n,v^n_x)v^n_{x_i}\Big)-\delta_i^j\Big]\Phi^{n,\lambda_n}_{x_ix_j}\\ \qquad+A_2(t,x,v^n,v^n_{x_i})+p_1\left[v^n-(v^n)^3\right]+\hat f_2(t,x)\Big\} & \text{in } Q\\ (\Phi^{n,\lambda_n}-\Phi^{n,\lambda})(t,x)=(\xi^{n,\lambda_n}-\xi^{n,\lambda})(t,x) & \text{on } \Sigma\\ (\Phi^{n,\lambda_n}-\Phi^{n,\lambda})(0,x)=(\lambda_n-\lambda)\varphi_0(x) & \text{in } \Omega\\ \dfrac{\partial}{\partial\mathbf n}(\Phi^{n,\lambda_n}-\Phi^{n,\lambda})+\dfrac{\partial}{\partial t}(\xi^{n,\lambda_n}-\xi^{n,\lambda})-\Delta_\Gamma(\xi^{n,\lambda_n}-\xi^{n,\lambda})\\ \qquad+c_0(\xi^{n,\lambda_n}-\xi^{n,\lambda})=(\lambda_n-\lambda)[-g_2(\bar v^n)+w_2(t,x)] & \text{on } \Sigma\\ (\xi^{n,\lambda_n}-\xi^{n,\lambda})(0,x)=(\lambda_n-\lambda)\xi_0(x) & \text{in } \Gamma.\end{cases} \tag{32}$$

Knowing that $\Phi^{n,\lambda_n}\in W_p^{1,2}(Q)$ and combining **Lemma 3.3** with relation $(12)_2$, we may conclude that the right-hand side in $(32)_1$ belongs to $L^p(Q)$. The embeddings $W_\infty^{2-\frac{2}{p}}(\Omega)\subset W_p^{2-\frac{2}{p}}(\Omega)$ and $W_\infty^{2-\frac{2}{p}}(\Sigma)\subset W_p^{2-\frac{2}{p}}(\Sigma)$ allow us to apply **Lemma 7.4** in Choban and Moroşanu [9, p. 114] with:

$$f_3=(\lambda_n-\lambda)\Big\{\Big[\tfrac{\partial}{\partial v^n_{x_j}}\Big(\Psi(t,x,v^n,v^n_x)v^n_{x_i}\Big)-\delta_i^j\Big]\Phi^{n,\lambda_n}_{x_ix_j}$$
$$+\left[A_2(t,x,v^n,v^n_{x_i})+p_1\left[v^n-(v^n)^3\right]+\hat f_2(t,x)\right]\Big\}\in L^p(Q),$$

$$g_3=(\lambda_n-\lambda)\left[-g_2(\bar v^n)+w_2(t,x)\right]\in L^p(\Sigma),$$

and so we get

$$\begin{aligned}
&\|\Phi^{n,\lambda_n}-\Phi^{n,\lambda}\|_{W_p^{1,2}(Q)}+\|\xi^{n,\lambda_n}-\xi^{n,\lambda}\|_{W_p^{1,2}(\Sigma)}\\
&\qquad\le C|\lambda_n-\lambda|\Bigg\{\bigg\|\Big[\frac{\partial}{\partial v^n_{x_j}}\Big(\Psi(t,x,v^n,v^n_x)v^n_{x_i}\Big)-\delta_i^j\Big]\Phi^{n,\lambda_n}_{x_ix_j}\bigg\|_{L^p(Q)}\\
&\qquad\quad+\|\varphi_0\|_{W_\infty^{2-\frac{2}{p}}(\Omega)}+\|\xi_0\|_{W_\infty^{2-\frac{2}{p}}(\Gamma)}\\
&\qquad\quad+\|A_2(t,x,v^n,v^n_{x_i})\|_{L^p(Q)}+p_1\|v^n-(v^n)^3\|_{L^p(\Omega)}\\
&\qquad\quad+\|\hat f_2\|_{L^p(Q)}+\|g_2(\bar v^n)\|_{L^p(\Sigma)}+\|w_2\|_{L^p(\Sigma)}\Bigg\},
\end{aligned}$$

for a positive constant C.

Owing to **Lemma 3.3**, we can conclude that $(v^n)^3$ is bounded in $L^p(Q)$, $\forall v^n\in W_p^{0,1}(Q)\cap L^{3p}(Q)$. Moreover, making use of inequality $(17)_2$ and knowing that $\Phi^{n,\lambda_n}_{x_ix_j}\in L^p(Q)$, we derive the boundedness in $L^p(Q)$ of

$$A_2(t,x,v^n,v^n_{x_i})\ \text{ and }\ \left[\frac{\partial}{\partial v^n_{x_j}}\Big(\Psi(t,x,v^n,v^n_x)v^n_{x_i}\Big)-\delta_i^j\right]\Phi^{n,\lambda_n}_{x_ix_j}.$$

Since $W_\infty^{2-\frac{2}{p}}(\Omega)\subset L^p(\Omega)$, it results that the remaining terms on the right-hand side from the above inequality are also bounded in $L^p(Q)$. Also, the sequence $\bar v^n$ is bounded in $L^{p'}(\Sigma)$, so that by (5) and (6) we derive the boundedness of $g_2(t,x,\bar v^n)$ in $L^p(\Sigma)$. Therefore, since $\lambda_n\to\lambda$, we obtain from the previous inequality

$$\|\Phi^{n,\lambda_n}-\Phi^{n,\lambda}\|_{W_p^{1,2}(Q)}+\|\xi^{n,\lambda_n}-\xi^{n,\lambda}\|_{W_p^{1,2}(\Sigma)}\to 0\quad\text{for } n\to\infty. \tag{33}$$

In order to evaluate the difference $H(v^n,\bar v^n,\lambda)-H(v,\bar v,\lambda)$, we will use again (27) and (28), so that

$$\begin{cases}
\dfrac{\partial}{\partial t}(\Phi^{n,\lambda}-\Phi^{\lambda})\\
-\left[\lambda\dfrac{\partial}{\partial v^n_{x_j}}(\Psi(t,x,v^n,v^n_x)v^n_{x_i})+(1-\lambda)\delta_i^j\right](\Phi^{n,\lambda}_{x_ix_j}-\Phi^{\lambda}_{x_ix_j})\\
=\lambda\Bigg\{\left[\dfrac{\partial}{\partial v^n_{x_j}}\Big(\Psi(t,x,v^n,v^n_x)v^n_{x_i}\Big)-\dfrac{\partial}{\partial v_{x_j}}\Big(\Psi(t,x,v,v_x)v_{x_i}\Big)\right]\Phi^{\lambda}_{x_ix_j}\\
+\Big[A_2(t,x,v^n,v^n_{x_i})-A_2(t,x,v,v_{x_i})\Big]+p_1\Big[(v^n-v)-\big((v^n)^3-v^3\big)\Big]\Bigg\}\qquad\text{in } Q,
\end{cases} \tag{34}$$

subject to the dynamic boundary conditions

$$\begin{cases}
(\Phi^{n,\lambda}-\Phi^{\lambda})(t,x)=(\xi^{n,\lambda}-\xi^{\lambda})(t,x),\\
\dfrac{\partial}{\partial \mathbf{n}}(\Phi^{n,\lambda}-\Phi^{\lambda})+\dfrac{\partial}{\partial t}(\xi^{n,\lambda}-\xi^{\lambda})-\Delta_\Gamma(\xi^{n,\lambda}-\xi^{\lambda})\\
\qquad+c_0(\xi^{n,\lambda}-\xi^{\lambda})=\lambda\Big[-g_2(\bar v^n)+g_2(\bar v)\Big]\qquad\text{on } \Sigma
\end{cases}$$

and the initial conditions

$$\begin{cases} (\Phi^{n,\lambda} - \Phi^{\lambda})(0,x) = 0 & \text{in } \Omega \\ (\xi^{n,\lambda} - \xi^{\lambda})(0,x) = 0 & \text{in } \Gamma. \end{cases}$$

Appling **Lemma 7.4** in Choban and Moroşanu [9, p. 114] to the linear in-homogeneous problem (34) with

$$f_3 = \lambda\Big\{\Big[\tfrac{\partial}{\partial v^n_{x_j}}\Big(\Psi(t,x,v^n,v^n_x)v^n_{x_i}\Big) - \tfrac{\partial}{\partial v_{x_j}}\Big(\Psi(t,x,v,v_x)v_{x_i}\Big)\Big]\Phi^{\lambda}_{x_i x_j}$$
$$+A_2(t,x,v^n,v^n_{x_i}) - A_2(t,x,v,v_{x_i}) + p_1\Big[(v^n - v) - \big((v^n)^3 - v^3\big)\Big]\Big\} \in L^p(Q),$$
$$g_3 = \lambda\left[-g_2(\bar{v}^n) + g_2(\bar{v})\right] \in L^p(\Sigma)$$

and $\varphi_0 = \xi_0 = 0$, $\lambda \in [0,1]$, we obtain

$$\|\Phi^{n,\lambda} - \Phi^{\lambda}\|_{W^{1,2}_p(Q)} + \|\xi^{n,\lambda} - \xi^{\lambda}\|_{W^{1,2}_p(\Sigma)}$$
$$\leq C\left\{\left\|\Big[\frac{\partial}{\partial v^n_{x_j}}\Big(\Psi(t,x,v^n,v^n_x)v^n_{x_i}\Big) - \frac{\partial}{\partial v_{x_j}}\Big(\Psi(t,x,v,v_x)v_{x_i}\Big)\Big]\Phi^{\lambda}_{x_i x_j}\right\|_{L^p(Q)}\right.$$
$$+\|A_2(t,x,v^n,v^n_{x_i}) - A_2(t,x,v,v_{x_i})\|_{L^p(Q)}$$
$$\left.+p_1\|(v^n - v) - \big((v^n)^3 - v^3\big)\|_{L^p(\Omega)} + \|g_2(\bar{v}^n) - g_2(\bar{v})\|_{L^p(\Sigma)}\right\},$$

for a positive constant C. Then, the convergences: $v^n \to v$ in $W^{0,1}_p(Q) \cap L^{3p}(Q)$, $\bar{v}^n \to \bar{v}$ in $L^{p'}(\Sigma)$, the continuity of the Nemytskij operator (see Moroşanu and Motreanu [24] and references therein) and the boundedness of the terms in right-hand side of above inequality helps us to conclude that

$$\|\Phi^{n,\lambda} - \Phi^{\lambda}\|_{W^{1,2}_p(Q)} + \|\xi^{n,\lambda} - \xi^{\lambda}\|_{W^{1,2}_p(\Sigma)} \to 0 \quad \text{for } n \to \infty. \tag{35}$$

Making use of the relations (33) and (35), we derive the continuity of the nonlinear operator H defined by (27). Moreover, the mapping H is compact, what can easily be seen by writting it as the composition

$$B^H \times [0,1] \to W^{1,2}_p(Q) \times W^{1,2}_p(\Sigma) \hookrightarrow B^H = W^{0,1}_p(Q) \cap L^{3p}(Q) \times L^{p'}(\Sigma),$$

where the second map is an compact inclusion due to Lions-Peeter embedding theorem (see Miranville and Moroşanu [17, p. 14] and references therein).

3.2. **The regularity of the solution** $\big(\Phi(t,x), \xi(t,x)\big)$**.** Now we establish the existence of a number $\delta > 0$ such that (see (27))

$$(\Phi,\xi,\lambda) \in B^H \times [0,1] \quad \text{with} \quad (\Phi,\xi) = H(\Phi,\xi,\lambda) \quad \Longrightarrow \quad \|(\Phi,\xi)\|_{B^H} < \delta. \tag{36}$$

The equality $(\Phi,\xi) = H(\Phi,\xi,\lambda)$ in (36) is equivalent to (see (24) and (28))

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi - \lambda \mathrm{div}\Big(\Psi(t,x,\Phi,\Phi_x)\nabla\Phi\Big) - (1-\lambda)\Delta\Phi \\ \qquad\qquad = \lambda\Big[p_1\left[\Phi(t,x) - \Phi^3(t,x)\right] + \hat{f}_2(t,x)\Big] & \text{in } Q \\ \Phi(t,x) = \xi(t,x) & \text{on } \Sigma \\ \Phi(0,x) = \lambda\varphi_0(x) & \text{on } \Omega \\ \dfrac{\partial}{\partial \mathbf{n}}\Phi + \dfrac{\partial}{\partial t}\xi - \Delta_\Gamma\xi + c_0\xi = \lambda\big[-g_2(\xi) + w_2(t,x)\big] & \text{on } \Sigma \\ \xi(0,x) = \lambda\xi_0(x) & x\in\Gamma. \end{cases} \tag{37}$$

Multiplying $(37)_1$ by $|\Phi(\tau,x)|^{3p-4}\Phi(\tau,x)$, integrating over $Q_t := (0,t)\times\Omega$, $t\in(0,T]$, we get

$$\begin{aligned} &\int_{Q_t} \frac{\partial}{\partial t}|\Phi(\tau,x)|^{3p-2}\, d\tau dx - \lambda\int_{Q_t} \mathrm{div}\Big(\Psi(\tau,x,\Phi,\Phi_x)\nabla\Phi\Big)|\Phi|^{3p-4}\Phi\, d\tau dx \\ &\quad -(1-\lambda)\int_{Q_t}\Delta\Phi\;|\Phi|^{3p-4}\Phi\, d\tau dx \\ &\qquad = \lambda p_1\int_{Q_t}\left[\Phi(\tau,x) - \Phi^3(\tau,x)\right]|\Phi(\tau,x)|^{3p-4}\Phi(\tau,x)\, d\tau dx \\ &\qquad\quad +\lambda\int_{Q_t}\hat{f}_2(\tau,x)|\Phi(\tau,x)|^{3p-4}\Phi(\tau,x)\, d\tau dx. \end{aligned} \tag{38}$$

In order to process the terms

$$\int_{Q_t}\mathrm{div}\big(\Psi(\tau,x,\Phi,\Phi_x)\nabla\Phi\big)|\Phi|^{3p-4}\Phi\, d\tau dx \quad\text{and}\quad \int_{Q_t}\Delta\Phi\;|\Phi|^{3p-4}\Phi\, d\tau dx,$$

we use Green's first identity and so we obtain

$$\begin{aligned} &-\lambda\int_{Q_t}\mathrm{div}\Big(\Psi(\tau,x,\Phi,\Phi_x)\nabla\Phi\Big)|\Phi|^{3p-4}\Phi\, d\tau dx \\ &= \lambda\int_{Q_t}\Psi(\tau,x,\Phi,\Phi_x)\nabla\Phi\cdot\nabla\big(|\Phi|^{3p-4}\Phi\big)\, d\tau dx + \lambda\int_{\Sigma_t}|\Phi|^{3p-4}\Phi\Big(-\frac{\partial}{\partial\mathbf{n}}\Phi\Big)d\tau d\gamma, \end{aligned} \tag{39}$$

$$\begin{aligned} &-(1-\lambda)\int_{Q_t}\Delta\Phi\;|\Phi|^{3p-4}\Phi\, d\tau dx \\ &= (1-\lambda)(p-1)\int_{Q_t}|\nabla\Phi|^2|\Phi|^{3p-4}d\tau dx + (1-\lambda)\int_{\Sigma_t}|\Phi|^{3p-4}\Phi\Big(-\frac{\partial}{\partial\mathbf{n}}\Phi\Big)d\tau d\gamma, \end{aligned} \tag{40}$$

where $\Sigma_t = (0,t)\times\partial\Omega$, $t\in(0,T]$ and

$$-\frac{\partial}{\partial \mathbf{n}}\Phi = \frac{\partial}{\partial t}\xi - \Delta_\Gamma \xi + c_0 \xi + \lambda g_2(\xi) - \lambda w_2(t,x)$$

(see $(37)_4$).

Combining the above equality with the boundary condition $(37)_2$ and, making use of the hypothesis $\mathbf{I}_2$, $\mathbf{g}_3$, as well as the relations (39), (40), then (38) leads us to the following inequality

$$\begin{aligned}
&\frac{1}{3p-2}\int_\Omega |\Phi(t,x)|^{3p-2}\, dx \\
&\quad +\lambda\frac{1}{3p-2}\int_\Gamma |\xi(t,x)|^{3p-2}\, d\gamma + (1-\lambda)\frac{1}{3p-2}\int_\Gamma |\xi(t,x)|^{3p-2}\, d\gamma \\
&\quad +\lambda\int_{Q_t} \Psi(\tau,x,\Phi,\Phi_x)\nabla\varphi\cdot\nabla\left(|\Phi|^{3p-4}\Phi\right)\, d\tau dx \\
&\quad +(1-\lambda)3(p-1)\int_{Q_t} |\nabla\Phi|^2|\Phi|^{3p-4}\, d\tau dx \\
&\quad +\lambda c_0\int_{\Sigma_t} |\xi|^{3p-2}\, d\tau d\gamma + (1-\lambda)c_0\int_{\Sigma_t} |\xi|^{3p-2}\, d\tau d\gamma \\
&\quad +\lambda\int_{\Sigma_t} \nabla_\Gamma\left(|\xi|^{3p-3}\right)\cdot\nabla_\Gamma\xi\, d\tau d\gamma \\
&\quad +(1-\lambda)\int_{\Sigma_t} \nabla_\Gamma\left(|\xi|^{3p-3}\right)\cdot\nabla_\Gamma\xi\, d\tau d\gamma + \lambda b_5\int_{\Sigma_t} |\xi|^{3p-2}\, d\tau d\gamma \\
&\leq \lambda\frac{1}{3p-2}\int_\Omega |\varphi_0(x)|^{3p-2}\, dx + \lambda\frac{1}{3p-2}\int_\Gamma |\xi_0(x)|^{3p-2} d\gamma \\
&\quad +(1-\lambda)\frac{1}{3p-2}\int_\Gamma |\xi_0(x)|^{3p-2} d\gamma + \lambda p_1\int_{Q_t} \left[\Phi-\Phi^3\right]|\Phi|^{3p-4}\Phi\, d\tau dx \\
&\quad +\lambda\int_{Q_t} \hat{f}_2|\Phi|^{3p-4}\Phi\, d\tau dx + \lambda\int_{\Sigma_t} w_2|\Phi|^{3p-4}\Phi\, d\tau d\gamma
\end{aligned} \tag{41}$$

for all $t\in(0,T]$. The Hölder and Cauchy inequalities, applied to the last terms in (41), give us

$\mathbf{i_1}$) $\displaystyle\int_{Q_t} \hat{f}_2|\Phi|^{3p-4}\Phi\, d\tau dx \leq \frac{p-1}{p}\varepsilon^{\frac{p}{p-1}}\int_{Q_t} |\Phi|^{3p}\, d\tau dx + \lambda\frac{1}{p}\varepsilon^{-p}\|\hat{f}_2\|^p_{L^p(Q)}$,

$\mathbf{i_2}$) $\displaystyle\lambda\int_{\Sigma_t} w_2|\Phi|^{3p-4}\Phi\, d\tau d\gamma \leq \frac{p-1}{p}\varepsilon^{\frac{p}{p-1}}\int_{\Sigma_t} |\Phi|^{3p}\, d\tau d\gamma + \lambda\frac{1}{p}\varepsilon^{-p}\int_{\Sigma_t} |w_2|^p\, d\tau d\gamma.$

By H_0 in [24, p. 189], relation (4) and Young's inequality, we obtain

$$\begin{aligned}
&\lambda p_1 \int\limits_{Q_t} \left[\Phi(\tau,x) - \Phi^3(\tau,x)\right] |\Phi(\tau,x)|^{3p-4}\Phi(\tau,x)\ d\tau dx \\
&\quad \le \lambda p_1 |\Omega| T + \lambda p_1 |\Omega| T \frac{1}{3p}\varepsilon^{-3p} \\
&\quad + \frac{3p-1}{3p}\varepsilon^{\frac{3p}{3p-1}} \int\limits_{Q_t} |\Phi(\tau,x)|^{3p}\ d\tau dx - \lambda p_1 \int\limits_{Q_t} |\Phi(\tau,x)|^{3p}\ d\tau dx.
\end{aligned}$$

Owing to the last three inequalities, from (41) we derive the following estimate

$$\begin{aligned}
&\frac{1}{3p-2}\left[\int\limits_{\Omega} |\Phi(t,x)|^{3p-2}\ dx + \int\limits_{\Gamma} |\xi(t,x)|^{3p-2}\ d\gamma\right] \\
&\quad + \lambda \int\limits_{Q_t} \Psi(\tau,x,\Phi,\Phi_x)\nabla\varphi\cdot\nabla\Big(|\Phi|^{3p-4}\Phi\Big)\ d\tau dx \\
&\quad + 3(1-\lambda)(p-1)\int\limits_{Q_t} |\nabla\Phi|^2|\Phi|^{3p-4}\ d\tau dx \\
&\quad + \lambda p_1 \int\limits_{Q_t} |\Phi(\tau,x)|^{3p}\ d\tau dx \\
&\quad + \left[c_0 + \lambda b_5\right]\int\limits_{\Sigma_t} |\xi|^{3p-2}\ d\tau d\gamma + \int\limits_{\Sigma_t} \nabla_\Gamma\Big(|\xi|^{3p-3}\Big)\cdot\nabla_\Gamma\xi\ d\tau d\gamma \\
&\le \frac{1}{3p-2}\left[\int\limits_{\Omega} |\Phi_0(x)|^{3p-2}\ dx + \int\limits_{\Gamma} |\xi_0(x)|^{3p-2} d\gamma\right] \\
&\quad + \left[\frac{3p-1}{3p}\varepsilon^{\frac{3p}{3p-1}} + 2\frac{p-1}{p}\varepsilon^{\frac{p}{p-1}}\right]\int\limits_{Q_t} |\Phi(\tau,x)|^{3p}\ d\tau dx \\
&\quad + \lambda p_1 |\Omega| T\left[1 + \frac{1}{3p}\varepsilon^{-3p}\right] + \frac{1}{p}\varepsilon^{-p}\|\hat{f}_2\|^p_{L^p(Q)} + \frac{1}{p}\varepsilon^{-p}\|w_2\|^{3p-2}_{L^{3p-2}(\Sigma_t)}
\end{aligned} \tag{42}$$

for all $t \in (0,T]$.

Taking ε small enough, inequality (42) yields

$$\begin{aligned}
&\lambda \||\Phi|^3\|^p_{L^p(Q)} \\
&\quad \le C_1\Big(1 + \|\varphi_0\|^{3p-2}_{L^{3p-2}(\Omega)} + \|\xi_0\|^{3p-2}_{L^{3p-2}(\Gamma)} + \|\hat{f}_2\|^p_{L^p(Q)} + \|w_2\|^{3p-2}_{L^{3p-2}(\Sigma_t)}\Big),
\end{aligned} \tag{43}$$

for a positive constant $C_1 = C(|\Omega|, T, n, p, p_1)$.

Further on, due to (43) and making use of the embedding $L^{3p-2}(\Sigma) \subset L^p(\Sigma)$ (see (4)), we deduce from (42) that

$$\begin{aligned} \|\xi\|^p_{L^p(\Sigma)} &\le C_2\|\xi\|^{3p-2}_{L^{3p-2}(\Sigma)} \\ &\le C_2\left(1+\|\varphi_0\|^{3p-2}_{L^{3p-2}(\Omega)}+\|\xi_0\|^{3p-2}_{L^{3p-2}(\Gamma)}+\|\hat{f}_2\|^p_{L^p(Q)}+\|w_2\|^{3p-2}_{L^{3p-2}(\Sigma_t)}\right), \end{aligned} \tag{44}$$

where $C_2 = C(|\Omega|, |\Gamma|, T, n, p, p_1, c_0, b_5) > 0$ denotes a new positive constant.

Moreover, using (43) and **Lemma 1.1** in Moroşanu and Motreanu [24], we get

$$\begin{aligned} &\|\Phi-\Phi^3\|_{L^p(Q)} \\ &\le C_1\left(1+\|\varphi_0\|^{\frac{3p-2}{p}}_{L^{3p-2}(\Omega)}+\|\xi_0\|^{\frac{3p-2}{p}}_{L^{3p-2}(\Gamma)}+\|\hat{f}_2\|_{L^p(Q)}+\|w_2\|^{\frac{3p-2}{p}}_{L^{3p-2}(\Sigma)}\right). \end{aligned} \tag{45}$$

Appling **Lemma 7.4** in Choban and Moroşanu [9, p. 114] to the linear inhomogeneous problem (37) with $K(t, x, v_x) = 1$,

$$f_3 = \lambda\left\{p_1\left[\Phi-\Phi^3\right]+\hat{f}_2(t,x)\right\} \in L^p(Q),$$

$$g_3 = \lambda\left[-g_2(\xi)+w_2(t,x)\right] \in L^p(\Sigma),$$

in conjunction with the embeddings $W^{2-\frac{2}{p}}_\infty(\Omega) \subset W^{2-\frac{2}{p}}_p(\Omega)$, $W^{2-\frac{2}{p}}_\infty(\Gamma) \subset W^{2-\frac{2}{p}}_p(\Gamma)$, we obtain

$$\begin{aligned} &\|\Phi\|_{W^{1,2}_p(Q)}+\|\xi\|_{W^{1,2}_p(\Sigma)} \\ &\quad\le C_3\left\{\|\varphi_0\|_{W^{2-\frac{2}{p}}_\infty(\Omega)}+\|\xi_0\|_{W^{2-\frac{2}{p}}_\infty(\Gamma)}+\lambda p_1\|\Phi-\Phi^3\|_{L^p(\Omega)}\right. \\ &\quad\quad\left.+\|\hat{f}_2\|_{L^p(Q)}+\lambda\|g_2(\xi)\|_{L^p(\Sigma)}+\|w_2\|_{L^p}(\Sigma)\right\}, \end{aligned} \tag{46}$$

for a constant $C_3 = C(|\Omega|, |\Gamma|, T, n, p, p_1) > 0$.

Using now (6) $(k = 2)$ and (45), then (46) becomes

$$\begin{aligned} &\|\Phi\|_{W^{1,2}_p(Q)}+\|\xi\|_{W^{1,2}_p(\Sigma)} \\ &\le C_4\left\{1+\|\varphi_0\|_{W^{2-\frac{2}{p}}_\infty(\Omega)}+\|\xi_0\|_{W^{2-\frac{2}{p}}_\infty(\Gamma)}+\|\varphi_0\|^{\frac{3p-2}{p}}_{L^{3p-2}(\Omega)}+\|\xi_0\|^{\frac{3p-2}{p}}_{L^{3p-2}(\Gamma)}\right. \\ &\quad\left.+\|\hat{f}_2\|_{L^p(Q)}+\|\xi\|_{L^{p'}(\Sigma)}+\|w_2\|_{L^p}(\Sigma)\right\}, \end{aligned} \tag{47}$$

for a constant $C_4 = C(|\Omega|, |\Gamma|, T, n, p, p_1, b_6) > 0$.

Owing to the embeddings (see and $(31)_2$)

$$W^{1,2}_p(\Sigma) \subset L^{p'}(\Sigma) \subset L^p(\Sigma),$$

a standard interpolation inequality (see [17, p. 14, (1.31)]) yields that $\forall\epsilon > 0$, $\exists C(\epsilon) > 0$ such that

$$\|\xi\|_{L^{p'}(\Sigma)} \le \epsilon\|\xi\|_{W^{1,2}_p(\Sigma)} + C(\epsilon)\|\xi\|_{L^p(\Sigma)},$$

and thus from (47), we get

$$\begin{aligned} &\|\Phi\|_{W_p^{1,2}(Q)} + (1-\varepsilon C_4)\|\xi\|_{W_p^{1,2}(\Sigma)} \\ &\quad \le C_5\Big\{1 + \|\varphi_0\|_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|\xi_0\|_{W_\infty^{2-\frac{2}{p}}(\Gamma)} + \|\varphi_0\|_{L^{3p-2}(\Omega)}^{\frac{3p-2}{p}} + \|\xi_0\|_{L^{3p-2}(\Gamma)}^{\frac{3p-2}{p}} \\ &\quad + \|\hat{f}_2\|_{L^p(Q)} + \|\xi\|_{L^p(\Sigma)} + \|w_2\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\Big\}, \end{aligned} \tag{48}$$

for a new constant $C_5 = C(\epsilon)C_4 > 0$. We specify that, in writing (48), we have used the embedding $W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma) \subset L^p(\Sigma)$.

The continuous embedding in (31) ensures that

$$\|\Phi\|_{L^p(Q)} + \|\xi\|_{L^p(\Sigma)} \le C\Big(\|\Phi\|_{W_p^{1,2}(Q)} + \|\xi\|_{W_p^{1,2}(\Sigma)}\Big)$$

wherefrom, for $\epsilon > 0$ with $1 - \varepsilon C_4 > 0$, and thanks to (44) and (48), we may conclude that a constant $\delta > 0$ can be found such that the property expressed in (36) is true.

Denoting $B_\delta^H := \Big\{(\Phi,\xi) \in B^H : \ \|(\Phi,\xi)\|_{B^H} < \delta\Big\}$, relation (27) implies that

$$(\Phi,\xi,\lambda) \neq (\Phi,\xi) \qquad \forall(\Phi,\xi) \in \partial B_\delta^H, \ \forall\lambda \in [0,1],$$

provided that $\delta > 0$ is sufficiently large. Furthermore, following the same reasoning as in [6], we conclude that problem (15) has a solution $(\Phi,\xi) \in W_p^{1,2}(Q) \times W_p^{1,2}(\Sigma)$ (for more details, see [20, p. 195]). Estimate (25) follows from (48) combined with (44). This completes the proof of the Theorem 3.2. □

Remark 3. The nonlinear operator H in (27) depends on $\lambda \in [0,1]$ and its fixed point for $\lambda = 1$ are solutions of problem (28).

4. The validity of the nonlinear second-order reaction-diffusion problem (14)-(15) in the class $W_p^{1,2}(Q) \times W_p^{1,2}(\Sigma)$, $W_\nu^{1,2}(Q) \times W_p^{1,2}(\Sigma)$.

Proof. In this section we will apply the Leray-Schauder principle in order to prove the first part of the result about problems (14)-(15) established by Theorem 2.2. On this line, taking positive integers p, p' as in (4) and (26), we consider the Banach space

$$B^S = W_p^{0,1}(Q) \times L^{p'}(\Sigma),$$

endowed with the norm $\|\cdot\|_{B^S}$, expresed by

$$\|(y,\bar{y})\|_{B^S} = \|y\|_{L^p(Q)} + \|y_x\|_{L^p(Q)} + \|\bar{y}\|_{L^{p'}(\Sigma)},$$

and a nonlinear operator $S : B^S \times [0,1] \to B^S$ defined by

$$(u,\alpha) = S(y,\bar{y},\lambda) = \Big(u(y,\bar{y},\lambda), \alpha(y,\bar{y},\lambda)\Big), \qquad \forall(y,\bar{y}) \in B^S, \ \forall\lambda \in [0,1], \tag{49}$$

where (u, α) is the unique solution to the following linear boundary value problem (see (14))

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) - \left[\lambda \dfrac{\partial}{\partial y_{x_j}}(K(t,x,y,y_x)y_{x_i}) - (1-\lambda)\delta_i^j\right] u_{x_i x_j} \\ \qquad = \lambda\Big[A_1(t,x,y,y_{x_i}) - \dfrac{\ell}{2}\dfrac{\partial}{\partial t}\Phi(t,x) + f_1(t,x)\Big] & \text{in } Q \\ u(t,x) = \alpha(t,x) & \text{on } \Sigma \\ \dfrac{\partial}{\partial \mathbf{n}}u + \dfrac{\partial}{\partial t}\alpha - \Delta_\Gamma \alpha + h\alpha = \lambda\big[-g_1(\bar{y}) + w_1(t,x)\big] & \text{on } \Sigma \\ u(0,x) = \lambda u_0(x) & \text{on } \Omega \\ \alpha(0,x) = \lambda\alpha_0(x) & x \in \Gamma, \end{cases} \tag{50}$$

and Φ represents the unique solution to the nonlinear parabolic boundary value problem (24) corresponding to $\hat{f}_2(t,x) = p_2 y(t,x) + f_2(t,x)$, i.e.

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi(t,x) - \dfrac{\partial}{\partial \Phi_{x_j}}\Big(\Psi(t,x,\Phi,\Phi_x)\Phi_{x_i}\Big)\Phi_{x_j x_i} \\ \qquad = A_2(t,x,\Phi,\Phi_{x_i}) + p_1\left[\Phi - \Phi^3\right] + p_2 y(t,x) + f_2(t,x) & \text{in } Q, \\ \Phi(t,x) = \xi(t,x) & \text{on } \Sigma \\ \dfrac{\partial}{\partial \mathbf{n}}\Phi + \dfrac{\partial}{\partial t}\xi - \Delta_\Gamma \xi + c_0\xi + g_2(\xi) = w_2(t,x) & \text{on } \Sigma \\ \Phi(0,x) = \varphi_0(x) & \text{on } \Omega \\ \xi(0,x) = \xi_0(x) & x \in \Gamma. \end{cases} \tag{51}$$

Let us recall that

$$f_1(t,x) \in L^p(Q),\ f_2(t,x) \in L^q(Q) \ \text{ and } \ w_1(t,x),\ w_2(t,x) \in W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)$$

are given functions, while p and q satisfy the relation (4).

4.1. **The properties of the homotopy** S **in** (49)**.** S **is well-defined.** Making use of (49), i.e. $y \in W_p^{0,1}(Q) \subset L^p(Q)$, and owing to the embedding $L^q(Q) \subset L^p(Q)$ (see (4)), we have that $p_2 y + f_2 \in L^p(Q)$. Applying Theorem 3.2 to the nonlinear parabolic problem (51), we deduce that there exists a unique solution $(\Phi, \xi) \in W_p^{1,2}(Q) \times W_p^{1,2}(\Sigma)$, which entitles us to conclude that $-\dfrac{\ell}{2}\dfrac{\partial}{\partial t}\Phi + f_1(t,x) \in L^p(Q)$. Now, using the assumptions $\mathbf{I}_1$, $\mathbf{I}_2$ and (8) (see **Lemma 3.3**), from the right-hand side of (50) it follows that $\forall y \in W_p^{0,1}(Q)$, then

$$\lambda\left[A_1(t,x,y,y_{x_i}) - \frac{\ell}{2}\frac{\partial}{\partial t}\Phi(t,x) + f_1(t,x)\right] \in L^p(Q).$$

Next, according to (6) ($k = 1$) we have that $g_1(\bar{y}) \in L^{\frac{p'}{r'}}(\Sigma)$ whenever $\bar{y} \in L^{p'}(\Sigma)$, i.e. $g_1(\bar{y}) \in L^p(\Sigma)$ (see (5)). Hence $\lambda\big[-g_1(\bar{y}) + w_1(t,x)\big] \in L^p(\Sigma)$, $\forall \bar{y} \in L^{p'}(\Sigma)$ (we have used the embedding $W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma) \subset L^p(\Sigma)$). Applying Theorem 2.1

in Choban and Moroşanu [9, p. 98], we can conclude that the solution (u, α) to the second-order boundary value problem (50) exists and is unique. Furthermore, $\forall (y, \bar{y}) \in W_p^{0,1}(Q) \times L^{p'}(\Sigma)$ and $\forall\ \lambda \in [0, 1]$,

$$(u, \alpha) = \Big(u(y, \bar{y}, \lambda), \alpha(y, \bar{y}, \lambda)\Big) = S(y, \bar{y}, \lambda) \in W_p^{1,2}(Q) \times W_p^{1,2}(\Sigma). \tag{52}$$

Owing to the continuous inclusions (see [17, p. 14])

$$W_p^{1,2}(Q) \subset W_p^{0,1}(Q) \subset L^p(Q) \tag{53}$$

and making use of $(31)_2$, we derive that $S(y, \bar{y}, \lambda) = (u, \alpha) \in W_p^{0,1}(Q) \times L^{p'}(\Sigma)$ for all $(y, \bar{y}) \in W_p^{0,1}(Q) \times L^{p'}(\Sigma)$ and $\forall\ \lambda \in [0, 1]$, which express that the mapping introduced in (49) is well defined.

The following lemmas highlight continuity and compactness properties of the nonlinear operator Φ in (49).

Lemma 4.1. *The mapping S in (49) has the following properties:*

i. *$S(\cdot, \lambda) : B^S \to B^S$ is compact for every $\lambda \in [0, 1]$, i.e., it is continuous and maps bounded sets into relatively compact sets;*

ii. *for every $\varepsilon > 0$ and every bounded set $A \subset W_p^{0,1}(Q)$, $\bar{A} \subset L^{p'}(\Sigma)$, there exists $\delta > 0$ such that*

$$\|S(y, \bar{y}, \lambda_n) - S(y, \bar{y}, \lambda)\|_{B^S} < \varepsilon$$

whenever $y \in A$, $\bar{y} \in \bar{A}$ and $|\lambda_n - \lambda| < \delta$.

Proof. **i.** Let us check the continuity of $S(\cdot, \lambda)$, $\forall \lambda \in [0, 1]$, at the point $(y, \bar{y}) \in B^S$. Let $y^n \to y$ in $W_p^{0,1}(Q)$, $\bar{y}^n \to \bar{y}$ in $L^{p'}(\Sigma)$ and we set $(u^{n,\lambda}, \alpha^{n,\lambda}) = S(y^n, \bar{y}^n, \lambda)$, $(u^\lambda, \alpha^\lambda) = S(y, \bar{y}, \lambda)$, for any $(y^n, \bar{y}^n), (y, \bar{y}) \in B^S$. Relation (49) and problem (50) enables us to write the difference $S(y^n, \bar{y}^n, \lambda) - S(y, \bar{y}, \lambda)$ as follows

$$\begin{cases} \dfrac{\partial}{\partial t}(u^{n,\lambda} - u^\lambda) \\ \quad - \Big[\lambda \dfrac{\partial}{\partial y^n_{x_j}}(K(t, x, y^n, y^n_x) y^n_{x_i}) + (1 - \lambda)\delta_i^j\Big](u^{n,\lambda}_{x_i x_j} - u^\lambda_{x_i x_j}) \\ \quad = \lambda \Big\{\Big[\dfrac{\partial}{\partial y^n_{x_j}}\Big(K(t, x, y^n, y^n_x) y^n_{x_i}\Big) - \dfrac{\partial}{\partial y_{x_j}}\Big(K(t, x, y, y_x) y_{x_i}\Big)\Big] u^\lambda_{x_i x_j} \\ \qquad + A_1(t, x, y^n, y^n_{x_i}) - A_1(t, x, y, y_{x_i}) - \dfrac{\ell}{2}\dfrac{\partial}{\partial t}(\Phi^n - \Phi)\Big\} \qquad \text{in } Q, \end{cases} \tag{54}$$

subject to the dynamic boundary conditions

$$\begin{cases} (u^{n,\lambda} - u^\lambda)(t, x) = (\alpha^{n,\lambda} - \alpha^\lambda)(t, x), \\ \dfrac{\partial}{\partial \mathbf{n}}(u^{n,\lambda} - u^\lambda) + \dfrac{\partial}{\partial t}(\alpha^{n,\lambda} - \alpha^\lambda) - \Delta_\Gamma(\alpha^{n,\lambda} - \alpha^\lambda) + h(\alpha^{n,\lambda} - \alpha^\lambda) \\ \quad = \lambda\big[-g_1(\bar{y}^n) + g_1(\bar{y})\big] \qquad \text{on } \Sigma \end{cases}$$

and the initial conditions

$$\begin{cases}(u^{n,\lambda}-u^{\lambda})(0,x)=0 & \text{in } \Omega\\ (\alpha^{n,\lambda}-\alpha^{\lambda})(0,x)=0 & \text{in } \Gamma.\end{cases}$$

The right-hand side in $(54)_1$ belongs to $L^p(Q)$, since $u^{\lambda}_{x_ix_j}\in W^{1,2}_p(Q)\subset L^p(Q)$. The embeddings $W^{2-\frac{2}{p}}_{\infty}(\Omega)\subset W^{2-\frac{2}{p}}_{p}(\Omega)$ and $W^{2-\frac{2}{p}}_{\infty}(\Sigma)\subset W^{2-\frac{2}{p}}_{p}(\Sigma)$ allow us to apply **Lemma 7.4** in Choban and Moroşanu [9, p. 114] with:

$$f_3=\lambda\Big\{\Big[\tfrac{\partial}{\partial y^n_{x_j}}\Big(K(t,x,y^n,y^n_x)y^n_{x_i}\Big)-\tfrac{\partial}{\partial y_{x_j}}\Big(K(t,x,y,y_x)y_{x_i}\Big)\Big]u^{\lambda}_{x_ix_j}$$
$$+A_1(t,x,y^n,y^n_{x_i})-A_1(t,x,y,y_{x_i})-\tfrac{\ell}{2}\tfrac{\partial}{\partial t}\left(\Phi^n-\Phi\right)\Big\}\in L^p(Q),$$

$$g_3=\lambda\left[-g_1(\bar{y}^n)+g_1(\bar{y})\right]\in L^p(\Sigma),$$

and so we get

$$\begin{aligned}&\|u^{n,\lambda}-u^{\lambda}\|_{W^{1,2}_p(Q)}+\|\alpha^{n,\lambda}-\alpha^{\lambda}\|_{W^{1,2}_p(\Sigma)}\\
&\le C\Big\{\Big\|\Big[\frac{\partial}{\partial y^n_{x_j}}\Big(K(t,x,y^n,y^n_x)y^n_{x_i}\Big)-\frac{\partial}{\partial y_{x_j}}\Big(K(t,x,y,y_x)y_{x_i}\Big)\Big]u^{\lambda}_{x_ix_j}\Big\|_{L^p(Q)}\\
&\quad+\|A_1(t,x,y^n,y^n_{x_i})-A_1(t,x,y,y_{x_i})\|_{L^p(Q)}+\frac{\ell}{2}\Big\|\frac{\partial}{\partial t}\left(\Phi^n-\Phi\right)\Big\|_{L^p(Q)}\\
&\quad+\|g_1(\bar{y}^n)-g_1(\bar{y})\|_{L^p(\Sigma)}\Big\},\end{aligned}\tag{55}$$

for a positive constant $C(|\Omega|,|\Gamma|,T,n,p)$.

Applying now Theorem 3.2 to problem (51) with $(\Phi^n-\Phi,\xi^n-\xi)$ in place of (Φ,ξ) and $\varphi^n_0-\varphi_0=0$, $\xi^n_0-\xi_0=0$, $w^n_2-w_2=0$, $\hat{f}_2(t,x)=p_2\left[y^n(t,x)-y(t,x)\right]$, we get (see (49))

$$\|\Phi^n-\Phi\|_{W^{1,2}_p(Q)}+\|\xi^n-\xi\|_{W^{1,2}_p(\Sigma)}\le C_1\|y^n-y\|_{L^p(Q)},\tag{56}$$

for a constant $C_1=C(|\Omega|,T,\ell,p_2)>0$. Combining relations (55) and (56) we obtain the estimate

$$\begin{aligned}&\|u^{n,\lambda}-u^{\lambda}\|_{W^{1,2}_p(Q)}+\|\alpha^{n,\lambda}-\alpha^{\lambda}\|_{W^{1,2}_p(\Sigma)}\\
&\le C\Big\{\Big\|\Big[\frac{\partial}{\partial y^n_{x_j}}\Big(K(t,x,y^n,y^n_x)y^n_{x_i}\Big)-\frac{\partial}{\partial y_{x_j}}\Big(K(t,x,y,y_x)y_{x_i}\Big)\Big]u^{\lambda}_{x_ix_j}\Big\|_{L^p(Q)}\\
&\quad+\|A_1(t,x,y^n,y^n_{x_i})-A_1(t,x,y,y_{x_i})\|_{L^p(Q)}+C_1\|y^n-y\|_{L^p(Q)}\\
&\quad+\|g_1(\bar{y}^n)-g_1(\bar{y})\|_{L^p(\Sigma)}\Big\},\end{aligned}\tag{57}$$

Then, the convergence $y^n\to y$ in $W^{0,1}_p(Q)$, the continuity of $A_1(t,x,y^n,y^n_{x_i})$ and $\dfrac{\partial}{\partial y^n_{x_j}}\Big(K(t,x,y^n,y^n_x)y^n_{x_i}\Big)$, as well as the continuity of the Nemytskii operator, combined with the inequality (57), permit us to conclude that

$$\|u^{n,\lambda}-u^{\lambda}\|_{W^{1,2}_p(Q)}+\|\alpha^{n,\lambda}-\alpha^{\lambda}\|_{W^{1,2}_p(\Sigma)}\to 0\quad\text{for } n\to\infty.\tag{58}$$

Making use of the continuous embedding $W_p^{1,2}(Q) \subset W_p^{0,1}(Q)$ and relations (56), (58), we derive the continuity of the map $S(\cdot,\lambda)$ at $(y,\bar{y}) \in B^S$, for each $\lambda \in [0,1]$. Furthermore, Φ is compact. Indeed, since $\mu_1 > p$ (see (22)), the inclusion $W_p^{1,2}(Q) \hookrightarrow W_p^{0,1}(Q)$ is compact (see [17, p. 14]). Owing that $S(\cdot,\lambda)$ is expressed as the composition (see (53))

$$B^S = W_p^{0,1}(Q) \times L^{p'}(\Sigma) \to W_p^{1,2}(Q) \times W_p^{1,2}(\Sigma) \hookrightarrow W_p^{0,1}(Q) \times L^{p'}(\Sigma) = B^S, \quad (59)$$

where $W_p^{1,2}(Q) \times W_p^{1,2}(\Sigma) \hookrightarrow W_p^{0,1}(Q) \times L^{p'}(\Sigma)$ is a compact inclusion due to the Lions-Peetre embedding Theorem (see [17, p. 14]), the compactness of $S(\cdot,\lambda)$ follows.

ii. Let us fix $\varepsilon > 0$, $A \subset W_p^{0,1}(Q)$ and $\bar{A} \subset L^{p'}(\Sigma)$. Consider $(u^n, \alpha^n, \lambda_n)$ and (u,α,λ) solving (50), corresponding to any $y \in A$ and any $\bar{y} \in \bar{A}$. We have

$$\begin{cases} \frac{\partial}{\partial t}(u^{n,\lambda_n} - u^{\lambda}) \\ \quad -\left[\lambda \frac{\partial}{\partial y_{x_j}}(K(t,x,y,y_x)y_{x_i}) + (1-\lambda)\delta_i^j\right](u^{n,\lambda_n}_{x_ix_j} - u^{\lambda}_{x_ix_j}) \\ \quad = (\lambda_n - \lambda)\Big\{\Big[\frac{\partial}{\partial y_{x_j}}\Big(K(t,x,y,y_x)y_{x_i}\Big) - \delta_i^j\Big] u^{n,\lambda_n}_{x_ix_j} \\ \qquad + A_1(t,x,y,y_{x_i}) + f_1(t,x)\Big\} - \frac{\ell}{2}\frac{\partial}{\partial t}(\lambda_n\Phi^n - \lambda\Phi) & \text{in } Q \\ (u^{n,\lambda_n} - u^{\lambda})(t,x) = (\alpha^{n,\lambda_n} - \alpha^{\lambda})(t,x), \\ \frac{\partial}{\partial \mathbf{n}}(u^{n,\lambda_n} - u^{\lambda}) + \frac{\partial}{\partial t}(\alpha^{n,\lambda_n} - \alpha^{\lambda}) - \Delta_{\Gamma}(\alpha^{n,\lambda_n} - \alpha^{\lambda}) + h(\alpha^{n,\lambda_n} - \alpha^{\lambda}) \\ \quad = (\lambda_n - \lambda)\big[-g_1(\bar{y}) + w_1(t,x)\big] & \text{on } \Sigma \\ (u^{n,\lambda_n} - u^{\lambda})(0,x) = (\lambda_n - \lambda)u_0(x) & \text{in } \Omega \\ (\alpha^{n,\lambda} - \alpha^{\lambda})(0,x) = (\lambda_n - \lambda)\alpha_0(x) & \text{in } \Gamma. \end{cases} \quad (60)$$

Applying now Theorem 3.2 to problem (51) with $(\Phi^n - \Phi, \xi^n - \xi)$ in place of (Φ,ξ) and $\varphi_0^n - \varphi_0 = 0$, $\xi_0^n - \xi_0 = 0$, $w_2^n - w_2 = 0$, $\hat{f}_2(t,x) = (\lambda_n - \lambda)\big[p_2 y(t,x) + f_2(t,x)\big]$, give us the estimate (see (49))

$$\|\Phi^n - \Phi\|_{W_p^{1,2}(Q)} + \|\xi^n - \xi\|_{W_p^{1,2}(\Sigma)} \le C_1|\lambda_n - \lambda|\|p_2 y + f_2\|_{L^p(Q)} \le C_2|\lambda_n - \lambda|, \quad (61)$$

where $C_2 = C(A, p_2, C_1) > 0$.

The right-hand side in $(60)_1$ belongs to $L^p(Q)$, since $u^{n,\lambda_n}_{x_ix_j} \in W_p^{1,2}(Q) \subset L^p(Q)$. Again, the embeddings $W_\infty^{2-\frac{2}{p}}(\Omega) \subset W_p^{2-\frac{2}{p}}(\Omega)$ and $W_\infty^{2-\frac{2}{p}}(\Sigma) \subset W_p^{2-\frac{2}{p}}(\Sigma)$ allow us to apply **Lemma 7.4** in Choban and Moroşanu [9, p. 114] with:

$$f_3 = (\lambda_n - \lambda)\Big\{\Big[\tfrac{\partial}{\partial y_{x_j}}\Big(K(t,x,y,y_x)y_{x_i}\Big) - \delta_i^j\Big] u^{n,\lambda_n}_{x_ix_j} \\ A_1(t,x,y,y_{x_i}) + f_1(t,x)\Big\} - \tfrac{\ell}{2}\tfrac{\partial}{\partial t}\big(\lambda_n\Phi^n - \lambda\Phi\big) \in L^p(Q),$$

$g_3 = (\lambda_n - \lambda)\left[-g_1(\bar{y}) + w_{_1}(t,x)\right] \in L^p(\Sigma)$,

and so we get

$$\begin{aligned} &\|u^{n,\lambda_n} - u^\lambda\|_{W_p^{1,2}(Q)} + \|\alpha^{n,\lambda_n} - \alpha^\lambda\|_{W_p^{1,2}(\Sigma)} \\ &\quad \leq C\Bigg\{\left\|\frac{\partial}{\partial t}\left(\lambda_n\Phi^n - \lambda\Phi\right)\right\|_{L^p(Q)} \\ &\qquad |\lambda_n - \lambda|\left\|\left[\frac{\partial}{\partial y_{x_j}}\Big(K(t,x,y,y_x)y_{x_i}\Big) - \delta_i^j\right]u^{n,\lambda_n}_{x_ix_j}\right\|_{L^p(Q)} \\ &\qquad +|\lambda_n - \lambda|\Bigg[\|A_1(t,x,y,y_{x_i})\|_{L^p(Q)} + \|f_{_1}\|_{L^p(Q)} \\ &\qquad +\|g_{_1}(\bar{y})\|_{L^p(\Sigma)} + \|w_1\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\Bigg\}, \end{aligned} \tag{62}$$

for a positive constant $C(|\Omega|, |\Gamma|, T, n, p, \ell)$.

We note that

$$\begin{aligned} &\left\|\frac{\partial}{\partial t}\left(\lambda_n\Phi^n - \lambda\Phi\right)\right\|_{L^p(Q)} \\ &\qquad = \left\|\frac{\partial}{\partial t}\left(\lambda_n\Phi^n - \lambda\Phi^n + \lambda\Phi^n - \lambda\Phi\right)\right\|_{L^p(Q)} \\ &\qquad = |\lambda_n - \lambda|\left\|\frac{\partial}{\partial t}\Phi^n\right\|_{L^p(Q)} + \lambda\left\|\frac{\partial}{\partial t}(\Phi^n - \Phi)\right\|_{L^p(Q)}. \end{aligned} \tag{63}$$

Estimate (25) in Theorem 3.2 ensures that $\left\|\frac{\partial}{\partial t}\Phi^n\right\|_{L^p(Q)}$ is uniformly bounded with respect to $y \in A$, because A is bounded. Also, $g_{_1}(\bar{y})$ is bounded in $L^p(\Sigma)$ because $\bar{y} \in \bar{A}$ and $\bar{A}$ is bounded in $L^{p'}(\Sigma) \subset L^p(\Sigma)$ (see relation (31)). Then, making use of (61) and (63), from (62) we conclude that there is a constant $C_3(|\Omega|, |\Gamma|, T, n, p, \ell, A, \bar{A}, C_2) > 0$ such that

$$\|u^{n,\lambda_n} - u^\lambda\|_{W_p^{1,2}(Q)} + \|\alpha^{n,\lambda_n} - \alpha^\lambda\|_{W_p^{1,2}(\Sigma)} \leq C_3|\lambda_n - \lambda|$$

which allows us to derive that the assertion **ii** is verified. □

4.2. **The regularity of the solution $\{(u,\alpha), (\varphi,\xi)\}$ in Theorem 2.2.** Now we establish the existence of a number $\delta_2 > 0$ such that (see (49))

$$(u,\alpha,\lambda) \in B^S \times [0,1] \quad \text{with} \quad (u,\alpha) = S(u,\alpha,\lambda) \implies \|(u,\alpha)\|_{B^S} < \delta_2. \tag{64}$$

The equality $(u,\alpha) = S(u,\alpha,\lambda)$ in (64) is equivalent to (see (7), (14) and (50))

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) - \lambda \operatorname{div}\Big[K(t,x,u,u_x)\nabla u(t,x)\Big] - (1-\lambda)\Delta u(t,x) \\ \qquad = \lambda\Big[-\dfrac{\ell}{2}\dfrac{\partial}{\partial t}\varphi(t,x) + f_1(t,x)\Big] & \text{in } Q \\ u(t,x) = \alpha(t,x) & \text{on } \Sigma \\ \dfrac{\partial}{\partial \mathbf{n}}u + \dfrac{\partial}{\partial t}\alpha - \Delta_\Gamma \alpha + h\alpha = \lambda\big[-g_1(\alpha) + w_1(t,x)\big] & \text{on } \Sigma \\ u(0,x) = \lambda u_0(x) & \text{on } \Omega \\ \alpha(0,x) = \lambda\alpha_0(x) & x \in \Gamma, \end{cases} \tag{65}$$

where $\varphi(t,x) = \Phi(t,x)$ is the unique solution to the nonlinear parabolic boundary value problem (51) with $\hat{f}_2(t,x) = p_2 u(t,x) + f_2(t,x)$, i.e.

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi(t,x) - \dfrac{\partial}{\partial \varphi_{x_j}}\Big[\Psi(t,x,\varphi,\varphi_x)\varphi_{x_i}\Big]\varphi_{x_j x_i} \\ \qquad = A_2(t,x,\varphi,\varphi_{x_i}) + p_1\left[\varphi - \varphi^3\right] + p_2 u(t,x) + f_2(t,x) & \text{in } Q \\ \varphi(t,x) = \xi(t,x) & \text{on } \Sigma \\ \dfrac{\partial}{\partial \mathbf{n}}\varphi + \dfrac{\partial}{\partial t}\xi - \Delta_\Gamma \xi + c_0\xi + g_2(\xi) = w_2(t,x) & \text{on } \Sigma \\ \varphi(0,x) = \varphi_0(x) & \text{on } \Omega \\ \xi(0,x) = \xi_0(x) & x \in \Gamma. \end{cases} \tag{66}$$

Theorem 3.2 guarantees that a unique solution $(\varphi,\xi) \in W_p^{1,2}(Q) \times W_p^{1,2}(\Sigma)$ exists for problem (66). In addition, the continuous inclusion $W_p^{1,2}(Q) \subset L^{\mu_1}(Q) \subset L^p(Q)$ (see (22)), as well as the estimate (25), imply that

$$\begin{aligned} \left\|\frac{\partial}{\partial t}\varphi\right\|_{L^p(Q)} &\le C\|\varphi\|_{W_p^{1,2}(Q)} \\ &\le \tilde{C}\left[1 + \|\varphi_0\|_{W_\infty^{2-\frac{2}{q}}(\Omega)}^{\frac{3p-2}{p}} + \|\xi_0\|_{W_\infty^{2-\frac{2}{p}}(\Gamma)}^{\frac{3p-2}{p}}\right. \\ &\qquad \left. + \|u\|_{L^p(Q)} + \|f_2\|_{L^p(Q)} + \|w_2\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\right], \end{aligned} \tag{67}$$

for a constant $\tilde{C} = C(|\Omega|, T, n, p, q, p_1, p_2, c_0, b_3, b_4, b_5, b_6) > 0$.

Multiplying (65))$_1$ by $|u|^{p-2}u$ and integrating over $Q_t := (0,t)\times\Omega$, $t\in(0,T]$, we get

$$\begin{aligned}&\frac{1}{p}\int\limits_{Q_t}\frac{\partial}{\partial t}|u(t,x)|^p\; d\tau dx\\ &\quad -\lambda\int\limits_{Q_t}\mathrm{div}\Big(K(t,x,u,u_x)\nabla u\Big)|u|^{p-2}u\; d\tau dx-(1-\lambda)\int\limits_{Q_t}\Delta u|u|^{p-2}u\; d\tau dx\\ &\quad = \lambda\int\limits_{Q_t}-\frac{\ell}{2}\frac{\partial}{\partial t}\varphi(t,x)|u|^{p-2}u\; d\tau dx+\lambda\int\limits_{Q_t}f_1|u|^{p-2}u\; d\tau dx.\end{aligned}\tag{68}$$

In order to process the terms

$$\int\limits_{Q_t}\mathrm{div}\big(K(t,x,u,u_x)\nabla u\big)|u|^{p-2}u\; d\tau dx\quad\text{and}\quad\int\limits_{Q_t}\Delta u\;|u|^{p-2}u\; d\tau dx,$$

we use Green's first identity (21)$_1$ and (21)$_2$, respectively, to obtain

$$\begin{aligned}&-\lambda\int\limits_{Q_t}\mathrm{div}\Big(K(t,x,u,u_x)\nabla u\Big)|u|^{p-2}u\; d\tau dx\\ &\qquad=\lambda\int\limits_{Q_t}K(t,x,u,u_x)\nabla u\cdot\nabla\Big(|u|^{p-2}u\Big)d\tau dx+\lambda\int\limits_{\Sigma_t}|u|^{p-2}u\Big(-\frac{\partial}{\partial\mathbf{n}}u\Big)d\tau d\gamma,\end{aligned}\tag{69}$$

$$\begin{aligned}&-(1-\lambda)\int\limits_{Q_t}\Delta u\;|u|^{p-2}u\; d\tau dx\\ &\qquad=(1-\lambda)(p-1)\int\limits_{Q_t}|\nabla u|^2|u|^{p-2}d\tau dx+(1-\lambda)\int\limits_{\Sigma_t}|u|^{p-2}u\Big(-\frac{\partial}{\partial\mathbf{n}}u\Big)d\tau d\gamma,\end{aligned}\tag{70}$$

where $\Sigma_t=(0,t)\times\partial\Omega$, $t\in(0,T]$ and (see (65)$_3$)

$$-\frac{\partial}{\partial\mathbf{n}}u=\frac{\partial}{\partial t}\alpha-\Delta_\Gamma\alpha+h\alpha+\lambda g_1(\alpha)-\lambda w_1(t,x).$$

Combining the above equality with the boundary condition (65)$_2$ and making use of the left inequality in (16)$_1$, the hypothesis $\mathbf{I}_2$, $(\mathbf{G})_3$ $(k=1)$, as well as the relations (69), (70), then (68) leads us to the following inequality

$$\begin{aligned}&\frac{1}{p}\int\limits_{\Omega}|u(t,x)|^p\; dx+\lambda\frac{1}{p}\int\limits_{\Gamma}|\alpha(t,x)|^p\; d\gamma+(1-\lambda)\frac{1}{p}\int\limits_{\Gamma}|\alpha(t,x)|^p\; d\gamma\\ &\quad+\lambda\int\limits_{Q_t}K(t,x,u,u_x)\nabla u\cdot\nabla\Big(|u|^{p-2}u\Big)d\tau dx+(1-\lambda)(p-1)\int\limits_{Q_t}|\nabla u|^2|u|^{p-2}d\tau dx\\ &\quad+\lambda h\int\limits_{\Sigma_t}|\alpha|^p\; d\tau d\gamma+(1-\lambda)h\int\limits_{\Sigma_t}|\alpha|^p\; d\tau d\gamma+\lambda b_5\int\limits_{\Sigma_t}|\alpha|^p\; d\tau d\gamma\end{aligned}\tag{71}$$

$$+\lambda\int_{\Sigma_t}\nabla_\Gamma\left(|u|^{p-1}\right)\cdot\nabla_\Gamma u\ d\tau d\gamma+(1-\lambda)\int_{\Sigma_t}\nabla_\Gamma\left(|u|^{p-1}\right)\cdot\nabla_\Gamma u\ d\tau d\gamma$$

$$\leq\lambda\frac{1}{p}\int_\Omega|u_0(x)|^p\ dx+\lambda\frac{1}{p}\int_\Gamma|\alpha_0(x)|^p d\gamma+(1-\lambda)\frac{1}{p}\int_\Gamma|\alpha_0(x)|^p d\gamma$$

$$+\lambda\int_{Q_t}-\frac{\ell}{2}\frac{\partial}{\partial t}\varphi(t,x)|u|^{p-2}ud\tau dx+\lambda\int_{Q_t}f_1|u|^{p-2}ud\tau dx+\lambda\int_{\Sigma_t}w_1|\alpha|^{p-2}\alpha\ d\tau d\gamma$$

for all $t\in(0,T]$. Hölder's and Cauchy's inequality, applied to the last terms in above inequality, give us

j$_1$. $\lambda\int_{Q_t}-\frac{\ell}{2}\frac{\partial}{\partial t}\varphi(t,x)|u|^{p-2}u\ d\tau dx\leq\frac{p-1}{p}\int_{Q_t}|u|^p\ d\tau dx+\lambda\frac{\ell}{2p}\int_{Q_t}\left|\frac{\partial}{\partial t}\varphi(t,x)\right|^p\ d\tau dx,$

j$_2$. $\lambda\int_{Q_t}f_1|u|^{p-2}u\ d\tau dx\leq\frac{p-1}{p}\int_{Q_t}|u|^p\ d\tau dx+\lambda\frac{1}{p}\int_{Q_t}|f_1|^p\ d\tau dx,$

j$_3$. $\lambda\int_{\Sigma_t}w_1|\alpha|^{p-2}\alpha\ d\tau d\gamma\leq\frac{p-1}{p}\int_{\Sigma_t}|\alpha|^p\ d\tau d\gamma+\lambda\frac{1}{p}\int_{\Sigma_t}|w_1|^p\ d\tau d\gamma.$

Combining **j**$_1$-**j**$_3$ and (67), after some simple computations in the left-hand side of (71), we get to the following estimate

$$\begin{aligned}&\int_\Omega|u(t,x)|^p\ dx+\int_\Gamma|\alpha(t,x)|^p\ d\gamma\\&+p\lambda\int_{Q_t}K(t,x,u,u_x)\nabla u\cdot\nabla\left(|u|^{p-2}u\right)d\tau dx+p(1-\lambda)(p-1)\int_{Q_t}|\nabla u|^2|u|^{p-2}d\tau dx\\&+ph\int_{\Sigma_t}|u|^p\ d\tau d\gamma+p\lambda b_5\int_{\Sigma_t}|\alpha|^p\ d\tau d\gamma+p\int_{\Sigma_t}\nabla_\Gamma\left(|u|^{p-1}\right)\cdot\nabla_\Gamma u\ d\tau d\gamma\\&\leq\int_\Omega|u_0(x)|^p\ dx+\int_\Gamma|\alpha_0(x)|^p d\gamma\\&+\frac{\ell}{2}\tilde{C}\left[1+\|\varphi_0\|^{\frac{3p-2}{p}}_{W^{2-\frac{2}{q}}_\infty(\Omega)}+\|\xi_0\|^{\frac{3p-2}{p}}_{W^{2-\frac{2}{p}}_\infty(\Gamma)}+\|f_2\|_{L^p(Q)}+\|w_2\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\right]\\&+\left[\frac{\ell}{2}\tilde{C}+3(p-1)\right]\int_{Q_t}|u|^p d\tau dx+(p-1)\int_{\Sigma_t}|\alpha|^p d\tau d\gamma+\int_{Q_t}|f_1|^p d\tau dx+\int_{\Sigma_t}|w_1|^p d\tau d\gamma\end{aligned}\tag{72}$$

for all $t\in(0,T]$.

In particular, it follows from (72) that

$$\begin{aligned}
&\int_{\Omega} |u(t,x)|^p \; dx + \int_{\Gamma} |\alpha(t,x)|^p \; d\gamma \\
&\leq C_4 \Bigg[1 + \|u_0\|^p_{L^p(\Omega)} + \|\alpha_0\|^p_{L^p(\Gamma)} + \|\varphi_0\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{q}}(\Omega)} + \|\xi_0\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{p}}(\Gamma)} \\
&\qquad + \|f_1\|_{L^p(Q)} + \|w_1\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)} + \|f_2\|_{L^p(Q)} + \|w_2\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)} \Bigg] \\
&\quad + C_4 \int_0^t \Big[\int_{\Omega} |u|^p dx + \int_{\Gamma} |\alpha|^p d\gamma \Big] d\tau
\end{aligned} \tag{73}$$

By Gronwall's lemma, from (73) we get

$$\begin{aligned}
&\|u\|_{L^p(Q)} + \|\alpha\|_{L^p(\Sigma)} \\
&\leq C(T, C_4) \Bigg[1 + \|u_0\|^p_{L^p(\Omega)} + \|\alpha_0\|^p_{L^p(\Gamma)} + \|\varphi_0\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{q}}(\Omega)} + \|\xi_0\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{p}}(\Gamma)} \\
&\qquad + \|f_1\|_{L^p(Q)} + \|w_1\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)} + \|f_2\|_{L^p(Q)} + \|w_2\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)} \Bigg]
\end{aligned} \tag{74}$$

Appling now **Lemma 7.4** in Choban and Moroşanu [9, p. 114] to the linear inhomogeneous problem (65) with

$$\begin{aligned}
f_3 &= \lambda \left[-\frac{\ell}{2} \frac{\partial}{\partial t} \varphi(t,x) + f_1(t,x) \right] \in L^p(Q), \\
g_3 &= \lambda \big[-g_1(\alpha) + w_1(t,x) \big] \in L^p(\Sigma),
\end{aligned} \tag{75}$$

we obtain

$$\begin{aligned}
&\|u\|_{W_p^{1,2}(Q)} + \|\alpha\|_{W_p^{1,2}(\Sigma)} \\
&\leq C_3 \Bigg\{ \|u_0\|_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|\alpha_0\|_{W_\infty^{2-\frac{2}{p}}(\Gamma)} \\
&\quad + \lambda \left\| \frac{\partial}{\partial t} \varphi \right\|_{L^p(Q)} + \lambda \|g_1(\alpha)\|_{L^p(\Sigma)} \\
&\quad + \|f_1\|_{L^p(Q)} + \|w_1\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)} \Bigg\},
\end{aligned}$$

for a constant $C_3 = C(|\Omega|, |\Gamma|, T, n, p, \ell) > 0$. Using now (6) ($k = 1$), (67) and the standard interpolation inequality (see [17, p. 14, (1.31)]) $\|\alpha\|_{L^{p'}(\Sigma)} \leq \epsilon \|\alpha\|_{W_p^{1,2}(\Sigma)} +$

$C(\epsilon)\|\alpha\|_{L^p(\Sigma)}$, then the above inequality becomes

$$\begin{aligned} &\|u\|_{W_p^{1,2}(Q)} + (1-\epsilon C_3)\|\alpha\|_{W_p^{1,2}(\Sigma)} \\ &\le C_4\Big\{\|u_0\|_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|\alpha_0\|_{W_\infty^{2-\frac{2}{p}}(\Gamma)} \\ &\quad +\tilde{C}\left[1+\|\varphi_0\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{q}}(\Omega)} + \|\xi_0\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{p}}(\Gamma)}\right. \\ &\quad \left.+\|f_2\|_{L^p(Q)} + \|w_2\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\right], \\ &\quad +\lambda\|u\|_{L^p(Q)} + \lambda\|\alpha\|_{L^p(\Sigma)} \\ &\quad +\|f_1\|_{L^p(Q)} + \|w_1\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\Big\}, \end{aligned} \tag{76}$$

for a new constant $C_5 = C(\epsilon)C_4 > 0$.

The continuous embedding $W_p^{1,2}(Q)\times W_p^{1,2}(\Sigma)\subset B^\Phi = W_p^{0,1}(Q)\times L^{p'}(\Sigma)$ ensures that

$$\|(u,\alpha)\|_{B^\Phi} \le C\Big(\|u\|_{W_p^{1,2}(Q)} + \|\alpha\|_{W_p^{1,2}(\Sigma)}\Big)$$

which, for $\epsilon > 0$ with $1-\epsilon C_4 > 0$, and thanks to (74) and (76), ensures that a constant $\delta_2 > 0$ can be found such that the property expressed in (64) is true.

Denoting $B^\Phi_{\delta_2} := \big\{(u,\alpha)\in B^\Phi : \ \|(u,\alpha)\|_{B^\Phi} < \delta_2\big\}$, Lemma 4.1, relation (64) and the homotopy invariance of the Leray-Schauder degree, enable us to conclude that problem (50) has a solution

$$(u,\alpha)\in W_p^{1,2}(Q)\times W_p^{1,2}(\Sigma)$$

which is determined by the unique solution $(\varphi,\xi)\in W_p^{1,2}(Q)\times W_p^{1,2}(\Sigma)$ to (66).

4.3. **The maximum regularity of the solution $\varphi(t,x)$ in Theorem 2.2.** Thanks to the embedding $W_p^{1,2}(Q)\subset L^{\mu_1}(Q)$ (see (22)), we can apply Theorem 3.2 for

$$\hat{f}(t,x) = p_2 u(t,x) + f_2(t,x) \in L^\nu(Q), \text{ where } \nu = \min\{q,\mu_1\}.$$

This ensures the existence of a solution

$$(\varphi,\xi)\in W_\nu^{1,2}(Q)\times W_p^{1,2}(\Sigma)$$

to problem (15). Corresponding, estimate (25) in Theorem 3.2 yields

$$\begin{aligned} \|\varphi\|_{W_\nu^{1,2}(Q)} + \|\xi\|_{W_p^{1,2}(\Sigma)} \le C &\left[1+\|\varphi_0\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{q}}(\Omega)} + \|\xi_0\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{p}}(\Gamma)}\right. \\ &\left. +\|u\|_{L^\nu(Q)} + \|f_2\|_{L^q(Q)} + \|w_2\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\right], \end{aligned} \tag{77}$$

where the embedding $L^q(Q)\subset L^\nu(Q)$ has been used too.

Lemma 7.4 in Choban and Moroşanu [9, p. 114] applied to the linear inhomogeneous problem (65) (see (75)), combined with the estimate (25) (expressed by (67)) and (74), implies the estimate

$$\begin{aligned}
&\|u\|_{W_p^{1,2}(Q)} + \|\alpha\|_{W_p^{1,2}(\Sigma)} \leq C_3\Big\{\|u_0\|_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|\alpha_0\|_{W_\infty^{2-\frac{2}{p}}(\Gamma)} \\
&\quad + \left\|\frac{\partial}{\partial t}\varphi\right\|_{L^p(Q)} + \|g_1(t,x,\alpha)\|_{L^p(\Sigma)} + \|f_1\|_{L^p(Q)} + \|w_1\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\Big\} \\
&\leq C_3\Big\{\|u_0\|_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|\alpha_0\|_{W_\infty^{2-\frac{2}{p}}(\Gamma)} \\
&\quad + \tilde{C}\Big[1 + \|\varphi_0\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{q}}(\Omega)} + \|\xi_0\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{p}}(\Gamma)} \\
&\qquad + \|u\|_{L^p(Q)} + \|f_2\|_{L^p(Q)} + \|w_2\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\Big] \\
&\quad + \|f_1\|_{L^p(Q)} + \|w_1\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\Big\},
\end{aligned} \tag{78}$$

for a new positive constant $C_5 = C(C_3, \ell, b_5, b_7)$.

On the basis of the interpolation inequality, written for the embeddings $W_p^{1,2}(Q) \subset L^\nu(Q) \subset L^p(Q)$, we have the following relation $(\varepsilon, C(\varepsilon) > 0)$

$$\|u\|_{L^\nu(Q)} \leq \varepsilon\|u\|_{W_p^{1,2}(Q)} + C(\varepsilon)\|u\|_{L^p(Q)}. \tag{79}$$

Adding (77)-(78) and making use of (76) and (79), we find that

$$\begin{aligned}
&(1-\varepsilon C_3)\,\|u\|_{W_p^{1,2}(Q)} + \|\varphi\|_{W_\nu^{1,2}(Q)} + (1-\varepsilon C_3)\,\|\alpha\|_{W_p^{1,2}(\Sigma)} + \|\xi\|_{W_p^{1,2}(\Sigma)} \\
&\leq C_5\Big[1 + \|u_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|\alpha_0\|_{W_p^{2-\frac{2}{p}}(\Gamma)} + \|\varphi_0\|^{\frac{3p-2}{p}}_{W_q^{2-\frac{2}{q}}(\Omega)} + \|\xi_0\|^{\frac{3p-2}{p}}_{W_p^{2-\frac{2}{p}}(\Gamma)} \\
&\quad + [C_3C(\varepsilon) + C_5]\,\|u\|_{L^p(Q)} \\
&\quad + \|f_1\|_{L^p(Q)} + \|f_2\|_{L^p(Q)} + \|w_1\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)} + \|w_2\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\Big].
\end{aligned} \tag{80}$$

Let us fix an $\varepsilon > 0$ such that $1 - \varepsilon C_3 > 0$. Then $C(\varepsilon)$ becomes a constant independent of ε and thus relation (80), combined with (74), implies estimate (18). This completes the proof of the first part in Theorem 2.2. □

4.4. **Proof of Theorem 2.2 continued.** In this subsection we prove the second part of Theorem 2.2 which comes down to checking the estimate (20) and the uniqueness of the solution to problem (1)-(3) (or (14)-(15)). For this goal we consider the solutions

$$u^1, u^2 \in W_p^{1,2}(Q), \quad \varphi^1, \varphi^2 \in W_\nu^{1,2}(Q), \quad \alpha^1, \alpha^2, \xi^1, \xi^2 \in W_p^{1,2}(\Sigma),$$

as in the statement of Theorem 2.2. Thus

$$U = u^1 - u^2 \in W_p^{1,2}(Q), \quad \Phi = \varphi^1 - \varphi^2 \in W_\nu^{1,2}(Q),$$

$$A = \alpha^1 - \alpha^2 \in W_p^{1,2}(\Sigma), \quad X = \xi^1 - \xi^2 \in W_p^{1,2}(\Sigma).$$

Subtracting the equations in (15)$_1$ corresponding to (φ^1, ξ^1), (φ^2, ξ^2) and then using the relation (25) in Theorem 3.2 corresponding to the settings

$$\hat{f}_2^a = p_2 u^1 + f_2^a,\ \hat{f}_2^b = p_2 u^2 + f_2^b \in L^\nu(Q),\ \ \varphi_0^1, \varphi_0^2,\ \ \xi_0^1, \xi_0^2,$$

as well as the embeddings $W_\nu^{1,2}(Q) \subset L^p(Q)$, $W_p^{1,2}(\Sigma) \subset L^p(\Sigma)$, we find that

$$\begin{aligned} &\left\|\frac{\partial}{\partial t}\Phi\right\|_{L^p(Q)} + \left\|\frac{\partial}{\partial t}X\right\|_{L^p(\Sigma)} \\ &\le C\Big[\|\Phi\|_{W_\nu^{1,2}(Q)} + \|X\|_{W_p^{1,2}(\Sigma)}\Big] \\ &\le C\Big[1 + \|\varphi_0^1 - \varphi_0^2\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{q}}(\Omega)} + \|\xi_0^1 - \xi_0^2\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{p}}(\Gamma)} \\ &\quad + \|U\|_{L^\nu(Q)} \\ &\quad + \|f_2^a - f_2^b\|_{L^\nu(Q)} + \|w_2^a - w_2^b\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\Big], \end{aligned} \tag{81}$$

for a new positive constant C.

Let us recall that

$$a_{ij}^1(t,x,u^1,u_x^1) = \frac{\partial}{\partial u_{x_j}} a_i^1(t,x,u^1,u_x^1) = \frac{\partial}{\partial u_{x_j}} K(t,x,u^1,u_x^1)u_{x_i}^1,\ i = 1,...,n;$$

$$a_{ij}^1(t,x,u^2,u_x^2) = \frac{\partial}{\partial u_{x_j}} a_i^1(t,x,u^2,u_x^2) = \frac{\partial}{\partial u_{x_j}} K(t,x,u^2,u_x^2)u_{x_i}^2,\ i = 1,...,n.$$

Following [2, p. 176], [9, p. 107], [11, p. 2268], [14, p. 137], [18, p. 241] and [26, 11] we write the increments of a_{ij}^1 and A_1 in the form

$$a_{ij}^1(t,x,u^1,u_x^1) - a_{ij}^1(t,x,u^2,u_x^2) = \int_0^1 \frac{d}{d\lambda} a_{i,j}^1\left(t,x,u^\lambda,u_x^\lambda\right) d\lambda,$$

$$A_1(t,x,u^1,u_x^1) - A_1(t,x,u^2,u_x^2) = \int_0^1 \frac{d}{d\lambda} A_1\left(t,x,u^\lambda,u_x^\lambda\right) d\lambda$$

where

$$u^\lambda(t,x) = \lambda u^1(t,x) + (1-\lambda)u^2(t,x),$$

$$u_x^\lambda(t,x) = \lambda u_x^1(t,x) + (1-\lambda)u_x^2(t,x),$$

$$A_1(t,x,u,u_{x_i}) = \frac{\partial}{\partial u} K(t,x,u,u_x)u_{x_i} + \frac{\partial}{\partial x_i} K(t,x,u,u_x)u_{x_i}.$$

Then

$$
\begin{aligned}
&a^1_{ij}(t,x,u^1,u^1_x)u^1_{x_ix_j} - a^1_{ij}(t,x,u^2,u^2_x)u^2_{x_ix_j}\\
&= a^1_{ij}(t,x,u^1,u^1_x)U_{x_ix_j}\\
&\quad + u^2_{x_ix_j}\Big[U_{x_i}\int_0^1 \frac{\partial}{\partial u^\lambda_{x_j}} a^1_{i,j}\left(t,x,u^\lambda,u^\lambda_x\right)d\lambda + U\int_0^1 \frac{\partial}{\partial u^\lambda} a^1_{i,j}\left(t,x,u^\lambda,u^\lambda_x\right)d\lambda\Big],
\end{aligned}
\tag{82}
$$

$$
\begin{aligned}
&A_1(t,x,u^1,u^1_x) - A_1(t,x,u^2,u^2_x)\\
&\quad = \int_0^1 \frac{d}{d\lambda} A_1\left(t,x,u^\lambda,u^\lambda_x\right)d\lambda\\
&\quad = U_{x_i}\int_0^1 \frac{\partial}{\partial u^\lambda_{x_j}} A_1\left(t,x,u^\lambda,u^\lambda_x\right)d\lambda + U\int_0^1 \frac{\partial}{\partial u^\lambda} A_1\left(t,x,u^\lambda,u^\lambda_x\right)d\lambda.
\end{aligned}
\tag{83}
$$

We subtract the equation (14) for $u^2(t,x)$ from the equations (14) for $u^1(t,x)$ and, owing to (82) and (83), we obtain the following linear problem endowed with nonlinear dynamic boundary conditions, that is

$$
\begin{cases}
\frac{\partial}{\partial t}U - \hat{a}^1_{ij}(t,x)U_{x_ix_j} = -\hat{a}^1_i(t,x)U_{x_i} - \hat{a}^1(t,x)U - \frac{\ell}{2}\frac{\partial}{\partial t}\Phi + f^a_1 - f^b_1 & \text{in } Q\\
U(t,x) = A(t,x) & \text{on } \Sigma\\
U(0,x) = (u^1_0 - u^2_0)(x) & \text{in } \Omega\\
\frac{\partial}{\partial \mathbf{n}}U + \frac{\partial}{\partial t}A - \Delta_\Gamma A + hA & \\
\qquad = -\big[g_1(\alpha^1) - g_1(\alpha^2)\big] + (w^a_1 - w^b_1) & \text{on } \Sigma\\
A(0,x) = (\alpha^1_0 - \alpha^2_0)(x) & \text{on } \Gamma,
\end{cases}
\tag{84}
$$

where

$$
\begin{aligned}
\hat{a}^1_{ij}(t,x) &= a^1_{ij}(t,x,u^1,u^1_x),\\
\hat{a}^1_i(t,x) &= -u^2_{x_ix_j}\int_0^1 \frac{\partial}{\partial u^\lambda_{x_j}} a^1_{i,j}\left(t,x,u^\lambda,u^\lambda_x\right)d\lambda + \int_0^1 \frac{\partial}{\partial u^\lambda_{x_j}} A_1\left(t,x,u^\lambda,u^\lambda_x\right)d\lambda,\\
\hat{a}^1(t,x) &= -u^2_{x_ix_j}\int_0^1 \frac{\partial}{\partial u^\lambda} a^1_{i,j}\left(t,x,u^\lambda,u^\lambda_x\right)d\lambda + \int_0^1 \frac{\partial}{\partial u^\lambda} A_1\left(t,x,u^\lambda,u^\lambda_x\right)d\lambda.
\end{aligned}
$$

By hypothesis we have $(u^1_0 - u^2_0) \in W^{2-\frac{2}{p}}_\infty(\Omega) \subset W^{2-\frac{2}{p}}_p(\Omega)$, $(\alpha^1_0 - \alpha^2_0) \in W^{2-\frac{2}{p}}_\infty(\Gamma) \subset W^{2-\frac{2}{p}}_p(\Gamma)$, $-\hat{a}^1_i(t,x)U_{x_i} - \hat{a}^1(t,x)U - \frac{\ell}{2}\frac{\partial}{\partial t}\Phi + (f^a_1 - f^b_1) \in L^p(Q)$ (recall that $U(t,x) \in W^{1,2}_p(Q)$, $\Phi(t,x) \in W^{1,2}_\nu(Q)$) and $-\big[g_1(t,x,\alpha^1) - g_1(t,x,\alpha^2)\big] \in L^p(\Sigma)$.

So, Theorem 2.1 in [6, relation (2.7)] applied to problem (84) for the unknown functions $U(t,x) = (u^1 - u^2)(t,x)$ and $A(t,x) = (\alpha^1 - \alpha^2)(t,x)$, gives the estimate

$$\begin{aligned} \|U\|^p_{W^{1,2}_p(Q)} + \|A\|^p_{W^{1,2}_p(\Sigma)} \leq \quad & C\Bigg[\|u^1_0 - u^2_0\|^p_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|\alpha^1_0 - \alpha^2_0\|^p_{W_\infty^{2-\frac{2}{p}}(\Gamma)} \\ & + \|\nabla U\|^p_{L^p(Q)} + \|U\|^p_{L^p(Q)} + \left\|\frac{\partial}{\partial t}\Phi\right\|^p_{L^p(Q)} \\ & + \|g_1(\alpha^1) - g_1(\alpha^2)\|^p_{L^p(\Sigma)} \\ & + \|f^a_1 - f^b_1\|^p_{L^p(Q)} + \|w^a_1 - w^b_1\|^p_{L^p(\Sigma)}\Bigg], \end{aligned} \tag{85}$$

where $C = C(|\Omega|, |\Gamma|, T, n, p, p_1, p_2, p_3) > 0$. Appling again Theorem 2.1 in [6, relation (2.4)] and making use of the embedding $W^{1,2}_p(Q) \subset L^p(Q)$, we have

$$\begin{aligned} \|\nabla U\|^p_{L^p(Q)} + \|U\|^p_{L^p(Q)} \quad & \leq \tilde{C}\Bigg[1 + \|u^1_0 - u^2_0\|^p_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|\alpha^1_0 - \alpha^2_0\|^p_{W_\infty^{2-\frac{2}{p}}(\Gamma)} \\ & + \|f^a_1 - f^b_1\|^p_{L^p(Q)} + \|w^a_1 - w^b_1\|^p_{W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\Sigma)}\Bigg]. \end{aligned} \tag{86}$$

Let us now focus our attention on the term $\|g_1(t,x,\alpha^1) - g_1(t,x,\alpha^2)\|^p_{L^p(\Sigma)}$ from the right-hand side of (84). Firstly, we recall that is true the following sequence of embeddings (see [17, p. 103, relation (2.198)]):

$$W^{1,2}_p(\Sigma) \subset L^{p'}(\Sigma) \subset L^{\ell_1}(\Sigma) \subset L^p(\Sigma) \subset L^2(\Sigma). \tag{87}$$

From $(\mathbf{G}_2)$, Hölder's inequality, relations (19) and (87), we derive that

$$\|g_1(t,x,\alpha^1) - g_1(t,x,\alpha^2)\|_{L^p(\Sigma)} \leq C_7\|\alpha^1 - \alpha^2\|_{L^{\ell_1}(\Sigma)}. \tag{88}$$

where $C_7 = C(|\Omega|, T, p, b_2)\Big(1 + 2M_4^2\Big)$. Using the embedding in (87), the standard interpolation inequalities (see [17, p. 14, (1.31)]) yield that $\forall \varepsilon > 0$, $\exists C(\varepsilon) > 0$ such that

$$\|y\|_{L^{\ell_1}(\Sigma)} \leq \varepsilon\|y\|_{W^{1,2}_p(\Sigma)} + C(\varepsilon)\|y\|_{L^p(\Sigma)}, \qquad \forall y \in W^{1,2}_p(\Sigma). \tag{89}$$

Combining (86), (88) and (89), estimate (85) leads to

$$\begin{aligned} & \|U\|^p_{W^{1,2}_p(Q)} + (1 - \epsilon C_7)\|A\|^p_{W^{1,2}_p(\Sigma)} \\ & \leq C\Bigg[\|u^1_0 - u^2_0\|^p_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|\alpha^1_0 - \alpha^2_0\|^p_{W_\infty^{2-\frac{2}{p}}(\Gamma)} \\ & \quad C_6\tilde{C}\Bigg[1 + \|u^1_0 - u^2_0\|^p_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|\alpha^1_0 - \alpha^2_0\|^p_{W_\infty^{2-\frac{2}{p}}(\Gamma)} \\ & \quad + \|f^a_1 - f^b_1\|^p_{L^p(Q)} + \|w^a_1 - w^b_1\|^p_{W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\Sigma)}\Bigg] \\ & \quad + C_8\|\alpha^1 - \alpha^2\|_{L^p(\Sigma)} + \|f^a_1 - f^b_1\|^p_{L^p(Q)} + \|w^a_1 - w^b_1\|^p_{L^p(\Sigma)}\Bigg], \end{aligned} \tag{90}$$

where $C_8 = C(\varepsilon)C_7 > 0$.

In order to handle the term $\|\alpha^1-\alpha^2\|_{L^p(\Sigma)}$, we rely on *a priori* estimates in $L^p(\Sigma)$. In this respect, we multiply $(84)_1$ by $|U|^{p-2}U = |u^1-u^2|^{p-2}(u^1-u^2)$. Integrating over Q_t, $t \in (0,T]$ and using Green's first identity as well as the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\frac{1}{p}\int\limits_{Q_t} \frac{\partial}{\partial t}|U(t,x)|^p \; d\tau dx \\
&\qquad +(p-1)\int\limits_{Q_t} |\nabla U|^2 \cdot \nabla\left(\hat{a}^1_{ij}(t,x)|U|^{p-2}\right) \; d\tau dx \\
&\qquad + \int\limits_{\Sigma_t} |U|^{p-1}\left(-\frac{\partial}{\partial \mathbf{n}}U\right) \; d\tau d\gamma \\
&\qquad \le \int\limits_{Q_t} |\hat{a}^1_i(t,x)| \; |\nabla U| \; |U|^{p-1} \; d\tau dx \\
&\qquad + p_2 \int\limits_{Q_t} |U|^p \; d\tau dx + \lambda \int\limits_{Q_t} |f^a_1 - f^b_1| \; |U|^{p-1} \; d\tau dx.
\end{aligned} \tag{91}$$

Using the boundary conditions $(84)_2$, $(84)_4$ and Hölder's inequality, then (91) becomes

$$\begin{aligned}
&\frac{1}{p}\int\limits_{Q_t} \frac{\partial}{\partial t}|U(t,x)|^p \; d\tau dx + \frac{1}{p}\int\limits_{\Sigma_t} \frac{\partial}{\partial t}|Z(t,x)|^p \; d\tau d\gamma \\
&\qquad +(p-1)\int\limits_{Q_t} |\nabla U|^2 \cdot \nabla\left(\hat{a}_{ij}(t,x)|U|^{p-2}\right) \; d\tau dx \\
&\qquad + p_3 \int\limits_{\Sigma_t} |Z(t,x)|^p \; d\tau d\gamma + (p-1)\int\limits_{\Sigma_t} |\nabla_\Gamma Z|^2 \; |Z|^{p-2} \; d\tau d\gamma \\
&\qquad + \int\limits_{\Sigma_t} \left[g_1(t,x,\alpha^1) - g_1(t,x,\alpha^2)\right]|Z|^{p-2}Z \; d\tau d\gamma \\
&\qquad\quad \le \tilde{C}\Big[\|\nabla(u^1-u^2)\|^p_{L^p(Q)} + \|u^1-u^2\|^p_{L^p(Q)} \\
&\qquad\qquad + \|u^1-u^2\|^p_{L^p(Q)} + \|\alpha^1-\alpha^2\|^p_{L^p(\Sigma)} \\
&\qquad\qquad + \|f^a_1 - f^b_1\|^p_{L^p(Q)} + \|w^a_1 - w^b_1\|^p_{L^p(\Sigma)}\Big]
\end{aligned} \tag{92}$$

where $\tilde{C} = C(|\Omega|, |\Gamma|, p, p_2, M_1, M_4) > 0$.

In particular, owing to hypothesis $(\mathbf{G}_1)$ and (86), it follows from (92) that

$$\begin{aligned}
&\int\limits_{\Omega} |u^1-u^2|^p \; dx + \int\limits_{\Gamma} |\alpha^1-\alpha^2|^p \; d\gamma \\
&\qquad \le C_9\Big[\|u^1_0 - u^2_0\|^p_{L^p(\Omega)} + \|\alpha^1_0 - \alpha^2_0\|^p_{L^p(\Gamma)}
\end{aligned} \tag{93}$$

$$+\|f_1^a - f_1^b\|^p_{L^p(Q)} + \|w_1^a - w_1^b\|^p_{L^p(\Sigma)}$$

$$+\int_0^t \left[\int_\Omega |u^1 - u^2|^p \, dx + \int_\Gamma |\alpha^1 - \alpha^2|^p \, d\gamma\right] d\tau\Bigg].$$

where $C_9 = C(\tilde{C}, p_3, b_1) > 0$.

Making uses of Gronwall's inequality, we can deduce from (93) that

$$\begin{aligned}
&\|U\|^p_{L^p(Q)} + \|A\|^p_{L^p(\Sigma)} \\
&\quad \le \exp^{C_9 T}\Big[\|u_0^1 - u_0^2\|^p_{L^p(\Omega)} + \|\alpha_0^1 - \alpha_0^2\|^p_{L^p(\Gamma)} \\
&\quad +\|f_1^a - f_1^b\|^p_{L^p(Q)} + \|w_1^a - w_1^b\|^p_{L^p(\Sigma)}\Big].
\end{aligned} \tag{94}$$

Making use of relations (94), from (90) we finally derive that

$$\begin{aligned}
&\|U\|_{W_p^{1,2}(Q)} + (1-\epsilon C_7)\|A\|_{W_p^{1,2}(\Sigma)} \\
&\quad \le C_{10}\Big[1 + \|u_0^1 - u_0^2\|_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|\alpha_0^1 - \alpha_0^2\|_{W_\infty^{2-\frac{2}{p}}(\Gamma)} \\
&\qquad +\|f_1^a - f_1^b\|_{L^p(Q)} + \|w_1^a - w_1^b\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\Big] \\
&\quad +C_8 \exp^{C_9 T}\Big[\|u_0^1 - u_0^2\|^p_{L^p(\Omega)} + \|\alpha_0^1 - \alpha_0^2\|^p_{L^p(\Gamma)} \\
&\quad +\|f_1^a - f_1^b\|^p_{L^p(Q)} + \|w_1^a - w_1^b\|^p_{L^p(\Sigma)}\Big].
\end{aligned} \tag{95}$$

Adding (81) with (95) and making use of the standard interpolation inequalities (written for $W_p^{1,2}(Q) \subset L^\nu(Q) \subset L^p(Q)$) combined with (94), we find that

$$\begin{aligned}
&(1-\epsilon C_7)\|U\|_{W_p^{1,2}(Q)} + (1-\epsilon C_7)\|A\|_{W_p^{1,2}(\Sigma)} + \|\Phi\|_{W_\nu^{1,2}(Q)} + \|X\|_{W_p^{1,2}(\Sigma)} \\
&\quad \le C_{10}\Big[1 + \|u_0^1 - u_0^2\|_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|\alpha_0^1 - \alpha_0^2\|_{W_\infty^{2-\frac{2}{p}}(\Gamma)} \\
&\qquad +\|\varphi_0^1 - \varphi_0^2\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{q}}(\Omega)} + \|\xi_0^1 - \xi_0^2\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{p}}(\Gamma)} \\
&\qquad +\|f_1^a - f_1^b\|_{L^p(Q)} + \|w_1^a - w_1^b\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)} \\
&\qquad +\|f_2^a - f_2^b\|_{L^q(Q)} + \|w_2^a - w_2^b\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\Big],
\end{aligned} \tag{96}$$

where the embedding $L^q(Q) \subset L^\nu(Q)$ has been used too.

For $\varepsilon > 0$ with $1-\varepsilon C_7 > 0$, the embedding $W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma) \subset L^p(\Sigma)$ and estimate (96) implies the estimate (20), which finishes the proof of Theorem 2.2. □

As a consequence, the uniqueness of solution to problem (1)-(3) (or (14)-(15)) is valid.

Corollary 1. *For the same initial conditions, the problem* (14)-(15) *possesses a unique classical solution.*

Proof. Let $f_1^a = f_1^b$, $f_2^a = f_2^b$, $w_1^a = w_1^b$ and $w_2^a = w_2^b$ in the Theorem 2.2. Then (20) demonstrates the corollary (see also [9, Corollary 5.1, p. 111]). □

5. **Conclusions.** The main problem addressed in this paper is a nonlinear second-order anisotropic phase-field transition system with principal part in *divergence* form, endowed with inhomogeneous dynamic boundary conditions (in both unknown $u(t,x)$ and $\varphi(t,x)$) and non-constant mobility $K(t,x,u(t,x),u_x(t,x))$, $\Psi(t,x,\varphi(t,x),\varphi_x(t,x))$. Under general assumptions on the boundary nonlinearities $g_k, k = 1,2$, and provided that the initial and boundary data meet appropriate regularity as well as compatibility conditions, it is proven the well-posedness (existence, estimate, uniqueness and regularity) of a classical solution to the phase-field transition system in this new formulation (Theorem 2.2). Precisely, the Leray-Schauder principle as well as the L^p theory of linear and quasi-linear parabolic equations, via **Lemma 7.4** (see [9]), are applied to prove the qualitative properties of solution (u,φ,α,ξ). In other words, we can not directly apply the L^p theory to the problem (1)-(3). Thus, it makes the result in **Lemma 7.4** in Choban and Moroşanu [9, p. 114] himself very important.

Moreover, the *a priori* estimates are made in $L^p(Q)$ and $L^{p'}(\Sigma)$ which permit to derive regularity properties of higher order for unknown functions, that is $(u,\varphi,\alpha,\xi) \in W_p^{1,2}(Q) \times W_\nu^{1,2}(Q) \times W_p^{1,2}(\Sigma) \times W_p^{1,2}(\Sigma)$. Thus, classical methods such as *bootstrap* (see Moroşanu and Motreanu [24]), can be avoided. This approach could be applied in future to study other kind of the first and second boundary value problems.

From the perspective of applicability, it is natural to find the suitable type of nonlinearities on the $\partial\Omega$, able to describe the complexity of many important physical phenomena, among which we mention *effect of surface tension*, *separating zone of solid and liquid states* etc. So, one of the most important characteristics of our improved mathematical model (1)-(3) is the nonlinear term $g_k, k = 1,2$, in the dynamic boundary conditions which allows to consider a nonlinearity with a larger growth exponent $r' \leq (n+2)/(n+2-2p)$ if $n+2 > 2p$ (see (5). It extends the already studied types of boundary conditions (see [6], [9], [11], [13], [14], [15]-[18], [20], [26]) and therefore makes the new formulation of model (1)-(3) to be more able to describe a wide variety of industrial applications, in particular, the interactions with the walls in confined systems (i.e. the *phase changes* at the boundary of Ω).

Let's also remark that, due to the presence of the terms $K(t,x,u(t,x),u_x(t,x))$, $\Psi(t,x,\varphi(t,x),\varphi_x(t,x))$, the nonlinear operator in (1) does not represent the gradient of the energy functional. Therefore, the new proposed second-order nonlinear problem can not be obtained from the minimisation of any energy cost functional, i.e. (1) is not a variational PDE model.

The qualitative results obtained here can be involved later in the quantitative approaches of the mathematical model (1)-(3) as well as in the study of distributed and/or boundary nonlinear optimal control problems governed by such a nonlinear problem. Amongst other things, we wish to be exploited all this in our future works.

At the end we want to underline the solutions dependence in Theorem 2.2 on physical parameters, which can be useful in future investigations regarding the error analysis and numerical simulations.

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