



Accelerated Contraction Algorithms for Split Variational Inclusion Problems

Kanokwan Sitthithakerngkiet¹, Abdulwahab Ahmad^{2,3}, Bashir Ali⁴, Vasile Berinde^{5,6,*}

¹ Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkuts University of Technology North Bangkok (KMUTNB), 1518, Wongsawang, Bangsue, 10800 Bangkok, Thailand
E-mail: kanokwan.s@sci.kmutnb.ac.th

² Center of Excellence in Theoretical and Computational Science (TaCS-CoE) & Fixed Point Research Laboratory, Room SCL 802, Fixed Point Laboratory, Science Laboratory Building, Departments of Mathematics, Faculty of Science, King Mongkuts University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road Thung Khru, Bang Mod, 10140, Bangkok, Thailand
E-mail: abumuhammaddm@gmail.com

³ Department of Mathematics, School of Secondary Education, Sciences, Federal College of Education, Katsina, 2041, Katsina, Nigeria

⁴ Department of Mathematical Sciences, Faculty of Science, Bayero University, Kano, (BUK), Kano, Nigeria
E-mail: bashiralik@yahoo.com

⁵ Department of Mathematics & Computer Science, Faculty of Science, North University Center at Baia Mare Technical University of Cluj-Napoca, Victoriei 76, Baia Mare, 430122, Romania

⁶ Academy of Romanian Scientists, 3 Ilfov, Bucharest, 050044, Romania
E-mail: vasile.berinde@mi.utcluj.ro

*Corresponding author.

Received: 2 April 2026 / Accepted: 13 July 2026

Abstract This work introduces four adaptive proximal viscosity-type and contraction algorithms designed to compute a unique viscosity-selected solution of a split variational inclusion problem in infinite-dimensional real Hilbert spaces. The proposed algorithms combine an alternating inertial technique with a three-term conjugate gradient-like direction to enhance their convergence performance. Unlike many existing methods, these schemes incorporate relaxation effects and simple adaptive step length strategies that are bounded away from zero, thereby eliminating the need for prior knowledge of the operator norm or costly line-search procedures. A further notable advantage lies in the adaptive step length strategy employed in the second and fourth algorithms, which is introduced for the first time in this work and is distinct from that used in the first and third algorithms. Furthermore, the conjugate gradient-based algorithms incorporate adaptive techniques for generating gradient-like parameters at each iteration, thereby avoiding the restrictive boundedness assumptions typically imposed in related methods. Under mild conditions, the strong convergence of all four algorithms to a unique viscosity-selected solution is established. Applications to split feasibility and split minimization problems are explored, providing a unified framework for solving inverse problems arising in areas such as image and signal recovery, machine learning, and other important domains. Numerical experiments demonstrate that the proposed algorithms are

Published online: 23 August 2026

© 2026 By TaCS-CoE, All rights reserved.

robust, efficient, and consistently outperform the comparison methods in terms of convergence speed, computational time, and accuracy across all considered examples.

MSC: 47J20; 47J25; 47J30; 65K15

Keywords: Alternating inertial techniques, Contraction algorithms, Split variational inclusion problem, Three-term conjugate gradient-like methods, Viscosity iteration

1. INTRODUCTION

This work focuses on approximating solutions to a split variational inclusion problem (SVIP):

$$\text{find } t \in \mathcal{H}_1 \text{ such that } 0 \in F_1(t) \text{ and } 0 \in F_2(Bt), \quad (1.1)$$

where $B : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a bounded linear operator, and $F_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$ and $F_2 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$ are multi-valued maximal monotone operators on real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. We denote the set of solutions to SVIP (1.1) by $\Omega_{SVIP} = \{u \in \mathcal{H}_1 : 0 \in F_1(u) \text{ and } 0 \in F_2(Bu)\}$. In particular, when $F_1 = N_{\mathcal{C}}$ and $F_2 = N_{\mathcal{Q}}$, where $N_{\mathcal{C}}$ and $N_{\mathcal{Q}}$ denote the normal cones associated with nonempty, closed and convex sets $\mathcal{C} \subseteq \mathcal{H}_1$ and $\mathcal{Q} \subseteq \mathcal{H}_2$, respectively, SVIP (1.1) reduces to split feasibility problem; see [12]. Moreover, when $F_2 \equiv 0$, SVIP (1.1) becomes the classical variational inclusion problem:

$$\text{find } t \in \mathcal{H}_1 \text{ such that } 0 \in F_1(t), \quad (1.2)$$

which further provides a unified approach for transforming and solving several important nonlinear problems, including variational inequalities, fixed point problems, and convex optimization problems. Generally speaking, since the inception of the SVIP (1.1) by Moudafi [31], its study has witnessed substantial growth due to its ability to encompass several problems of interest, such as split variational inequality problems, split feasibility problems, split equilibrium problems, fixed point problems, variational inequality problems, variational inclusion problems, and convex optimization problems; see [12, 13, 25] and the references therein, to mention only a few, as particular cases. The SVIP (1.1) considered in this work is motivated by its wide range of applications to many real-world problems, including radiation therapy [33], machine learning [2, 7, 8], medical image reconstruction, signal processing, jointly constrained Nash equilibrium problems, and many other areas; see [9, 17]. Recently, the study of robust and efficient numerical methods for solving the SVIP (1.1) has grown substantially and has consequently become one of the most active areas of mathematical research. Using the equivalent fixed point formulation:

$$t = J_{\zeta}^{F_1}(I - \lambda B^*(I - J_{\zeta}^{F_2})B)t, \quad (1.3)$$

where $J_{\zeta}^{F_1}$ and $J_{\zeta}^{F_2}$ are the resolvent operators associated with F_1 and F_2 , respectively, for $\zeta > 0$, B^* denotes the adjoint of B , I is the identity operator, and λ is a positive constant. Using the particular Fréchet differentiable real-valued function $g : \mathcal{H}_1 \rightarrow \mathbb{R}$ defined by $g(t) = \frac{1}{2}\|(I - J_{\zeta}^{F_2})Bt\|^2$, whose gradient is given by $\nabla g(t) = B^*(I - J_{\zeta}^{F_2})Bt$, Byrne et al. [11] were the first to develop two iterative algorithms for solving the SVIP

(1.1), with the following recursive relations: for $t_0 \in \mathcal{H}_1$,

$$t_{n+1} = J_{\zeta}^{F_1}(t_n - \lambda \nabla g(t_n)), \quad \forall n \geq 0 \quad (1.4)$$

and

$$t_{n+1} = \alpha_n t_0 + (1 - \alpha_n) J_{\zeta}^{F_1}(t_n - \lambda \nabla g(t_n)), \quad \forall n \geq 0, \quad (1.5)$$

where $\lambda \in (0, \frac{2}{\|B\|^2})$ is the step length, and Scheme (1.5) is motivated by the Halpern iteration; see [21]. Under suitable conditions on the sequence $\{\alpha_n\} \subset (0, 1)$, they established the weak and strong convergence of the sequence $\{t_n\}$ generated by Schemes (1.4) and (1.5) to a point in Ω_{SVIP} , respectively. Despite the effectiveness of Algorithms (1.4) and (1.5), they require prior knowledge or an estimate of the operator norm of B , which is often difficult to obtain and consequently limits their applicability in practice. To overcome these difficulties and further improve the convergence properties of Algorithms (1.4) and (1.5), several adaptive algorithms have been developed; see [16, 36, 37] and the references therein.

Currently, considerable attention has been devoted to constructing algorithms with fast convergence properties, as such algorithms are highly desirable for solving various problems arising in many real-world applications [14, 26]. In this regard, one of the most widely studied extrapolation techniques for accelerating iterative processes is Polyak's inertial method; see [34]. Let t_{n-1} and t_n be two successive iterates generated by an iterative process. The Polyak inertial term is defined by

$$u_n = t_n + \beta(t_n - t_{n-1}), \quad \forall n \geq 1, \quad \beta > 0, \quad (1.6)$$

which is commonly referred to as a one-step inertial term. Subsequently, several numerical algorithms based on (1.6) have been developed to solve the SVIP (1.1) and other related problems; see [6, 19, 22] and the references therein. Recently, Tan et al. [39] introduced the following strongly convergent adaptive inertial proximal and contraction algorithm based on viscosity iteration; see [30], together with an Armijo-like step length strategy.

Algorithm 1 Tang, Qin and Yao Algorithm (TQY)

Initialization: Take $\zeta > 0$, $\eta_1 > 0$, $\gamma > 0$, $\varepsilon \in (0, 1)$, $\chi \in (0, 1)$, $\sigma \in (0, 2)$, $\{\varsigma_n\} \subset [0, +\infty)$ and $\{\alpha_n\} \subset (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \rightarrow \infty} \frac{\varsigma_n}{\alpha_n} = 0$.

Choose $t_0, t_1 \in \mathcal{H}_1$ and a c -contractive mapping $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$. Set $n = 1$.

Step 1. Compute

$$u_n = t_n + \beta_n(t_n - t_{n-1}), \text{ where } \beta_n = \begin{cases} \min \left\{ \frac{\varsigma_n}{\|t_n - t_{n-1}\|}, \eta_1 \right\} & \text{if } t_n \neq t_{n-1}, \\ \eta_1, & \text{otherwise.} \end{cases} \quad (1.7)$$

Step 2. Compute $h_n = J_{\zeta}^{F_1}(u_n - \lambda_n \nabla g(u_n))$, where $\lambda_n = \gamma \chi^{r_n}$ and r_n is the least nonnegative integer such that

$$\lambda_n \|\nabla g(u_n) - \nabla g(h_n)\| \leq \varepsilon \|u_n - h_n\|. \quad (1.8)$$

If $h_n = u_n$, stop the iteration and $h_n \in \Omega_{SVIP}$. Else, move to step 3.



Step 3. Compute $m_n = u_n - \sigma\theta_n f_n$, where

$$\begin{cases} f_n = u_n - h_n - \lambda_n(\nabla g(u_n) - \nabla g(h_n)), \\ \theta_n = \frac{\langle u_n - h_n, f_n \rangle}{\|f_n\|^2}. \end{cases} \quad (1.9)$$

Step 4. Compute $t_{n+1} = \alpha_n f(t_n) + (1 - \alpha_n)m_n$. Set $n := n + 1$ and go back to Step 1.

However, in several scenarios, it has been observed that determining an appropriate step length at each iteration using line-search strategies requires multiple search procedures, which may lead to computational inefficiency and reduced algorithmic performance. Several studies have also reported that some methods incorporating the inertial term (1.6) appear to converge more slowly than their non-inertial counterparts; see [10, 28] and the references therein. Recently, for any two successive iterates t_{n-1} and t_n generated by an iterative process, Mu and Peng [32] introduced the following alternating inertial term:

$$u_n = \begin{cases} t_n, & \text{if } n \text{ is even,} \\ t_n + \beta_n(t_n - t_{n-1}), & \text{if } n \text{ is odd,} \end{cases} \quad (1.10)$$

which is a modification of (1.6), where the sequence $\{\beta_n\}$ satisfies certain assumptions. One of the notable advantages of algorithms based on the alternating inertial technique (1.10) is that, in certain settings, the even subsequence satisfies a Fejér monotonicity property, thereby improving stability, while the odd subsequence provides inertial acceleration. However, in viscosity-type iteration framework, the viscosity step generally precludes such a Fejér monotonicity property. Nevertheless, the alternating inertial strategy remains an essential component of our convergence analysis, where the separation of odd and even iterations enables effective control of the inertial perturbation. However, an improperly chosen sequence $\{\beta_n\}$ may cause slight oscillations. Recently, several researchers have incorporated the alternating inertial term (1.10) into their algorithms to improve convergence performance; see [1–3, 8] and the references therein for particular cases of the SVIP (1.1). However, to the best of our knowledge, algorithms based on the alternating inertial technique for the general form of the SVIP (1.1) have not yet been investigated.

Moreover, from an optimization perspective, we observe that all the aforementioned algorithms for the SVIP (1.1), including (1.4), (1.5), Algorithm 1 (referred to as TQY), and those proposed in [16, 36–38], are hybrid steepest descent-type methods. However, in light of the results reported in [27], the performance of these methods may be further improved by incorporating either of the following search directions. For a sequence $\{t_n\}$ generated by an iterative process, define

$$\gamma_n = -\nabla g(t_n) + \xi_n^{(1)}\gamma_{n-1} - \xi_n^{(2)}w_n, \quad \forall n \geq 1, \quad (1.11)$$

and

$$\gamma_n = -\nabla g(t_n) + \xi_n^{(1)}\gamma_{n-1}, \quad \forall n \geq 1, \quad (1.12)$$

where $\{w_n\}$ is an arbitrary sequence in $\mathcal{H}_1$ and, for each $i = 1, 2$, $\xi_n^{(i)} \in [0, +\infty)$ satisfies certain assumptions. The directions (1.11) and (1.12) are referred to as the three-term conjugate gradient-like (TTCG-like) and two-term conjugate gradient-like (CG-like) directions, respectively. In [27], a TTCG-like algorithm based on the direction (1.11), with



$\xi_n^{(2)} w_n \neq 0$, was proposed and shown numerically to outperform its CG-like counterpart based on (1.12), provided that the sequence $\{w_n\}$ is bounded and, for each $i = 1, 2$, $\lim_{n \rightarrow \infty} \xi_n^{(i)} = 0$. Under these assumptions, the TTCG-like direction always generates a descent direction regardless of the choice of $\xi_n^{(1)}$, whereas, for the CG-like direction (1.12), this property is guaranteed only when $\xi_n^{(1)}$ is appropriately chosen. Motivated by these findings, many researchers have improved their iterative algorithms for various problems by incorporating either the CG-like direction (1.12) or the TTCG-like direction (1.11); see [4, 5, 18, 23].

Very recently, Zhang and Ma [43] introduced the following strongly convergent inertial CG-like algorithm, which incorporates the inertial term (1.6), the direction (1.12), and an adaptive step length strategy to avoid the multiple search procedures required by line-search methods.

Algorithm 2 Zhang and Ma Algorithm (ZM)

Initialization: Take $\lambda_0 > 0$, $\zeta_n > 0$, $\eta_1 \in (0, 1)$, $\gamma > 0$, $\varepsilon \in (0, \mu) \subset (0, \frac{1}{2})$, $\sigma \in (0, 2)$, $\bar{\nu}_n \in [0, 1]$, $\{\alpha_n\} \subset (0, 1)$, $\{\varsigma_n\} \subset [0, +\infty)$, $\{\phi_n\} \subset [1, +\infty)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n \rightarrow \infty} \frac{\varsigma_n}{\alpha_n} = 0$ and $\sum_{n=1}^{\infty} (\phi_n - 1) < \infty$. Choose $t_0, t_1 \in \mathcal{H}_1$ and $\gamma_0 = 0 \in \mathcal{H}_1$. Set $n = 1$.

Step 1. Compute $\{u_n\}$ and $\{\beta_n\}$ by (1.7), and

$$\gamma_n = -B^*(I - J_{\zeta_n}^{F_2})Bu_n + \xi_n \gamma_{n-1}.$$

Step 2. Compute $h_n = J_{\zeta_n}^{F_1}((1 - \alpha_n \lambda_n)u_n - \lambda_n B^*(I - J_{\zeta_n}^{F_2})Bu_n + \bar{\nu}_n \gamma_n)$. If $h_n = u_n$, stop the iteration and $h_n \in \Omega_{SVIP}$. Else, move to step 3.

Step 3. Compute $t_{n+1} = u_n - \theta_n f_n$, where θ_n and f_n are defined by

$$f_n = u_n - h_n - \lambda_n (B^*(I - J_{\zeta_n}^{F_2})Bu_n - B^*(I - J_{\zeta_n}^{F_2})Bh_n),$$

and (1.9), and update λ_{n+1} by

$$\lambda_{n+1} = \begin{cases} \min \left\{ \frac{\varepsilon \|u_n - h_n\|^2}{p_n}, \phi_n \lambda_n \right\} & \text{if } p_n > 0, \\ \phi_n \lambda_n, & \text{otherwise,} \end{cases} \quad (1.13)$$

where $p_n = \langle B^*(I - J_{\zeta_n}^{F_2})Bu_n - B^*(I - J_{\zeta_n}^{F_2})Bh_n, u_n - h_n \rangle$. Set $n := n + 1$ and go back to Step 1.

Remark 1.1. Despite the computational attractiveness of the inertial CG-like Algorithm 2, we make the following observations.

- (i) The convergence results of Algorithm 2, as well as many other CG-like algorithms, were established under the assumption that the sequence $\{t_n\}$ is bounded. This assumption appears to be restrictive, and it would be desirable to dispense with it.
- (ii) To the best of our knowledge, the behavior of algorithms incorporating the TTCG-like direction (1.11) for the general case of the SVIP (1.1) has not yet been investigated.
- (iii) It is not difficult to verify that the stopping criterion used in Algorithm 2 (Zhang and Ma [43], Algorithm 1) does not guarantee that whenever it is satisfied, the



resulting point (namely, h_n for some $n \geq 1$) belongs to the solution set of the SVIP (1.1). Indeed, if $h_n = u_n$ in Step 2 of Algorithm 2, one easily sees that

$$h_n = J_{\zeta_n}^{F_1}(h_n - \lambda_n B^*(I - J_{\zeta_n}^{F_2})Bh_n + (\bar{\nu}_n \gamma_n - \alpha_n \lambda_n h_n)). \quad (1.14)$$

Thus, the presence of $(\bar{\nu}_n \gamma_n - \alpha_n \lambda_n h_n)$ in (1.14) does not allow one to generally claim from Lemma 2.5 that $h_n \in \Omega$ if $h_n = u_n$ for some $n \geq 1$. However, to ensure the feasibility of the termination process in Algorithm 2, we instead employ the stopping criterion $E_n = \|h_n - u_n\| + \|\bar{\nu}_n \gamma_n - \alpha_n \lambda_n u_n\| = 0$, in which case the iteration is terminated and $h_n \in \Omega_{SVIP}$ for some $n \geq 1$. Indeed, it is straightforward to verify that the simpler stopping criterion $\|h_n - J_{\zeta_n}^{F_1}(h_n - \lambda_n \nabla g_n(h_n))\| = 0$ follows immediately from the nonexpansiveness of $J_{\zeta_n}^{F_1}$, the Lipschitz continuity of $\nabla g_n = B^*(I - J_{\zeta_n}^{F_2})B$, and the proposed stopping criterion for Algorithm 2 as follows:

$$\begin{aligned} \|h_n - J_{\zeta_n}^{F_1}(h_n - \lambda_n \nabla g_n(h_n))\| &= \\ \|J_{\zeta_n}^{F_1}(u_n - \lambda_n \nabla g_n(u_n) + (\bar{\nu}_n \gamma_n - \alpha_n \lambda_n u_n)) - J_{\zeta_n}^{F_1}(h_n - \lambda_n \nabla g_n(h_n))\| \\ &\leq (1 + \lambda_n \|B\|)(\|u_n - h_n\| + \|\bar{\nu}_n \gamma_n - \alpha_n \lambda_n u_n\|) = 0, \end{aligned} \quad (1.15)$$

thus, $h_n = J_{\zeta_n}^{F_1}(h_n - \lambda_n \nabla g_n(h_n))$. Hence it follows from Lemma 2.5 that $h_n \in \Omega$.

- (iv) It is also noted that division by zero may occur in Step 3 of Algorithms 1 and 2, since the case where $f_n = 0$ while $h_n \neq u_n$ has not been examined and no provision has been made in either algorithm to handle the case when $f_n = 0$

Motivated by the results in [5, 27, 30, 32, 39, 43], this work proposes four efficient adaptive viscosity-type proximal and contraction algorithms for solving split variational inclusion problems in the framework of infinite-dimensional real Hilbert spaces. Each of the proposed algorithms incorporates the alternating inertial technique (1.10) and the TTCG-like direction (1.11), both of which are recently developed mechanisms for accelerating the convergence of iterative processes. In contrast to several existing methods, the proposed algorithms incorporate relaxation effects, which significantly improve their numerical performance, together with a simple adaptive step length strategies that differ from (1.13) and are bounded away from zero. Consequently, the proposed algorithms do not require prior knowledge of the operator norm of B or the use of a line-search procedure at each iteration. To the best of our knowledge, the adaptive step length strategy employed in the second and fourth algorithms is introduced for the first time in this work and is different from that used in the first and third algorithms. Furthermore, one of the main advantages and distinctive features of the proposed three-term conjugate gradient-like algorithms is that they eliminate the restrictive boundedness assumption highlighted in Remark 1.1(i), which is commonly imposed in most existing CG-like algorithms based on direction (1.12); see [43] and the references therein. Under mild and easily verifiable conditions, we prove that each of the proposed algorithms converges strongly to the unique viscosity-selected solution of the SVIP (1.1) in infinite-dimensional real Hilbert spaces. Moreover, we investigate applications of the proposed algorithms to split feasibility and split minimization problems, thereby providing a unified framework for transforming and solving important inverse problems arising in practical applications, including image and signal recovery, machine learning, and many other related areas. We also present numerical experiments to evaluate the performance of the proposed algorithms. The results demonstrate that the proposed algorithms are robust and efficient, consistently outperforming the selected



comparison algorithms in terms of convergence speed, computational time, stability and accuracy across all the examples considered. Finally, the proposed algorithms extend, generalize, and improve several existing methods in the literature; see [16, 27, 30, 36–38, 43] and the references therein. The framework developed in this work encompasses variational inequality problems, equilibrium problems, optimization problems, and systems of equations as special cases, together with many other related problems arising in practical applications.

2. PRELIMINARIES

In this work, $\mathcal{H}$ and $\mathbb{R}$ represent a real Hilbert space and the set of real numbers, respectively, and $t_n \rightarrow t$ (resp., $t_n \rightharpoonup t$) denote the strong (resp., weak) convergence of $\{t_n\}$ to t . For any $t, s \in \mathcal{H}$ and $\kappa \in \mathbb{R}$, the following hold.

$$\|t + s\|^2 = 2\langle t, s \rangle + \|t\|^2 + \|s\|^2, \quad (2.1)$$

$$2ts \leq t^2 + s^2 \quad (2.2)$$

and

$$\|\kappa t + (1 - \kappa)s\|^2 = \kappa\|t\|^2 + (1 - \kappa)\|s\|^2 - \kappa(1 - \kappa)\|t - s\|^2. \quad (2.3)$$

Additionally, for any nonempty closed and convex set $\mathcal{C} \subseteq \mathcal{H}$, $t \in \mathcal{H}$ and $s \in \mathcal{C}$, we have

$$\langle t - P_{\mathcal{C}}t, s - P_{\mathcal{C}}t \rangle \leq 0, \quad (2.4)$$

which is equivalent to

$$\|P_{\mathcal{C}}t - s\|^2 \leq \|t - s\|^2 - \|t - P_{\mathcal{C}}t\|^2, \quad (2.5)$$

where $P_{\mathcal{C}}$ is the metric projection; see [20].

Definition 2.1. Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a mapping. Then A is said to be:

(i) Lipschitz continuous, if there exists $L > 0$ satisfying

$$\|As - At\| \leq L\|s - t\|, \quad \forall s, t \in \mathcal{H}; \quad (2.6)$$

(ii) nonexpansive (resp., contraction), if (2.6) holds with $L = 1$ (resp., $L \in (0, 1)$);

(iii) firmly nonexpansive, if

$$\|As - At\|^2 \leq \langle s - t, As - At \rangle, \quad \forall s, t \in \mathcal{H}.$$

Definition 2.2. Let $F : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be a set-valued operator and $D(F) = \{t \in \mathcal{H} : F(t) \neq \emptyset\}$ denotes the domain of F . Then F is said to be:

(i) monotone, if $\forall t, s \in \mathcal{H}$, $x \in F(t)$ and $y \in F(s)$,

$$\langle t - s, x - y \rangle \geq 0;$$

(ii) maximal monotone, if it is monotone and its graph

$$\text{Graph}(F) := \{(t, x) : t \in D(F), x \in F(t)\}$$

is not contained in the graph of any other monotone operator.

As noted from [35], if $F : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ is a multi-valued maximal monotone operator, then for any $t \in \mathcal{H}$ and $\zeta > 0$, there exists a unique $t^* \in \mathcal{H}$ such that $t \in (I + \zeta F)t^*$,



where I denotes the identity operator in $\mathcal{H}$. In other words, it follows from the maximal monotonicity of F that the range $R(I + \zeta F) = \mathcal{H}$ and the resolvent operator $J_\zeta^F : \mathcal{H} \rightarrow \mathcal{H}$ is defined by

$$J_\zeta^F(t) := (I + \zeta F)^{-1}(t) = \{s \in \mathcal{H} : t \in s + \zeta F(s)\}, \forall t \in \mathcal{H}, \zeta > 0.$$

Lemma 2.3. [29] *Let $F : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be a set-valued maximal monotone mapping and $\zeta > 0$. Then the following hold:*

- (i) J_ζ^F is single-valued and firmly nonexpansive;
- (ii) $D(J_\zeta^F) = \mathcal{H}$ and $\text{Fix}(J_\zeta^F) = F^{-1}(0) = \{t \in D(F) : 0 \in F(t)\}$;
- (iii) $(1 - J_\zeta^F)$ is firmly nonexpansive;
- (iv) If $F^{-1}(0) \neq \emptyset$. Then

$$\|J_\zeta^F(t) - z\|^2 \leq \|t - z\|^2 - \|J_\zeta^F(t) - t\|^2, \forall t \in \mathcal{H}, z \in F^{-1}(0) \quad (2.7)$$

and

$$\langle t - J_\zeta^F(t), J_\zeta^F(t) - z \rangle \geq 0, \forall t \in \mathcal{H}, z \in F^{-1}(0). \quad (2.8)$$

Lemma 2.4. [40] *Let $h : \mathcal{H}_1 \rightarrow \mathbb{R}$ and $g : \mathcal{H}_1 \rightarrow \mathbb{R}$ be two functions defined by*

$$h(t) = \frac{1}{2} \|(I - J_\zeta^{F_1})t\|^2 \text{ and } g(t) = \frac{1}{2} \|(I - J_\zeta^{F_2})Bt\|^2.$$

Then

- (i) h and g are convex, differentiable, and weakly lower semicontinuous;
- (ii) the functions $\nabla h(t) = (I - J_\zeta^{F_1})t$ and $\nabla g(t) = B^*(I - J_\zeta^{F_2})Bt$ are nonexpansive and Lipschitz continuous with constant $L = \|B\|^2$, respectively.

Lemma 2.5. ([15], Lemma 3.1) *Let $B : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear operator, B^* be the adjoint operator of B , $F_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$ and $F_2 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$ be set-valued maximal monotone operators on real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Let $J_\zeta^{F_1}$ and $J_\zeta^{F_2}$ be the resolvent operators associated with F_1 and F_2 for $\zeta > 0$, respectively, and suppose that $\Omega_{SVIP} \neq \emptyset$. Then for any $t \in \mathcal{H}_1$ and $\lambda > 0$, $t \in \Omega$ if and only if $t = J_\zeta^{F_1}(I - \lambda B^*(I - J_\zeta^{F_2})B)t$.*

Lemma 2.6. [24] *Let $\{t_n\}$ be a sequence of real nonnegative numbers such that*

$$t_{n+1} \leq (1 - \mu_n)t_n + \mu_n\eta_n \text{ and } t_{n+1} \leq t_n - \chi_n + \Phi_n, \forall n \geq 1$$

where $\chi_n \in [0, +\infty)$, $\mu_n \in (0, 1)$ and $\eta_n, \Phi_n \in (-\infty, +\infty)$ such that

- (i) $\lim_{n \rightarrow \infty} \mu_n = 0$ and $\sum_{n=1}^{\infty} \mu_n = \infty$;
- (ii) $\lim_{n \rightarrow \infty} \Phi_n = 0$;
- (iii) $\lim_{j \rightarrow \infty} \chi_{n_j} = 0 \Rightarrow \limsup_{r \rightarrow \infty} \eta_{n_j} \leq 0$,

then $\lim_{n \rightarrow \infty} t_n = 0$.

3. ALGORITHMS AND THEIR STRONG CONVERGENCE RESULTS.

In this section, we present four adaptive proximal and contraction algorithms for solving the SVIP (1.1). The proposed algorithms incorporate the alternating inertial term (1.10) and the TTCG-like direction (1.11). For the convergence analysis of these algorithms, we make the following assumptions.



Assumption 3.1. (B1) $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ is a c -contraction mapping, $F_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$ and $F_2 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$ are two set-valued maximal monotone mappings and $B : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a bounded linear operator with B^* as its adjoint operator;

(B2) The solution set $\Omega = \Omega_{SVIP} \neq \emptyset$;

(B3) $\lambda_1 > 0$, $\zeta > 0$, $\varepsilon \in (0, 1)$, $\tau \in (0, \frac{1}{\varepsilon})$, $\sigma \in (0, 2)$, $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{\alpha_n\}$ is nonincreasing, and $\delta_n, \mu_n \in [a, 1 - a]$ for some $a \in (0, \frac{1}{2})$ such that $\alpha_n + \delta_n + \mu_n = 1$, $\forall n \geq 1$.

Algorithm 3

Initialization: Take $\lambda_1, \zeta, \varepsilon, \sigma, \tau, \{\delta_n\}, \{\alpha_n\}, \{\mu_n\}$ such that condition (B3) holds. Select $\xi_n^{(i)}, \beta_n \in [0, +\infty)$ for $i = 1, 2$. Choose $t_0, t_1 \in \mathcal{H}_1$, $\gamma_0 = -\nabla g(t_0)$ and $\{w_n\} \subset \mathcal{H}_1$. Set $n = 1$.

Step 1. Compute

$$u_n = \begin{cases} t_n, & \text{if } n \text{ is even,} \\ t_n + \beta_n(t_n - t_{n-1}), & \text{if } n \text{ is odd.} \end{cases} \quad (3.1)$$

Step 2. Compute $h_n = J_{\zeta}^{F_1}(u_n - \tau \lambda_n \nabla g(u_n))$. If $h_n = u_n$, stop the iteration and $h_n \in \Omega_{SVIP}$. Else, move to step 3.

Step 3. Compute

$$f_n = u_n - h_n - \tau \lambda_n (\nabla g(u_n) - \nabla g(h_n)). \quad (3.2)$$

If $f_n = 0$, stop the iteration and $h_n \in \Omega_{SVIP}$. Else, compute $m_n = u_n - \sigma \theta_n f_n + \varpi_n$, where θ_n and ϖ_n are defined as follows:

$$\begin{cases} \theta_n = \frac{\langle u_n - h_n, f_n \rangle + 2\tau \lambda_n g(h_n)}{\|f_n\|^2}, \\ \varpi_n = J_{\zeta}^{F_1}(u_n + \gamma_n) - J_{\zeta}^{F_1}(u_n - \lambda_n \nabla g(h_n)), \end{cases} \quad (3.3)$$

and

$$\gamma_n = -\lambda_n \nabla g(h_n) + \xi_n^{(1)} \gamma_{n-1} - \xi_n^{(2)} w_n. \quad (3.4)$$

Step 4. Compute $t_{n+1} = \alpha_n f(u_n) + \delta_n u_n + \mu_n m_n$, update λ_{n+1} by

$$\lambda_{n+1} = \begin{cases} \min \left\{ \frac{\varepsilon \|u_n - h_n\|}{\|\nabla g(u_n) - \nabla g(h_n)\|}, \lambda_n \right\} & \text{if } \|\nabla g(u_n) - \nabla g(h_n)\| \neq 0, \\ \lambda_n, & \text{otherwise,} \end{cases} \quad (3.5)$$

set $n := n + 1$ and go back to Step 1.

In addition to the conditions of the Assumption 3.1, to establish the convergence of the Algorithm 3, the following assumptions are required.

Assumption 3.2. We choose the parameters β_n and $\xi_n^{(i)}$ for all $n \geq 1$ and $i = 1, 2$ by

$$\beta_n := \begin{cases} \min \left\{ \frac{\zeta_n}{\|t_n - t_{n-1}\|}, \eta_1 \right\} & \text{if } t_n \neq t_{n-1}, \\ \eta_1, & \text{otherwise} \end{cases} \quad (3.6)$$

and

$$\xi_n^{(1)} = \frac{\zeta_n^{(1)}}{\max\{\|\gamma_{n-1}\|, \eta_2\}}, \quad \xi_n^{(2)} = \frac{\zeta_n^{(2)}}{\max\{\|w_n\|, \eta_3\}}, \quad (3.7)$$



where $\eta_1, \eta_2, \eta_3 > 0$, $\varsigma_n, \zeta_n^{(i)} \in [0, +\infty)$ such that $\lim_{n \rightarrow \infty} \frac{\varsigma_n}{\alpha_n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\zeta_n^{(i)}}{\alpha_n} = 0$, $i = 1, 2$.

We observe from (3.6) and (3.7) that $\forall n \geq 1$ and $i = 1, 2$,

$$\beta_n \|t_n - t_{n-1}\| \leq \varsigma_n, \quad \xi_n^{(1)} \|\gamma_{n-1}\| \leq \zeta_n^{(1)} \text{ and } \xi_n^{(2)} \|w_n\| \leq \zeta_n^{(2)}. \quad (3.8)$$

Since $\lim_{n \rightarrow \infty} \frac{\varsigma_n}{\alpha_n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\zeta_n^{(i)}}{\alpha_n} = 0$, $i = 1, 2$, then (3.8) yields

$$\lim_{n \rightarrow \infty} \frac{\beta_n \|t_n - t_{n-1}\|}{\alpha_n} = 0, \quad \lim_{n \rightarrow \infty} \frac{\xi_n^{(1)} \|\gamma_{n-1}\|}{\alpha_n} = 0, \quad \lim_{n \rightarrow \infty} \frac{\xi_n^{(2)} \|w_n\|}{\alpha_n} = 0. \quad (3.9)$$

Moreover, from the assignment of ϖ_n in Algorithm 3 and the nonexpansiveness of $J_\zeta^{F_1}$, we have

$$\begin{aligned} \|\varpi_n\| &= \|J_\zeta^{F_1}(u_n + \gamma_n) - J_\zeta^{F_1}(u_n - \lambda_n \nabla g(h_n))\| \\ &\leq \xi_n^{(1)} \|\gamma_{n-1}\| + \xi_n^{(2)} \|w_n\|, \end{aligned} \quad (3.10)$$

thus, from (3.9) and the fact that $\alpha_n \in (0, 1)$, we obtain $\lim_{n \rightarrow \infty} \frac{\|\varpi_n\|}{\alpha_n} = 0$.

We first prove the following lemma.

Lemma 3.3. Suppose that the sequence $\{\lambda_n\}$ is produced by (3.5). Then it is bounded away from zero and there exists $\lambda > 0$ such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda$.

Proof. If $\|\nabla g(u_n) - \nabla g(h_n)\| \neq 0$, then $\|B\| > 0$. Moreover, $\|\nabla g(u_n) - \nabla g(h_n)\| \leq \|B\|^2 \|u_n - h_n\|$ (since ∇g is $\|B\|^2$ -Lipschitz continuous). Therefore, we deduce

$$\frac{\varepsilon \|u_n - h_n\|}{\|\nabla g(u_n) - \nabla g(h_n)\|} \geq \frac{\varepsilon \|u_n - h_n\|}{\|B\|^2 \|u_n - h_n\|} = \frac{\varepsilon}{\|B\|^2}. \quad (3.11)$$

From (3.5), we see that $\lambda_{n+1} \geq \min\{\lambda_n, \frac{\varepsilon}{\|B\|^2}\}$. By induction, we obtain that $\lambda_n \geq \min\{\lambda_1, \frac{\varepsilon}{\|B\|^2}\}$. One also sees from (3.5) that $\lambda_{n+1} \leq \lambda_n$ for all $n \geq 1$. Thus the monotonicity and the existence of the lower bound of the sequence $\{\lambda_n\}$ implies $\lim_{n \rightarrow \infty} \lambda_n$ exists. Since $\min\{\lambda_1, \frac{\varepsilon}{\|B\|^2}\} > 0$ is the lower bound of the sequence $\{\lambda_n\}$, there exists $\lambda > 0$ such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda$. ■

Next, we verify the stopping criterion of Algorithm 3.

Lemma 3.4. If $h_n = u_n$ or $f_n = 0$ for some $n \geq 1$ in Algorithm 3, then $h_n \in \Omega$.

Proof. First, we show that $h_n = u_n$ if and only if $f_n = 0$. From the assignments of f_n and λ_{n+1} in (3.2) and (3.5), respectively, we obtain

$$\begin{aligned} \|f_n\| &\geq \|u_n - h_n\| - \tau \lambda_n \|\nabla g(u_n) - \nabla g(h_n)\| \\ &\geq \left(1 - \frac{\tau \varepsilon \lambda_n}{\lambda_{n+1}}\right) \|u_n - h_n\| \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} \|f_n\| &\leq \|u_n - h_n\| + \tau \lambda_n \|\nabla g(u_n) - \nabla g(h_n)\| \\ &\leq \left(1 + \frac{\tau \varepsilon \lambda_n}{\lambda_{n+1}}\right) \|u_n - h_n\|. \end{aligned} \quad (3.13)$$



In connection with Lemma 3.3 and the fact that $\tau \in (0, \frac{1}{\varepsilon})$, we have

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\tau \varepsilon \lambda_n}{\lambda_{n+1}}\right) = 1 - \tau \varepsilon > 0. \quad (3.14)$$

Combining (3.12), (3.13) and (3.14), one sees that $h_n = u_n$ if and only if $f_n = 0$. Now, let $h_n = u_n$ or $f_n = 0$, then we obtain from the definitions of h_n in Algorithm 3 that $h_n = J_\zeta^{F_1}(h_n - \tau \lambda_n \nabla g(h_n))$. Thus, it follows from Lemma 2.5 that $h_n \in \Omega$. Moreover, from (3.14), one easily sees that there exists $\mathcal{X} \in \mathbb{N}$ such that $(1 - \pi_n) > 0$ and $(1 + \pi_n) > 0$, $\forall n \geq \mathcal{X}$, where $\pi_n = \frac{\tau \varepsilon \lambda_n}{\lambda_{n+1}}$, $\forall n \geq 1$. ■

Next, we establish the boundedness of the even subsequence $\{t_{2n}\}$ generated by Algorithm 3.

Lemma 3.5. *Let the sequence $\{t_n\}$ be generated by Algorithm 3. Then, an even subsequence $\{\|t_{2n} - z\|\}$ of $\{\|t_n - z\|\}$ is bounded for any point $z \in \Omega$.*

Proof. We observe from Lemma 3.3, the assignments of f_n , θ_n and λ_{n+1} in (3.2), (3.3) and (3.5), respectively, and the fact that $\tau > 0$ and $g(h_n) \geq 0$, $\forall n \geq 1$ that

$$\begin{aligned} \theta_n \|f_n\|^2 &\geq \langle f_n, u_n - h_n \rangle \\ &= \langle u_n - h_n, u_n - h_n \rangle - \tau \lambda_n \langle \nabla g(u_n) - \nabla g(h_n), u_n - h_n \rangle \\ &\geq \|u_n - h_n\|^2 - \tau \lambda_n \|\nabla g(u_n) - \nabla g(h_n)\| \|u_n - h_n\| \\ &\geq (1 - \pi_n) \|u_n - h_n\|^2. \end{aligned} \quad (3.15)$$

Combining (3.13) and (3.15), we have

$$\theta_n \geq \frac{1 - \pi_n}{(1 + \pi_n)^2}. \quad (3.16)$$

From the inequalities (3.15) and (3.16) and the fact that there exists $\mathcal{X} \in \mathbb{N}$ such that $1 - \pi_n > 0$ and $1 + \pi_n > 0$, $\forall n \geq \mathcal{X}$, one sees that $\theta_n > 0$, $\forall n \geq \mathcal{X}$ and

$$\|u_n - h_n\|^2 \leq \left(\frac{1 + \pi_n}{1 - \pi_n}\right)^2 \theta_n^2 \|f_n\|^2, \quad \forall n \geq \mathcal{X}. \quad (3.17)$$

Now, let $z \in \Omega$. From the assignment of h_n in Algorithm 3 and the resolvent property (2.8), we have

$$\langle u_n - \tau \lambda_n \nabla g(u_n) - h_n, h_n - z \rangle \geq 0. \quad (3.18)$$

Moreover, combining Lemma 2.3(iii), the assignment of f_n in (3.2), the inequality (3.18) and the fact that for any $z \in \Omega$, $\nabla g(z) = 0$, we obtain

$$\begin{aligned} \langle f_n, h_n - z \rangle &= \langle u_n - \tau \lambda_n \nabla g(u_n) - h_n, h_n - z \rangle + \langle \tau \lambda_n \nabla g(h_n), h_n - z \rangle \\ &\geq \langle \tau \lambda_n \nabla g(h_n), h_n - z \rangle \\ &= \tau \lambda_n \left\langle (I - J_\zeta^{F_2}) B h_n - (I - J_\zeta^{F_2}) B z, B h_n - B z \right\rangle \\ &\geq 2\tau \lambda_n g(h_n). \end{aligned} \quad (3.19)$$

We also find from the assignment of θ_n in (3.3) and inequality (3.19) that

$$\begin{aligned} \langle f_n, u_n - z \rangle &= \langle f_n, u_n - h_n \rangle + \langle f_n, h_n - z \rangle \\ &\geq \langle f_n, u_n - h_n \rangle + 2\tau \lambda_n g(h_n) \\ &= \theta_n \|f_n\|^2. \end{aligned} \quad (3.20)$$



Combining (3.20) and setting $q_n = u_n - \sigma\theta_n f_n$, $\forall n \geq 1$, we get

$$\begin{aligned} \|q_n - z\|^2 &= \|u_n - z\|^2 + \sigma^2 \theta_n^2 \|f_n\|^2 - 2\sigma\theta_n \langle f_n, u_n - z \rangle \\ &\leq \|u_n - z\|^2 - \left(\frac{2-\sigma}{\sigma}\right) \sigma^2 \theta_n^2 \|f_n\|^2. \end{aligned} \quad (3.21)$$

From (3.14), the facts that $\sigma \in (0, 2)$ and there exists $\mathcal{X} \in \mathbb{N}$ such that $\theta_n > 0$, $\forall n \geq \mathcal{X}$ and (3.21), we have

$$\|q_n - z\| \leq \|u_n - z\|, \forall n \geq \mathcal{X}. \quad (3.22)$$

From the assignments of m_n , q_n and (3.22), we deduce

$$\begin{aligned} \|m_n - z\| &= \|q_n - z + \varpi_n\| \\ &\leq \|q_n - z\| + \|\varpi_n\| \\ &\leq \|u_n - z\| + \|\varpi_n\|, \forall n \geq \mathcal{X}. \end{aligned} \quad (3.23)$$

Setting $c_n := \frac{1}{1-\alpha_n}(\delta_n u_n + \mu_n m_n)$, we find from (3.23) and the fact that $\alpha_n + \delta_n + \mu_n = 1$, $\forall n \geq 1$ that

$$\begin{aligned} \|c_n - z\| &\leq \frac{\delta_n}{1-\alpha_n} \|u_n - z\| + \frac{\mu_n}{1-\alpha_n} \|m_n - z\| \\ &\leq \frac{\delta_n}{1-\alpha_n} \|u_n - z\| + \frac{\mu_n}{1-\alpha_n} \|\varpi_n\| + \frac{\mu_n}{1-\alpha_n} \|u_n - z\| \\ &\leq \|u_n - z\| + \frac{\mu_n}{1-\alpha_n} \|\varpi_n\|, \forall n \geq \mathcal{X}. \end{aligned} \quad (3.24)$$

From Algorithm 3, the inequality (3.24) and the c -contractiveness of f , we obtain

$$\|t_{n+1} - z\| \leq \alpha_n \|f(u_n) - f(z)\| + (1-\alpha_n) \|c_n - z\| + \alpha_n \|f(z) - z\| \quad (3.25)$$

$$\begin{aligned} &\leq \alpha_n c \|u_n - z\| + (1-\alpha_n) \|u_n - z\| + \alpha_n \|f(z) - z\| + \mu_n \|\varpi_n\| \\ &= (1-\alpha_n(1-c)) \|u_n - z\| + \alpha_n \|f(z) - z\| \\ &\quad + \mu_n \|\varpi_n\|, \forall n \geq \mathcal{X}. \end{aligned} \quad (3.26)$$

In view of the inequalities (3.8), (3.26) and the assignment of u_n in (3.1), we have

$$\begin{aligned} \|t_{2n+1} - z\| &\leq (1-\alpha_{2n}(1-c)) \|t_{2n} - z\| + \mu_{2n} \|\varpi_{2n}\| \\ &\quad + \alpha_{2n} \|f(z) - z\|, \forall n \geq \mathcal{X}. \end{aligned} \quad (3.27)$$

and

$$\begin{aligned} \|u_{2n+1} - z\| &\leq \|t_{2n+1} - z\| + \beta_{2n+1} \|t_{2n+1} - t_{2n}\| \\ &\leq \|t_{2n+1} - z\| + \varsigma_{2n+1}. \end{aligned} \quad (3.28)$$



From (3.10), (3.26), (3.27), (3.28) and the facts that the sequence $\{\alpha_n\}$ is nonincreasing, $\alpha_n(1-c) \in (0, 1)$, and $\mu_n \in [a, 1-a], \forall n \geq 1, a \in (0, \frac{1}{2})$, we find that

$$\begin{aligned}
 \|t_{2n+2} - z\| &\leq \|u_{2n+1} - z\| + \alpha_{2n+1}\|f(z) - z\| + \mu_{2n+1}\|\varpi_{2n+1}\| \\
 &\leq \|t_{2n+1} - z\| + \alpha_{2n+1}\|f(z) - z\| + \mu_{2n+1}\|\varpi_{2n+1}\| + \varsigma_{2n+1} \\
 &\leq (1 - \alpha_{2n}(1-c))\|t_{2n} - z\| + \alpha_{2n}\|f(z) - z\| + \alpha_{2n+1}\|f(z) - z\| \\
 &\quad + \mu_{2n}\|\varpi_{2n}\| + \mu_{2n+1}\|\varpi_{2n+1}\| + \varsigma_{2n+1} \\
 &\leq (1 - \alpha_{2n}(1-c))\|t_{2n} - z\| + \alpha_{2n}(1-c)\left(\frac{2}{1-c}\|f(z) - z\| \right. \\
 &\quad \left. + \frac{\varsigma_{2n+1}}{\alpha_{2n}(1-c)} + \frac{\Theta_{2n}}{\alpha_{2n}(1-c)}\right), \forall n \geq \mathcal{X},
 \end{aligned} \tag{3.29}$$

where $\Theta_{2n} = (1-a)(\xi_{2n}^{(1)}\|\gamma_{2n-1}\| + \xi_{2n}^{(2)}\|w_{2n}\| + \xi_{2n+1}^{(1)}\|\gamma_{2n}\| + \xi_{2n+1}^{(2)}\|w_{2n+1}\|)$.

In view (3.8) and the facts that $\lim_{n \rightarrow \infty} \frac{\varsigma_n}{\alpha_n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\varsigma_n^{(i)}}{\alpha_n} = 0, i = 1, 2$ in Assumption 3.2, take $M_1, M_2 > 0$ such that $\frac{\varsigma_n^{(i)}}{\alpha_n} \leq M_1$ and $\frac{\varsigma_n}{\alpha_n} \leq M_2, \forall n \geq 1, i = 1, 2$. Then, since the sequence $\{\alpha_n\}$ is nonincreasing, by denoting $M := \frac{2}{1-c}(\|f(z) - z\| + 2M_1) + \frac{M_2}{1-c}$, (3.29) yields

$$\begin{aligned}
 \|t_{2n+2} - z\| &\leq (1 - \alpha_{2n}(1-c))\|t_{2n} - z\| + \alpha_{2n}(1-c)M \\
 &\leq \max\{\|t_{2n} - z\|, M\} \\
 &\quad \vdots \\
 &\leq \max\{\|t_0 - z\|, M\}, \forall n \geq \mathcal{X},
 \end{aligned} \tag{3.30}$$

which implies that a subsequence $\{\|t_{2n} - z\|\}$ of $\{\|t_n - z\|\}$ is bounded. So, the even subsequence $\{t_{2n}\}$ generated by Algorithm 3 is bounded. So as the subsequences $\{m_{2n}\}, \{h_{2n}\}, \{f_{2n}\}$ and $\{u_{2n}\}$. Moreover, one easily sees from (3.27), (3.28), the Assumption 3.2 and the assignment of h_{2n+1} that the odd subsequences $\{t_{2n+1}\}, \{u_{2n+1}\}$ and $\{h_{2n+1}\}$ are bounded. Consequently, from (3.8), (3.10), (3.13), the assignment of m_{2n+1} and the facts that $\lim_{n \rightarrow \infty} \frac{\varsigma_n^{(i)}}{\alpha_n} = 0, i = 1, 2$, we see that the odd subsequences $\{f_{2n+1}\}$ and $\{m_{2n+1}\}$ are bounded. $\blacksquare$

Next, we establish the convergence of Algorithm 3.

Theorem 3.6. Suppose that Assumptions 3.1 and 3.2 hold and $\{t_n\}$ is a sequence produced by Algorithm 3. Then it strongly converges to a point $z = P_{\Omega_{SVIP}}f(z)$.

Proof. Applying the inequality (3.21) and the assignment of m_n in Algorithm 3, we obtain

$$\begin{aligned}
 \|m_n - z\|^2 &= \|q_n + \varpi_n - z\|^2 \\
 &\leq (1 + \|\varpi_n\|)\|q_n - z\|^2 + \|\varpi_n\| + \|\varpi_n\|^2 \\
 &\leq \|u_n - z\|^2 - (1 + \|\varpi_n\|)\left(\frac{2-\sigma}{\sigma}\right)\sigma^2\theta_n^2\|f_n\|^2 \\
 &\quad + \|\varpi_n\|(1 + \|\varpi_n\| + \|u_n - z\|^2).
 \end{aligned} \tag{3.31}$$



From (2.3), (3.31) and the fact that $\alpha_n + \delta_n + \mu_n = 1$, one finds

$$\begin{aligned}
 \|c_n - z\|^2 &= \frac{\delta_n}{1 - \alpha_n} \|u_n - z\|^2 + \frac{\mu_n}{1 - \alpha_n} \|m_n - z\|^2 - \frac{\delta_n \mu_n}{(1 - \alpha_n)^2} \|u_n - m_n\|^2 \\
 &\leq \frac{\delta_n}{1 - \alpha_n} \|u_n - z\|^2 + \frac{\mu_n}{1 - \alpha_n} \|u_n - z\|^2 - \frac{\delta_n \mu_n}{(1 - \alpha_n)^2} \|u_n - m_n\|^2 \\
 &\quad + \frac{\mu_n}{1 - \alpha_n} \|\varpi_n\| (1 + \|\varpi_n\| + \|u_n - z\|^2) \\
 &\quad - \frac{\mu_n}{1 - \alpha_n} (1 + \|\varpi_n\|) \left(\frac{2 - \sigma}{\sigma} \right) \sigma^2 \theta_n^2 \|f_n\|^2 \\
 &= \frac{\mu_n}{1 - \alpha_n} \|\varpi_n\| (1 + \|\varpi_n\| + \|u_n - z\|^2) - \frac{\delta_n \mu_n}{(1 - \alpha_n)^2} \|u_n - m_n\|^2 \\
 &\quad + \|u_n - z\|^2 - \frac{\mu_n}{1 - \alpha_n} (1 + \|\varpi_n\|) \left(\frac{2 - \sigma}{\sigma} \right) \sigma^2 \theta_n^2 \|f_n\|^2. \quad (3.32)
 \end{aligned}$$

In view of (3.25) and (3.32), we deduce

$$\begin{aligned}
 \|t_{n+1} - z\|^2 &\leq \alpha_n^2 c^2 \|u_n - z\|^2 + \alpha_n^2 \|f(z) - z\|^2 + (1 - \alpha_n)^2 \|c_n - z\|^2 + \Lambda_n \\
 &\leq \alpha_n c \|u_n - z\|^2 + \alpha_n^2 \|f(z) - z\|^2 - \delta_n \mu_n \|u_n - m_n\|^2 + \Lambda_n \\
 &\quad + (1 - \alpha_n) \|u_n - z\|^2 + \mu_n (1 - \alpha_n) \|\varpi_n\| (1 + \|\varpi_n\| + \|u_n - z\|^2) \\
 &\quad - \mu_n (1 - \alpha_n) (1 + \|\varpi_n\|) \left(\frac{2 - \sigma}{\sigma} \right) \sigma^2 \theta_n^2 \|f_n\|^2 \\
 &= (1 - \alpha_n (1 - c)) \|u_n - z\|^2 + \Phi_n + \Lambda_n - \Psi_n. \quad (3.33)
 \end{aligned}$$

where

$$\Phi_n := \alpha_n^2 \|f(z) - z\|^2 + \mu_n (1 - \alpha_n) \|\varpi_n\| (1 + \|\varpi_n\| + \|u_n - z\|^2),$$

$$\Lambda_n := 2\alpha_n c \|u_n - z\| (\alpha_n \|f(z) - z\| + (1 - \alpha_n) \|c_n - z\|) + 2\alpha_n (1 - \alpha_n) \|f(z) - z\| \|c_n - z\|$$

and

$$\Psi_n := \mu_n (1 - \alpha_n) (1 + \|\varpi_n\|) \left(\frac{2 - \sigma}{\sigma} \right) \sigma^2 \theta_n^2 \|f_n\|^2 + \delta_n \mu_n \|u_n - m_n\|^2.$$

From (3.33) and similar arguments in the derivation of (3.27), we get

$$\|t_{2n+1} - z\|^2 \leq (1 - \alpha_{2n} (1 - c)) \|t_{2n} - z\|^2 + \Phi_{2n} + \Lambda_{2n} - \Psi_{2n}. \quad (3.34)$$

Since $\alpha_n (1 - c) \in (0, 1)$, $\forall n \geq 1$, using (3.28), (3.33), (3.34) and the same arguments used to deduce (3.29), we obtain

$$\begin{aligned}
 \|t_{2n+2} - z\|^2 &\leq (1 - \alpha_{2n+1} (1 - c)) \|t_{2n+1} - z\|^2 + \Phi_{2n+1} + \Lambda_{2n+1} - \Psi_{2n+1} \\
 &\leq \|t_{2n+1} - z\|^2 + \varsigma_{2n+1} (\varsigma_{2n+1} + 2 \|t_{2n+1} - z\|) \\
 &\quad + \Phi_{2n+1} + \Lambda_{2n+1} - \Psi_{2n+1} \\
 &\leq \|t_{2n} - z\|^2 + \Phi_{2n} + \Phi_{2n+1} + \Lambda_{2n} + \Lambda_{2n+1} \\
 &\quad + \varsigma_{2n+1} (\varsigma_{2n+1} + 2 \|t_{2n+1} - z\|) - (\Psi_{2n} + \Psi_{2n+1}). \quad (3.35)
 \end{aligned}$$



On the other hand, in view of the inequality (3.24) and the fact that f is c -contractive, we have

$$\begin{aligned} \|t_{n+1} - z\|^2 &= \|\alpha_n(f(u_n) - f(z)) + (1 - \alpha_n)(c_n - z) + \alpha_n(f(z) - z)\|^2 \\ &\leq \|\alpha_n(f(u_n) - f(z)) + (1 - \alpha_n)(c_n - z)\|^2 + 2\alpha_n \langle f(z) - z, t_{n+1} - z \rangle \\ &\leq \alpha_n \|f(u_n) - f(z)\|^2 + (1 - \alpha_n) \|c_n - z\|^2 + 2\alpha_n \langle f(z) - z, t_{n+1} - z \rangle \\ &\leq (1 - \alpha_n(1 - c)) \|u_n - z\|^2 + \mu_n \|\varpi_n\| \left(\frac{\mu_n}{1 - \alpha_n} \|\varpi_n\| + 2\|u_n - z\| \right) \\ &\quad + 2\alpha_n \langle f(z) - z, t_{n+1} - z \rangle, \quad \forall n \geq \mathcal{X}. \end{aligned} \quad (3.36)$$

In view of (3.36) and similar arguments used in deducing (3.27), we deduce

$$\begin{aligned} \|t_{2n+1} - z\|^2 &\leq (1 - \alpha_{2n}(1 - c)) \|t_{2n} - z\|^2 + 2\alpha_{2n} \langle f(z) - z, t_{2n+1} - z \rangle \\ &\quad + \mu_{2n} \|\varpi_{2n}\| \left(\frac{\mu_{2n}}{1 - \alpha_{2n}} \|\varpi_{2n}\| + 2\|t_{2n} - z\| \right), \quad \forall n \geq \mathcal{X}. \end{aligned} \quad (3.37)$$

Similarly, from the inequalities (3.28), (3.36), (3.37) and the same arguments in obtaining (3.29), we have

$$\begin{aligned} \|t_{2n+2} - z\|^2 &\leq \|u_{2n+1} - z\|^2 + 2\alpha_{2n+1} \langle f(z) - z, t_{2n+2} - z \rangle \\ &\quad + \mu_{2n+1} \|\varpi_{2n+1}\| \left(\frac{\mu_{2n+1}}{1 - \alpha_{2n+1}} \|\varpi_{2n+1}\| + 2\|u_{2n+1} - z\| \right) \\ &\leq \|t_{2n+1} - z\|^2 + \mu_{2n+1} \|\varpi_{2n+1}\| \left(\frac{\mu_{2n+1}}{1 - \alpha_{2n+1}} \|\varpi_{2n+1}\| + 2\|u_{2n+1} - z\| \right) \\ &\quad + \varsigma_{2n+1} (\varsigma_{2n+1} + 2\|t_{2n+1} - z\|) + 2\alpha_{2n+1} \langle f(z) - z, u_{2n+2} - z \rangle \\ &\leq (1 - \alpha_{2n}(1 - c)) \|t_{2n} - z\|^2 + 2\alpha_{2n+1} \langle f(z) - z, t_{2n+2} - z \rangle \\ &\quad + \varsigma_{2n+1} (\varsigma_{2n+1} + 2\|t_{2n+1} - z\|) + 2\alpha_{2n} \langle f(z) - z, t_{2n+1} - z \rangle \\ &\quad + \mu_{2n+1} \|\varpi_{2n+1}\| \left(\frac{\mu_{2n+1}}{1 - \alpha_{2n+1}} \|\varpi_{2n+1}\| + 2\|u_{2n+1} - z\| \right) \\ &\quad + \mu_{2n} \|\varpi_{2n}\| \left(\frac{\mu_{2n}}{1 - \alpha_{2n}} \|\varpi_{2n}\| + 2\|t_{2n} - z\| \right), \quad \forall n \geq \mathcal{X}. \end{aligned} \quad (3.38)$$

Consequently, we obtain from (3.35) and (3.38) that for all $n \geq \mathcal{X}$,

$$\|t_{2n+2} - z\|^2 \leq \|t_{2n} - z\|^2 + \Gamma_{2n} - \hat{\chi}_{2n}$$

and

$$\|t_{2n+2} - z\|^2 \leq (1 - \alpha_{2n}(1 - c)) \|t_{2n} - z\|^2 + \alpha_{2n}(1 - c) \Upsilon_{2n}, \quad (3.39)$$

where

$$\hat{\chi}_{2n} := \Psi_{2n} + \Psi_{2n+1},$$

$$\Gamma_{2n} := \Phi_{2n} + \Phi_{2n+1} + \Lambda_{2n} + \Lambda_{2n+1} + \varsigma_{2n+1} (\varsigma_{2n+1} + 2\|t_{2n+1} - z\|)$$

and

$$\begin{aligned} \Upsilon_{2n} &:= \frac{1}{\alpha_{2n}(1 - c)} \left(2\alpha_{2n+1} \langle f(z) - z, t_{2n+2} - z \rangle + \varsigma_{2n+1} (\varsigma_{2n+1} + 2\|t_{2n+1} - z\|) \right. \\ &\quad + 2\alpha_{2n} \langle f(z) - z, t_{2n+1} - z \rangle + \mu_{2n+1} \|\varpi_{2n+1}\| \left(\frac{\mu_{2n+1}}{1 - \alpha_{2n+1}} \|\varpi_{2n+1}\| + 2\|u_{2n+1} - z\| \right) \\ &\quad \left. + \mu_{2n} \|\varpi_{2n}\| \left(\frac{\mu_{2n}}{1 - \alpha_{2n}} \|\varpi_{2n}\| + 2\|t_{2n} - z\| \right) \right). \end{aligned}$$



From the facts that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} \frac{\varsigma_n}{\alpha_n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\varsigma_n^{(i)}}{\alpha_n} = 0$, $i = 1, 2$ in Assumption 3.2, it follows from (3.8) and (3.10) that $\lim_{n \rightarrow \infty} \|\varpi_n\| = 0$. Thus, $\lim_{n \rightarrow \infty} \Gamma_{2n} = 0$. Therefore, to use Lemma 2.6, we show that for every subsequence $\{\hat{\chi}_{2n_r}\}$ of $\{\hat{\chi}_{2n}\}$, the following holds:

$$\lim_{r \rightarrow \infty} \hat{\chi}_{2n_r} = 0 \Rightarrow \limsup_{r \rightarrow \infty} \Upsilon_{2n_r} \leq 0.$$

Now, suppose that $\{\hat{\chi}_{2n_r}\}$ is a subsequence of $\{\hat{\chi}_{2n}\}$ such that $\lim_{r \rightarrow \infty} \hat{\chi}_{2n_r} = 0$, then from the facts that $\sigma \in (0, 2)$, $\alpha_n \in (0, 1)$, δ_n , $\mu_n \in [a, 1 - a]$ for some $a \in (0, \frac{1}{2})$ and $\theta_n > 0$, $\forall n \geq \mathcal{X}$, we have

$$\lim_{r \rightarrow \infty} \|f_{2n_r}\| = 0, \lim_{r \rightarrow \infty} \|f_{2n_r+1}\| = 0, \lim_{r \rightarrow \infty} \|u_{2n_r} - m_{2n_r}\| = 0$$

and

$$\lim_{r \rightarrow \infty} \|u_{2n_r+1} - m_{2n_r+1}\| = 0. \quad (3.40)$$

Moreover, from the assignment of c_n and (3.40), one sees that

$$\|c_{2n_r} - m_{2n_r}\| = \frac{\delta_n}{1 - \alpha_n} \|u_{2n_r} - m_{2n_r}\| \rightarrow 0 \text{ as } r \rightarrow \infty \quad (3.41)$$

and

$$\|c_{2n_r+1} - m_{2n_r+1}\| \leq \frac{\delta_{2n_r+1}}{1 - \alpha_{2n_r+1}} \|u_{2n_r+1} - m_{2n_r+1}\| \rightarrow 0 \text{ as } r \rightarrow \infty. \quad (3.42)$$

Furthermore, from (3.40), (3.41), (3.42), the assignment of t_{n+1} in Algorithm 3, the facts that $\lim_{n \rightarrow \infty} \alpha_n = 0$, f is c -contractive, $\{u_{2n}\}$, $\{m_{2n}\}$, $\{u_{2n+1}\}$, $\{m_{2n+1}\}$ are bounded, $\lim_{n \rightarrow \infty} \|\varpi_n\| = 0$, and there exists $\mathcal{X} \in \mathbb{N}$ such that $\theta > 0$, $\forall n \geq \mathcal{X}$, we find that

$$\begin{aligned} \|t_{2n_r+1} - m_{2n_r}\| &\leq \alpha_{2n_r} \|f(u_{2n_r}) - m_{2n_r}\| \\ &\quad + (1 - \alpha_{2n_r}) \|c_{2n_r} - m_{2n_r}\| \rightarrow 0 \text{ as } r \rightarrow \infty, \end{aligned} \quad (3.43)$$

$$\|m_{2n_r+1} - t_{2n_r+1}\| \leq \sigma \theta_{2n_r+1} \|f_{2n_r+1}\| + \|\varpi_{2n_r+1}\| \rightarrow 0 \text{ as } r \rightarrow \infty \quad (3.44)$$

and

$$\begin{aligned} \|t_{2n_r+2} - m_{2n_r+1}\| &\leq \alpha_{2n_r+1} \|f(u_{2n_r+1}) - m_{2n_r+1}\| \\ &\quad + (1 - \alpha_{2n_r+1}) \|c_{2n_r+1} - m_{2n_r+1}\| \rightarrow 0 \text{ as } r \rightarrow \infty. \end{aligned} \quad (3.45)$$

Then, since the subsequence $\{t_{2n}\}$ of $\{t_n\}$ generated by Algorithm 3 is bounded, there exists a subsequence $\{t_{2n_r}\}$ of $\{t_{2n}\}$ such that $t_{2n_r} \rightharpoonup t^*$ as $r \rightarrow \infty$. Consequently, from (3.40), (3.41), (3.43), (3.44), (3.45) and the assignment of u_{2n_r} in Algorithm 3, we see that $m_{2n_r} \rightharpoonup t^*$, $c_{2n_r} \rightharpoonup t^*$, $t_{2n_r+1} \rightharpoonup t^*$, $m_{2n_r+1} \rightharpoonup t^*$ and $t_{2n_r+2} \rightharpoonup t^*$ as $r \rightarrow \infty$. Combining (3.17) and (3.40), we have

$$\lim_{r \rightarrow \infty} \|u_{2n_r} - h_{2n_r}\| = 0. \quad (3.46)$$

In connection with the assignment of θ_{2n} and (3.15), we get

$$\begin{aligned} \theta_{2n_r}^2 \|f_{2n_r}\|^2 &= \theta_{2n_r} (\langle u_{2n_r} - h_{2n_r}, f_{2n_r} \rangle + 2\tau \lambda_{2n_r} g(h_{2n_r})) \\ &\geq \theta_{2n_r} ((1 - \pi_{2n_r}) \|u_{2n_r} - h_{2n_r}\|^2 + 2\tau \lambda_{2n_r} g(h_{2n_r})). \end{aligned} \quad (3.47)$$



Combining (3.40), (3.46) and (3.47), we obtain

$$\lim_{r \rightarrow \infty} g(h_{2n_r}) = \lim_{r \rightarrow \infty} \frac{1}{2} \|(I - J_\zeta^{F_2})Bh_{2n_r}\|^2 = 0. \quad (3.48)$$

Since B is bounded linear operator, one sees from (3.48) that

$$\begin{aligned} \|\nabla g(h_{2n_r})\|^2 &= \|B^*(I - J_\zeta^{F_2})Bh_{2n_r}\|^2 \\ &\leq \|B\|^2 \|(I - J_\zeta^{F_2})Bh_{2n_r}\|^2 \\ &= 2\|B\|^2 g(h_{2n_r}) \rightarrow 0 \text{ as } r \rightarrow \infty. \end{aligned} \quad (3.49)$$

Moreover, it follows from (3.5), (3.46) and Lemma 3.3 that

$$\|\nabla g(u_{2n_r}) - \nabla g(h_{2n_r})\| \leq \frac{\varepsilon}{\lambda_{2n_r+1}} \|u_{2n_r} - h_{2n_r}\| \rightarrow 0 \text{ as } r \rightarrow \infty. \quad (3.50)$$

Combining (3.49) and (3.50), we obtain that

$$\|\nabla g(u_{2n_r})\| \leq \|\nabla g(u_{2n_r}) - \nabla g(h_{2n_r})\| + \|\nabla g(h_{2n_r})\| \rightarrow 0 \text{ as } r \rightarrow \infty. \quad (3.51)$$

Using (3.51), Lemma 3.3, the definitions of h_{2n} , s_{2n} and the nonexpansiveness of $J_\zeta^{F_1}$, we have

$$\begin{aligned} \|h_{2n_r} - J_\zeta^{F_1}u_{2n_r}\| &= \|J_\zeta^{F_1}(u_{2n_r} - \tau\lambda_{2n_r}\nabla g(u_{2n_r})) - J_\zeta^{F_1}u_{2n_r}\| \\ &\leq \tau\lambda_{2n_r}\|\nabla g(u_{2n_r})\| \rightarrow 0 \text{ as } r \rightarrow \infty. \end{aligned} \quad (3.52)$$

Combining (3.46) and (3.52), we see that

$$\|u_{2n_r} - J_\zeta^{F_1}u_{2n_r}\| \leq \|u_{2n_r} - h_{2n_r}\| + \|h_{2n_r} - J_\zeta^{F_1}u_{2n_r}\| \rightarrow 0 \text{ as } r \rightarrow \infty. \quad (3.53)$$

Since $u_{2n_r} \rightharpoonup t^*$ and $h_{2n_r} \rightharpoonup t^*$ as $r \rightarrow \infty$, then by the weak lower semicontinuity of g and h , together with (3.48) and (3.53) imply that

$$0 \leq g(t^*) \leq \liminf_{r \rightarrow \infty} g(h_{2n_r}) = 0,$$

and

$$0 \leq h(t^*) \leq \liminf_{r \rightarrow \infty} h(u_{2n_r}) = 0. \quad (3.54)$$

Therefore,

$$\|(I - J_\zeta^{F_2})Bt^*\| = 0 \quad \text{and} \quad \|(I - J_\zeta^{F_1})u^*\| = 0,$$

which imply that

$$0 \in F_1(t^*) \quad \text{and} \quad 0 \in F_2(Bt^*),$$

thus, $t^* \in \Omega_{SVIP}$. From (2.4), the definition of z and the fact that $t_{2n_r+1} \rightharpoonup t^*$ and $t_{2n_r+2} \rightharpoonup t^*$ as $r \rightarrow \infty$, we see that

$$\begin{aligned} \limsup_{r \rightarrow \infty} \langle f(z) - z, t_{2n_r+1} - z \rangle &= \lim_{j \rightarrow \infty} \langle f(z) - z, t_{2n_{r_j}+1} - z \rangle \\ &= \langle f(z) - z, t^* - z \rangle \leq 0 \end{aligned} \quad (3.55)$$

and

$$\begin{aligned} \limsup_{r \rightarrow \infty} \langle f(z) - z, t_{2n_r+2} - z \rangle &= \lim_{j \rightarrow \infty} \langle f(z) - z, t_{2n_{r_j}+2} - z \rangle \\ &= \langle f(z) - z, t^* - z \rangle \leq 0. \end{aligned} \quad (3.56)$$



Combining (3.55), (3.56), Assumptions 3.1 and Assumption 3.2, we have $\limsup_{r \rightarrow \infty} \Upsilon_{2n_r} \leq 0$. Therefore, from Lemma 2.6, we see that $\lim_{n \rightarrow \infty} \|t_{2n} - z\| = 0$. Thus, $t_{2n} \rightarrow z$ as $n \rightarrow \infty$.

Finally, since $\lim_{n \rightarrow \infty} \|t_{2n} - z\| = 0$, $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\lim_{n \rightarrow \infty} \|\varpi_n\| = 0$, (3.27) yields $t_{2n+1} \rightarrow z$ as $n \rightarrow \infty$. Then $\{t_n\}$ produced by Algorithm 3 strongly converges to z , which marks the end of the proof. ■

Next, we present the first modification of Algorithm 3, which differs from Algorithm 3 in Step 4. In the first modified version, a new adaptive step length strategy that is different from the one provided in (3.5) is suggested.

Algorithm 4

Initialization: Take λ_1 , ζ , ε , σ , τ , $\{\delta_n\}$, $\{\alpha_n\}$, $\{\mu_n\}$, $\{\beta_n\}$, and $\{\xi_n^{(i)}\}$ for $i = 1, 2$ such that condition (B3) and Assumption 3.2 hold. Select $\varepsilon_2 \in (0, \frac{4}{\tau})$. Choose t_0 , $t_1 \in \mathcal{H}_1$, $\gamma_0 = -\nabla g(t_0)$ and $\{w_n\} \subset \mathcal{H}_1$. Set $n = 1$.

Step 1. Compute u_n by (3.1).

Step 2. Compute $h_n = J_{\zeta}^{F_1}(u_n - \tau \lambda_n \nabla g(u_n))$. If $h_n = u_n$, stop the iteration and $h_n \in \Omega_{SVIP}$. Else, move to step 3.

Step 3. Compute f_n by (3.2). If $f_n = 0$, stop the iteration and $h_n \in \Omega_{SVIP}$. Else, compute $m_n = u_n - \sigma \theta_n f_n + \varpi_n$, where θ_n , ϖ_n and γ_n are defined by (3.3) and (3.4), respectively.

Step 4. Compute $t_{n+1} = \alpha_n f(h_n) + \delta_n h_n + \mu_n m_n$, update λ_{n+1} by

$$\lambda_{n+1} = \begin{cases} \min \left\{ \frac{\varepsilon \|u_n - h_n\|}{\|A_n\|}, \frac{\varepsilon_2 g(u_n)}{\|\nabla g(u_n)\|^2}, \lambda_n \right\} & \text{if } A_n \neq 0 \text{ and } \nabla g(u_n) \neq 0, \\ \min \left\{ \frac{\varepsilon \|u_n - h_n\|}{\|A_n\|}, \lambda_n \right\} & \text{if } A_n \neq 0 \text{ and } \nabla g(u_n) = 0, \\ \min \left\{ \frac{\varepsilon_2 g(u_n)}{\|\nabla g(u_n)\|^2}, \lambda_n \right\} & \text{if } A_n = 0 \text{ and } \nabla g(u_n) \neq 0, \\ \lambda_n, & \text{otherwise,} \end{cases} \quad (3.57)$$

where $A_n = \nabla g(u_n) - \nabla g(h_n)$. Set $n := n + 1$ and go back to Step 1.

It is observed that the convergence analysis of Algorithm 4 holds under the conditions provided in Assumption 3.1 and Assumption 3.2. However, we first prove the following lemma.

Lemma 3.7. Suppose that $\{\lambda_n\}$ is a sequence of step lengths generated by (3.57). Then it is bounded away from zero and there exists $\lambda > 0$ such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda$.

Proof. Following the same arguments, one can easily derive (3.11). However, we additionally observe that if $\nabla g(u_n) \neq 0$, we have

$$\begin{aligned} \|\nabla g(u_n)\|^2 &= \left\langle (I - J_{\zeta}^{F_2})Bu_n, BB^*(I - J_{\zeta}^{F_2})Bu_n \right\rangle \\ &\leq \|B\|^2 \|(I - J_{\zeta}^{F_2})Bu_n\|^2 \\ &= 2\|B\|^2 g(u_n), \end{aligned} \quad (3.58)$$



thus, from (3.57) and (3.58), we deduce

$$\frac{\varepsilon g(u_n)}{\|\nabla g(u_n)\|^2} \geq \frac{\varepsilon_2 g(u_n)}{2\|B\|^2 g(u_n)} = \frac{\varepsilon_2}{2\|B\|^2}. \quad (3.59)$$

Similarly, in view of (3.11), (3.57) and (3.59), one finds that either $\lambda_{n+1} \geq \min\{\lambda_n, \frac{\varepsilon_2}{2\|B\|^2}\}$, $\lambda_{n+1} \geq \min\{\lambda_n, \frac{\varepsilon}{\|B\|^2}, \frac{\varepsilon_2}{2\|B\|^2}\}$ or $\lambda_{n+1} \geq \min\{\lambda_n, \frac{\varepsilon}{\|B\|^2}\}$. For the remaining part of the proof, refer to the proof of Lemma 3.3. ■

Similarly, we show the boundedness of the even subsequence $\{t_{2n}\}$ of $\{t_n\}$ produced by Algorithm 4.

Lemma 3.8. *Let $\{t_n\}$ be the sequence generated by Algorithm 4. Then, for any point $z \in \Omega$, the even subsequence $\{\|t_{2n} - z\|\}$ of $\{\|t_n - z\|\}$ is bounded.*

Proof. Proceeding as in the proof of Lemma 3.5, one easily obtains the same results as in (3.15)-(3.23). Since $z \in \Omega_{SVIP}$ and $Bz = J_\zeta^{F_2}(Bz)$, then $\nabla g(z) = B^*(I - J_\zeta^{F_2})Bz = 0$. Consequently, from the firmly nonexpansiveness of $(I - J_\zeta^{F_2})$, we have

$$\begin{aligned} \langle \nabla g(u_n), u_n - z \rangle &= \langle B^*(I - J_\zeta^{F_2})Bu_n, u_n - z \rangle \\ &= \langle (I - J_\zeta^{F_2})Bu_n - (I - J_\zeta^{F_2})Bz, Bu_n - Bz \rangle \\ &\geq \|(I - J_\zeta^{F_2})Bu_n\|^2 = 2g(u_n). \end{aligned} \quad (3.60)$$

Moreover, from (3.57), (3.60), the nonexpansiveness of $J_\zeta^{F_1}$ and the assignment of h_n in Algorithm 4, we derive

$$\begin{aligned} \|h_n - z\|^2 &= \|J_\zeta^{F_1}(u_n - \tau\lambda_n \nabla g(u_n)) - z\|^2 \\ &\leq \|(u_n - \tau\lambda_n \nabla g(u_n)) - z\|^2 \\ &= \|u_n - z\|^2 + \tau^2 \lambda_n^2 \|\nabla g(u_n)\|^2 - 2\tau\lambda_n \langle \nabla g(u_n), u_n - z \rangle \\ &\leq \|u_n - z\|^2 - \tau\lambda_n \sigma_n g(u_n), \end{aligned} \quad (3.61)$$

where

$$\sigma_n = 4 - \frac{\varepsilon_2 \tau \lambda_n}{\lambda_{n+1}}, \quad \forall n \geq 1. \quad (3.62)$$

Observe that for any $\tau \in (0, \frac{1}{\varepsilon})$ and $\varepsilon_2 \in (0, \frac{4}{\tau})$, we see from Lemma 3.7 and (3.62) that there exists $\sigma^* = 4 - \tau\varepsilon_2 > 0$ such that $\lim_{n \rightarrow \infty} \sigma_n = \sigma^*$. Thus, a positive integer $\mathcal{R}$ exists that satisfies $\sigma_n > 0$, $\forall n \geq \mathcal{R}$. Therefore, we deduce from (3.61) that

$$\|h_n - z\| \leq \|u_n - z\|, \quad \forall n \geq \mathcal{R}. \quad (3.63)$$

Setting $\check{c}_n := \frac{1}{1-\alpha_n}(\delta_n h_n + \mu_n m_n)$, from (3.23), (3.63) and the fact that $\alpha_n + \delta_n + \mu_n = 1$, $\forall n \geq 1$, we find that

$$\begin{aligned} \|\check{c}_n - z\| &\leq \frac{\delta_n}{1-\alpha_n} \|h_n - z\| + \frac{\mu_n}{1-\alpha_n} \|m_n - z\| \\ &\leq \frac{\delta_n}{1-\alpha_n} \|u_n - z\| + \frac{\mu_n}{1-\alpha_n} \|u_n - z\| + \frac{\mu_n}{1-\alpha_n} \|\varpi_n\| \\ &\leq \|u_n - z\| + \frac{\mu_n}{1-\alpha_n} \|\varpi_n\|, \quad \forall n \geq \mathcal{N}, \end{aligned} \quad (3.64)$$



where $\mathcal{N} = \max\{\mathcal{X}, \mathcal{R}\}$. From (3.63), (3.64), the c -contractiveness of f , and the assignment of t_{n+1} in Algorithm 4, one finds

$$\|t_{n+1} - z\| \leq \alpha_n \|f(h_n) - f(z)\| + (1 - \alpha_n) \|\check{c}_n - z\| + \alpha_n \|f(z) - z\| \quad (3.65)$$

$$\begin{aligned} &\leq \alpha_n c \|h_n - z\| + (1 - \alpha_n) \|u_n - z\| + \alpha_n \|f(z) - z\| + \mu_n \|\varpi_n\| \\ &= (1 - \alpha_n(1 - c)) \|u_n - z\| + \alpha_n \|f(z) - z\| \\ &\quad + \mu_n \|\varpi_n\|, \forall n \geq \mathcal{N}. \end{aligned} \quad (3.66)$$

Then the rest of the proof follows from the proof of Lemma 3.5. This completes the proof. ■

Next, we establish the convergence of Algorithm 4.

Theorem 3.9. *Suppose that Assumptions 3.1 and 3.2 hold and that the sequence $\{t_n\}$ is generated by Algorithm 4. Then it strongly converges to a point $z = P_{\Omega_{SVIP}} f(z)$.*

Proof. From (3.31), (3.61) and the fact that $\alpha_n + \delta_n + \mu_n = 1$, one obtains

$$\begin{aligned} \|\check{c}_n - z\|^2 &= \frac{\delta_n}{1 - \alpha_n} \|h_n - z\|^2 + \frac{\mu_n}{1 - \alpha_n} \|m_n - z\|^2 - \frac{\delta_n \mu_n}{(1 - \alpha_n)^2} \|h_n - m_n\|^2 \\ &\leq \frac{\delta_n}{1 - \alpha_n} \|u_n - z\|^2 + \frac{\mu_n}{1 - \alpha_n} \|u_n - z\|^2 - \frac{\delta_n \mu_n}{(1 - \alpha_n)^2} \|h_n - m_n\|^2 \\ &\quad + \frac{\mu_n}{1 - \alpha_n} \|\varpi_n\| (1 + \|\varpi_n\| + \|u_n - z\|^2) - \frac{\tau \delta_n \lambda_n \sigma_n}{1 - \alpha_n} g(u_n) \\ &\quad - \frac{\mu_n}{1 - \alpha_n} (1 + \|\varpi_n\|) \left(\frac{2 - \sigma}{\sigma} \right) \sigma^2 \theta_n^2 \|f_n\|^2 \\ &= \frac{\mu_n}{1 - \alpha_n} \|\varpi_n\| (1 + \|\varpi_n\| + \|u_n - z\|^2) - \frac{\delta_n \mu_n}{(1 - \alpha_n)^2} \|h_n - m_n\|^2 \\ &\quad + \|u_n - z\|^2 - \frac{\mu_n}{1 - \alpha_n} (1 + \|\varpi_n\|) \left(\frac{2 - \sigma}{\sigma} \right) \sigma^2 \theta_n^2 \|f_n\|^2 \\ &\quad - \frac{\tau \delta_n \lambda_n \sigma_n}{1 - \alpha_n} g(u_n). \end{aligned} \quad (3.67)$$

In view of (3.61), (3.65), (3.67) and the fact that $\alpha_n, c \in (0, 1)$, we deduce

$$\begin{aligned} \|t_{n+1} - z\|^2 &\leq \alpha_n^2 c^2 \|h_n - z\|^2 + \alpha_n^2 \|f(z) - z\|^2 + (1 - \alpha_n)^2 \|\check{c}_n - z\|^2 + \Lambda_n \\ &\leq \alpha_n c \|u_n - z\|^2 + \alpha_n^2 \|f(z) - z\|^2 - \delta_n \mu_n \|h_n - m_n\|^2 + \Lambda_n \\ &\quad + (1 - \alpha_n) \|u_n - z\|^2 + \mu_n (1 - \alpha_n) \|\varpi_n\| (1 + \|\varpi_n\| + \|u_n - z\|^2) \\ &\quad - \mu_n (1 - \alpha_n) (1 + \|\varpi_n\|) \left(\frac{2 - \sigma}{\sigma} \right) \sigma^2 \theta_n^2 \|f_n\|^2 \\ &\quad - \tau \lambda_n \sigma_n (\alpha_n^2 c^2 + (1 - \alpha_n) \delta_n) g(u_n) \\ &= (1 - \alpha_n(1 - c)) \|u_n - z\|^2 + \Phi_n + \hat{\Lambda}_n - \hat{\Psi}_n. \end{aligned} \quad (3.68)$$

where

$$\hat{\Lambda}_n := 2\alpha_n c \|h_n - z\| (\alpha_n \|f(z) - z\| + (1 - \alpha_n) \|c_n - z\|) + 2\alpha_n (1 - \alpha_n) \|f(z) - z\| \|c_n - z\|$$

and

$$\begin{aligned} \hat{\Psi}_n &:= \mu_n (1 - \alpha_n) (1 + \|\varpi_n\|) \left(\frac{2 - \sigma}{\sigma} \right) \sigma^2 \theta_n^2 \|f_n\|^2 + \delta_n \mu_n \|h_n - m_n\|^2 \\ &\quad + \tau \lambda_n \sigma_n (\alpha_n^2 c^2 + (1 - \alpha_n) \delta_n) g(u_n). \end{aligned}$$



Proceeding as in the proof of Theorem 3.6, it is not difficult to obtain the updated versions of (3.34) and (3.35). Moreover, from the inequalities (3.63), (3.64) and the fact that f is c -contractive, we have

$$\begin{aligned} \|t_{n+1} - z\|^2 &\leq \alpha_n \|f(h_n) - f(z)\|^2 + 2\alpha_n \langle f(z) - z, t_{n+1} - z \rangle + (1 - \alpha_n) \|\check{c}_n - z\|^2 \\ &\leq (1 - (1 - c)\alpha_n) \|u_n - z\|^2 + \mu_n \|\varpi_n\| \left(\frac{\mu_n}{1 - \alpha_n} \|\varpi_n\| + 2\|u_n - z\| \right) \\ &\quad + 2\alpha_n \langle f(z) - z, t_{n+1} - z \rangle, \quad \forall n \geq \mathcal{N}. \end{aligned} \quad (3.69)$$

Similarly, it is easy to obtain the updated versions of (3.37)-(3.39). Furthermore, the rest of the proof follows from the proof of Theorem 3.6. This completes the proof. ■

Next, we introduce the second modification of Algorithm 3. In the second modified version, we compute Steps 1 – 4 differently from Algorithm 3.

Algorithm 5

Initialization: Take $\lambda_1, \zeta, \varepsilon, \sigma, \tau, a, \{\delta_n\}, \{\alpha_n\}, \{\mu_n\}, \{\xi_n^{(1)}\}$, and $\{\beta_n\}$ such that (B3) and Assumption 3.2 hold. Select $\xi_n^{(2)} \in (-\infty, +\infty)$. Choose $t_0, t_1 \in \mathcal{H}_1$, $\gamma_0 = -\nabla g(t_0)$ and $\{w_n\} \subset \mathcal{H}_1$. Set $n = 1$.

Step 1. Compute u_n by (3.1) and

$$z_n = u_n + \xi_n^{(1)} \gamma_{n-1}.$$

Step 2. Compute $h_n = J_{\zeta}^{F_1}(z_n - \tau \lambda_n \nabla g(z_n))$. If $h_n = z_n$, stop the iteration and $h_n \in \Omega_{SVIP}$. Else, move to step 3.

Step 3. Compute

$$f_n = z_n - h_n - \tau \lambda_n (\nabla g(z_n) - \nabla g(h_n)). \quad (3.70)$$

If $f_n = 0$, stop the iteration and $h_n \in \Omega_{SVIP}$. Else, compute $m_n = z_n - \sigma \theta_n f_n$, where θ_n is defined as follows:

$$\theta_n = \frac{\langle z_n - h_n, f_n \rangle + 2\tau \lambda_n g(h_n)}{\|f_n\|^2}. \quad (3.71)$$

Step 4. Compute $t_{n+1} = \alpha_n f(z_n) + \delta_n z_n + \mu_n m_n$, update λ_{n+1} and γ_n by

$$\lambda_{n+1} = \begin{cases} \min \left\{ \frac{\varepsilon \|z_n - h_n\|}{\|\nabla g(z_n) - \nabla g(h_n)\|}, \lambda_n \right\} & \text{if } \|\nabla g(z_n) - \nabla g(h_n)\| \neq 0, \\ \lambda_n, & \text{otherwise} \end{cases} \quad (3.72)$$

and

$$\gamma_n = -\tau \lambda_n \nabla g(z_n) + \xi_n^{(1)} \gamma_{n-1} - \xi_n^{(2)} w_n. \quad (3.73)$$

Set $n := n + 1$ and go back to step 1.

Observe that, in Algorithm 5, we relax some of the assumptions imposed on the parameter $\xi_n^{(2)}$ in Assumption 3.2 by expanding its admissible range to the interval $(-\infty, +\infty)$, rather than restricting it to the set of nonnegative real numbers as in Algorithms 3 and 4. Nevertheless, its convergence analysis remains valid under the same conditions as in the Assumptions 3.1 and 3.2, except for the sequence $\{\xi_n^{(2)}\}$. Furthermore, from the point of view of computational complexity and cost, Algorithm 5 improves upon Algorithm 3 by



requiring only a single evaluation of the resolvent operator $J_{\zeta}^{F_1}$ at each iteration.

Similarly, we validate the stopping criterion of Algorithm 5.

Lemma 3.10. *If $h_n = z_n$ or $f_n = 0$ for some $n \geq 1$ in Algorithm 5, then $h_n \in \Omega$.*

Proof. Similarly, we first show that $h_n = z_n$ if and only if $f_n = 0$. From the assignments of f_n and λ_{n+1} in (3.70) and (3.72), respectively, and following the same lines of proof of (3.12) and (3.13), one easily sees that

$$\|f_n\| \geq \left(1 - \frac{\tau\varepsilon\lambda_n}{\lambda_{n+1}}\right)\|z_n - h_n\| \quad (3.74)$$

and

$$\|f_n\| \leq \left(1 + \frac{\tau\varepsilon\lambda_n}{\lambda_{n+1}}\right)\|z_n - h_n\|. \quad (3.75)$$

Thus, from (3.14), (3.74) and (3.75), we similarly see that $h_n = z_n$ if and only if $f_n = 0$. Then the remaining part of the proof follows from the proof of Lemma 3.4. ■

Next, we similarly establish that the even subsequence $\{t_{2n}\}$ of $\{t_n\}$ generated by Algorithm 5 is bounded.

Lemma 3.11. *Let $\{t_n\}$ be the sequence generated by Algorithm 5. Then, for any point $z \in \Omega$, the even subsequence $\{\|t_{2n} - z\|\}$ of $\{\|t_n - z\|\}$ is bounded.*

Proof. We observe from (2.2), Lemma 3.3, the assignments of θ_n and λ_{n+1} in (3.71) and (3.72), respectively, and following the same lines of proof of (3.15) that

$$\theta_n \|f_n\|^2 \geq \left(1 - \frac{\tau\varepsilon\lambda_n}{\lambda_{n+1}}\right)\|z_n - h_n\|^2. \quad (3.76)$$

Applying (3.16), (3.76) and the same arguments used in the derivation of (3.17), one sees that

$$\|z_n - h_n\|^2 \leq \left(\frac{1 + \pi_n}{1 - \pi_n}\right)^2 \theta_n^2 \|f_n\|^2, \quad \forall n \geq \mathcal{X}. \quad (3.77)$$

Now, let $z \in \Omega$. From the assignment of h_n in Algorithm 5 and the resolvent property (2.8), we have

$$\langle z_n - \tau\lambda_n \nabla g(z_n) - h_n, h_n - z \rangle \geq 0. \quad (3.78)$$

In view of (3.78) and similar arguments, we easily derive the result in (3.19). We also find from (3.19) and the assignment of θ_n and z_n in (3.71) and Algorithm 5 that

$$\begin{aligned} \langle f_n, z_n - z \rangle &= \langle f_n, z_n - h_n \rangle + \langle f_n, h_n - z \rangle \\ &\geq \langle f_n, z_n - h_n \rangle + 2\tau\lambda_n g(h_n) \\ &= \theta_n \|f_n\|^2 \end{aligned} \quad (3.79)$$

and

$$\|z_n - z\| \leq \|u_n - z\| + \xi_n^{(1)} \|\gamma_{n-1}\|. \quad (3.80)$$

Moreover, applying (3.79) and assignment of m_n in Algorithm 5, we obtain

$$\begin{aligned} \|m_n - z\|^2 &= \|z_n - z\|^2 + \sigma^2 \theta_n^2 \|f_n\|^2 - 2\sigma \theta_n \langle f_n, z_n - z \rangle \\ &\leq \|z_n - z\|^2 - \left(\frac{2 - \sigma}{\sigma}\right) \sigma^2 \theta_n^2 \|f_n\|^2. \end{aligned} \quad (3.81)$$



Combining (3.14), (3.80), (3.81), the facts that $\sigma \in (0, 2)$ and there exists $\mathcal{X} \in \mathbb{N}$ such that $\theta_n > 0$, $\forall n \geq \mathcal{X}$, we have

$$\|m_n - z\| \leq \|u_n - z\| + \xi_n^{(1)} \|\gamma_{n-1}\|, \quad \forall n \geq \mathcal{X}. \quad (3.82)$$

Similarly, letting $\bar{c}_n := \frac{1}{1-\alpha_n}(\delta_n z_n + \mu_n m_n)$, we find from (3.80), (3.82) and the fact that $\alpha_n + \delta_n + \mu_n = 1$, $\forall n \geq 1$ that

$$\begin{aligned} \|\bar{c}_n - z\| &\leq \frac{\delta_n}{1-\alpha_n} \|z_n - z\| + \frac{\mu_n}{1-\alpha_n} \|m_n - z\| \\ &\leq \|u_n - z\| + \xi_n^{(1)} \|\gamma_{n-1}\|, \quad \forall n \geq \mathcal{X}. \end{aligned} \quad (3.83)$$

Additionally, from Algorithm 5, the inequality (3.83) and the same arguments in the derivation of (3.66), we get

$$\begin{aligned} \|t_{n+1} - z\| &\leq (1 - \alpha_n(1-c)) \|u_n - z\| + \alpha_n \|f(z) - z\| \\ &\quad + (1 - \alpha_n(1-c)) \xi_n^{(1)} \|\gamma_{n-1}\|, \quad \forall n \geq \mathcal{X}. \end{aligned} \quad (3.84)$$

So, from the inequalities (3.8), (3.84) and the assignment of u_n in (3.1), we have

$$\begin{aligned} \|t_{2n+1} - z\| &\leq (1 - \alpha_{2n}(1-c)) \|t_{2n} - z\| + \alpha_{2n} \|f(z) - z\| \\ &\quad + (1 - \alpha_{2n}(1-c)) \zeta_{2n}^{(1)}, \quad \forall n \geq \mathcal{X}. \end{aligned} \quad (3.85)$$

Using (3.8), (3.28), (3.84), (3.85) and the facts that the sequence $\{\alpha_n\}$ is nonincreasing and $\alpha_n(1-c) \in (0, 1)$, $\forall n \geq 1$, we find that

$$\begin{aligned} \|t_{2n+2} - z\| &\leq \|u_{2n+1} - z\| + \alpha_{2n+1} \|f(z) - z\| + (1 - \alpha_{2n+1}(1-c)) \zeta_{2n+1}^{(1)} \\ &\leq \|t_{2n+1} - z\| + \alpha_{2n+1} \|f(z) - z\| + (1 - \alpha_{2n+1}(1-c)) \zeta_{2n+1}^{(1)} + \varsigma_{2n+1} \\ &\leq (1 - \alpha_{2n}(1-c)) \|t_{2n} - z\| + \alpha_{2n} \|f(z) - z\| + \alpha_{2n+1} \|f(z) - z\| \\ &\quad + (1 - \alpha_{2n}(1-c)) \zeta_{2n}^{(1)} + (1 - \alpha_{2n+1}(1-c)) \zeta_{2n+1}^{(1)} + \varsigma_{2n+1} \\ &\leq (1 - \alpha_{2n}(1-c)) \|t_{2n} - z\| + \alpha_{2n}(1-c) \left(\frac{2}{1-c} \|f(z) - z\| \right. \\ &\quad \left. + \frac{\varsigma_{2n+1}}{\alpha_{2n}(1-c)} + \frac{\Theta_{2n}}{\alpha_{2n}(1-c)} \right), \quad \forall n \geq \mathcal{X}, \end{aligned} \quad (3.86)$$

where $\Theta_{2n} = (1 - \alpha_{2n}(1-c)) \zeta_{2n}^{(1)} + (1 - \alpha_{2n+1}(1-c)) \zeta_{2n+1}^{(1)}$.

From (3.8) and the facts that $\lim_{n \rightarrow \infty} \frac{\varsigma_n}{\alpha_n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\zeta_n^{(1)}}{\alpha_n} = 0$ in Assumption 3.2, select

$M_1, M_2 > 0$ such that $\frac{\zeta_n^{(1)}}{\alpha_n} \leq M_1$ and $\frac{\varsigma_n}{\alpha_n} \leq M_2$, $\forall n \geq 1$. Then, by denoting $M := \frac{2}{1-c} (\|f(z) - z\| + M_1) + \frac{M_2}{1-c}$, (3.86) yields

$$\begin{aligned} \|t_{2n+2} - z\| &\leq (1 - \alpha_{2n}(1-c)) \|t_{2n} - z\| + \alpha_{2n}(1-c) M \\ &\leq \max \{ \|t_{2n} - z\|, M \} \\ &\vdots \\ &\leq \max \{ \|t_0 - z\|, M \}, \quad \forall n \geq \mathcal{X}, \end{aligned} \quad (3.87)$$

which implies that the subsequence $\{\|t_{2n} - z\|\}$ of $\{\|t_n - z\|\}$ is bounded. So, the even subsequence $\{t_{2n}\}$ generated by Algorithm 5 is bounded. So, as the subsequences $\{m_{2n}\}$, $\{h_{2n}\}$, $\{z_{2n}\}$, $\{f_{2n}\}$ and $\{u_{2n}\}$. Moreover, one easily sees from (3.28), (3.75), (3.80),



(3.85), Assumption 3.2 and the assignments of h_{2n+1} and m_{2n+1} that the odd subsequences $\{t_{2n+1}\}$, $\{u_{2n+1}\}$, $\{z_{2n+1}\}$, $\{h_{2n+1}\}$, $\{f_{2n+1}\}$ and $\{m_{2n+1}\}$ are bounded. ■

Next, we show the convergence of Algorithm 5.

Theorem 3.12. *Suppose that Assumptions 3.1 and 3.2 hold and $\{t_n\}$ is a sequence produced by Algorithm 5. Then it strongly converges to a point $z = P_{\Omega_{SVIP}} f(z)$.*

Proof. Similarly, applying (3.8), (3.80) and (3.81), we obtain

$$\begin{aligned} \|m_n - z\|^2 &\leq \|z_n - z\|^2 - \left(\frac{2-\sigma}{\sigma}\right) \sigma^2 \theta_n^2 \|f_n\|^2 \\ &\leq \|u_n - z\|^2 + \zeta_n^{(1)} (\zeta_n^{(1)} + 2\|u_n - z\|) - \left(\frac{2-\sigma}{\sigma}\right) \sigma^2 \theta_n^2 \|f_n\|^2. \end{aligned} \quad (3.88)$$

Additionally, in view of (3.8), (3.80), (3.88) and the fact that $\alpha_n + \delta_n + \mu_n = 1$, one sees that

$$\begin{aligned} \|\bar{c}_n - z\|^2 &= \frac{\delta_n}{1-\alpha_n} \|z_n - z\|^2 + \frac{\mu_n}{1-\alpha_n} \|m_n - z\|^2 - \frac{\delta_n \mu_n}{(1-\alpha_n)^2} \|z_n - m_n\|^2 \\ &\leq \frac{\delta_n}{1-\alpha_n} \|u_n - z\|^2 - \frac{\delta_n \mu_n}{(1-\alpha_n)^2} \|u_n - m_n\|^2 + \frac{\mu_n}{1-\alpha_n} \|u_n - z\|^2 \\ &\quad + \left(\frac{\delta_n}{1-\alpha_n} + \frac{\mu_n}{1-\alpha_n}\right) \zeta_n^{(1)} (\zeta_n^{(1)} + 2\|u_n - z\|) \\ &\quad - \frac{\mu_n}{1-\alpha_n} \left(\frac{2-\sigma}{\sigma}\right) \sigma^2 \theta_n^2 \|f_n\|^2 \\ &= \|u_n - z\|^2 + \zeta_n^{(1)} (\zeta_n^{(1)} + 2\|u_n - z\|) - \frac{\delta_n \mu_n}{(1-\alpha_n)^2} \|u_n - m_n\|^2 \\ &\quad - \frac{\mu_n}{1-\alpha_n} \left(\frac{2-\sigma}{\sigma}\right) \sigma^2 \theta_n^2 \|f_n\|^2. \end{aligned} \quad (3.89)$$

Moreover, from the fact that $\alpha_n, c \in (0, 1)$, $\forall n \geq 1$, (3.8), (3.65), (3.89) and the assignment of t_{n+1} in Algorithm 5, we deduce

$$\begin{aligned} \|t_{n+1} - z\|^2 &\leq \alpha_n^2 c^2 \|z_n - z\|^2 + \alpha_n^2 \|f(z) - z\|^2 + (1-\alpha_n)^2 \|c_n - z\|^2 + \Lambda_n \\ &\leq \alpha_n c \|u_n - z\|^2 + \alpha_n^2 \|f(z) - z\|^2 - \delta_n \mu_n \|z_n - m_n\|^2 + \Lambda_n \\ &\quad + (1-\alpha_n) \|u_n - z\|^2 + \left((1-\alpha_n)^2 + \alpha_n^2 c^2\right) \zeta_n^{(1)} (\zeta_n^{(1)} + 2\|u_n - z\|) \\ &\quad - \mu_n (1-\alpha_n) \left(\frac{2-\sigma}{\sigma}\right) \sigma^2 \theta_n^2 \|f_n\|^2 \\ &= (1-\alpha_n(1-c)) \|u_n - z\|^2 + \check{\Phi}_n + \Lambda_n - \check{\Psi}_n. \end{aligned} \quad (3.90)$$

where

$$\check{\Phi}_n := \alpha_n^2 \|f(z) - z\|^2 + ((1-\alpha_n)^2 + \alpha_n^2 c^2) \zeta_n^{(1)} (\zeta_n^{(1)} + 2\|u_n - z\|),$$

$$\Lambda_n := 2\alpha_n c \|z_n - z\| (\alpha_n \|f(z) - z\| + (1-\alpha_n) \|c_n - z\|) + 2\alpha_n (1-\alpha_n) \|f(z) - z\| \|c_n - z\|$$

and

$$\check{\Psi}_n := \mu_n (1-\alpha_n) \left(\frac{2-\sigma}{\sigma}\right) \sigma^2 \theta_n^2 \|f_n\|^2 + \delta_n \mu_n \|z_n - m_n\|^2.$$

From (3.90) and similar arguments in the derivation of (3.27), we get

$$\|t_{2n+1} - z\|^2 \leq (1-\alpha_{2n}(1-c)) \|t_{2n} - z\|^2 + \check{\Phi}_{2n} + \Lambda_{2n} - \check{\Psi}_{2n}. \quad (3.91)$$



Combining (3.28), (3.90), (3.91) and the same arguments in the derivation of (3.29), we have

$$\begin{aligned} \|t_{2n+2} - z\|^2 &\leq \|t_{2n} - z\|^2 + \check{\Phi}_{2n+1} + \check{\Phi}_{2n} + \Lambda_{2n} + \Lambda_{2n+1} \\ &\quad + \varsigma_{2n+1}(\varsigma_{2n+1} + 2\|t_{2n+1} - z\|) - (\check{\Psi}_{2n} + \check{\Psi}_{2n+1}). \end{aligned} \quad (3.92)$$

Since f is c -contractive, we find from (3.8), (3.80), (3.83) and the assignment of t_{n+1} in Algorithm 5 that

$$\begin{aligned} \|t_{n+1} - z\|^2 &\leq (1 - \alpha_n(1 - c))\|u_n - z\|^2 + ((1 - \alpha_n) + \alpha_n c^2)\zeta_n^{(1)}(\zeta_n^{(1)} + 2\|u_n - z\|) \\ &\quad + 2\alpha_n \langle f(z) - z, t_{n+1} - z \rangle, \quad \forall n \geq \mathcal{X}. \end{aligned} \quad (3.93)$$

Using (3.93) and similar arguments used in (3.27), we get

$$\begin{aligned} \|t_{2n+1} - z\|^2 &\leq (1 - \alpha_{2n}(1 - c))\|t_{2n} - z\|^2 + 2\alpha_{2n} \langle f(z) - z, t_{2n+1} - z \rangle \\ &\quad + ((1 - \alpha_{2n}) + \alpha_{2n} c^2)\zeta_{2n}^{(1)}(\zeta_{2n}^{(1)} + 2\|u_{2n} - z\|), \quad \forall n \geq \mathcal{X}. \end{aligned} \quad (3.94)$$

Similarly, from the inequalities (3.28), (3.93), (3.94) and the same arguments applied in (3.29), one finds

$$\begin{aligned} \|t_{2n+2} - z\|^2 &\leq (1 - \alpha_{2n}(1 - c))\|t_{2n} - z\|^2 + 2\alpha_{2n+1} \langle f(z) - z, t_{2n+2} - z \rangle \\ &\quad + \varsigma_{2n+1}(\varsigma_{2n+1} + 2\|t_{2n+1} - z\|) + 2\alpha_{2n} \langle f(z) - z, t_{2n+1} - z \rangle \\ &\quad + ((1 - \alpha_{2n+1}) + \alpha_{2n+1} c^2)\zeta_{2n+1}^{(1)}(\zeta_{2n+1}^{(1)} + 2\|u_{2n+1} - z\|) \\ &\quad + ((1 - \alpha_{2n}) + \alpha_{2n} c^2)\zeta_{2n}^{(1)}(\zeta_{2n}^{(1)} + 2\|u_{2n} - z\|), \quad \forall n \geq \mathcal{X}. \end{aligned} \quad (3.95)$$

Furthermore, in view of (3.92) and (3.95), we obtain that for all $n \geq \mathcal{X}$,

$$\|t_{2n+2} - z\|^2 \leq \|t_{2n} - z\|^2 + \Gamma_{2n} - \hat{\chi}_{2n}$$

and

$$\|t_{2n+2} - z\|^2 \leq (1 - \alpha_{2n}(1 - c))\|t_{2n} - z\|^2 + \alpha_{2n}(1 - c)\Upsilon_{2n}, \quad (3.96)$$

where

$$\check{\chi}_{2n} := \check{\Psi}_{2n} + \check{\Psi}_{2n+1},$$

$$\check{\Gamma}_{2n} := \check{\Phi}_{2n} + \check{\Phi}_{2n+1} + \Lambda_{2n} + \Lambda_{2n+1} + \varsigma_{2n+1}(\varsigma_{2n+1} + 2\|t_{2n+1} - z\|)$$

and

$$\begin{aligned} \check{\Upsilon}_{2n} &:= \frac{1}{\alpha_{2n}(1-c)} \left(2\alpha_{2n+1} \langle f(z) - z, t_{2n+2} - z \rangle + 2\alpha_{2n} \langle f(z) - z, t_{2n+1} - z \rangle + \varsigma_{2n+1}(\varsigma_{2n+1} + 2\|t_{2n+1} - z\|) \right. \\ &\quad \left. + ((1 - \alpha_{2n+1}) + \alpha_{2n+1} c^2)\zeta_{2n+1}^{(1)}(\zeta_{2n+1}^{(1)} + 2\|u_{2n+1} - z\|) + ((1 - \alpha_{2n}) + \alpha_{2n} c^2)\zeta_{2n}^{(1)}(\zeta_{2n}^{(1)} + 2\|u_{2n} - z\|) \right). \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} \frac{\varsigma_n}{\alpha_n} = 0$ and $\lim_{n \rightarrow \infty} \frac{\zeta_n^{(1)}}{\alpha_n} = 0$, we have $\lim_{n \rightarrow \infty} \check{\Gamma}_{2n} = 0$. Therefore, to apply Lemma 2.6, we only establish that for any $\{\check{\chi}_{2n_r}\}$ of $\{\check{\chi}_{2n}\}$, we have

$$\lim_{r \rightarrow \infty} \check{\chi}_{2n_r} = 0 \Rightarrow \limsup_{r \rightarrow \infty} \check{\Upsilon}_{2n_r} \leq 0.$$



Now, assume that the subsequence $\{\check{\chi}_{2n_r}\}$ of $\{\check{\chi}_{2n}\}$ satisfies $\lim_{r \rightarrow \infty} \check{\chi}_{2n_r} = 0$, then from the facts that $\sigma \in (0, 2)$, $\alpha_n \in (0, 1)$, δ_n , $\mu_n \in [a, 1 - a]$ for some $a \in (0, \frac{1}{2})$ and $\theta_n > 0$, $\forall n \geq \mathcal{X}$, we have

$$\lim_{r \rightarrow \infty} \|f_{2n_r}\| = 0, \lim_{r \rightarrow \infty} \|f_{2n_r+1}\| = 0, \lim_{r \rightarrow \infty} \|z_{2n_r} - m_{2n_r}\| = 0$$

and

$$\lim_{r \rightarrow \infty} \|z_{2n_r+1} - m_{2n_r+1}\| = 0. \quad (3.97)$$

Moreover, from the assignment of c_n and (3.97), one sees that

$$\|c_{2n_r} - m_{2n_r}\| = \frac{\delta_{2n_r}}{1 - \alpha_{2n_r}} \|z_{2n_r} - m_{2n_r}\| \rightarrow 0 \text{ as } r \rightarrow \infty \quad (3.98)$$

and

$$\|c_{2n_r+1} - m_{2n_r+1}\| \leq \frac{\delta_{2n_r+1}}{1 - \alpha_{2n_r+1}} \|z_{2n_r+1} - m_{2n_r+1}\| \rightarrow 0 \text{ as } r \rightarrow \infty. \quad (3.99)$$

Combining (3.97), (3.98), (3.99), the assignments of z_n , t_{n+1} and u_n in Algorithm 5, the facts that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} \frac{s_n}{\alpha_n} = 0$, $\lim_{n \rightarrow \infty} \frac{\zeta_n^{(1)}}{\alpha_n} = 0$, f is c -contractive, $\{z_{2n}\}$, $\{m_{2n}\}$, $\{u_{2n+1}\}$, $\{z_{2n+1}\}$, $\{m_{2n+1}\}$ are bounded, and there exists $\mathcal{X} \in \mathbb{N}$ such that $\theta_n > 0$, $\forall n \geq \mathcal{X}$, we find that

$$\|z_{2n_r} - t_{2n_r}\| \leq \xi_{2n_r}^{(1)} \|\gamma_{2n_r-1}\| \rightarrow 0 \text{ as } r \rightarrow \infty, \quad (3.100)$$

$$\begin{aligned} \|t_{2n_r+1} - m_{2n_r}\| &\leq \alpha_{2n_r} \|f(z_{2n_r}) - m_{2n_r}\| \\ &\quad + (1 - \alpha_{2n_r}) \|c_{2n_r} - m_{2n_r}\| \rightarrow 0 \text{ as } r \rightarrow \infty, \end{aligned} \quad (3.101)$$

$$\|u_{2n_r+1} - t_{2n_r+1}\| \leq \beta_{2n_r+1} \|t_{2n_r+1} - t_{2n_r}\| \rightarrow 0 \text{ as } r \rightarrow \infty, \quad (3.102)$$

$$\|z_{2n_r+1} - u_{2n_r+1}\| \leq \xi_{2n_r+1}^{(1)} \|\gamma_{2n_r}\| \rightarrow 0 \text{ as } r \rightarrow \infty, \quad (3.103)$$

and

$$\begin{aligned} \|t_{2n_r+2} - m_{2n_r+1}\| &\leq (1 - \alpha_{2n_r+1}) \|c_{2n_r+1} - m_{2n_r+1}\| \\ &\quad + \alpha_{2n_r+1} \|f(z_{2n_r+1}) - m_{2n_r+1}\| \rightarrow 0 \text{ as } r \rightarrow \infty. \end{aligned} \quad (3.104)$$

Then, since the subsequence $\{t_{2n}\}$ of $\{t_n\}$ generated by Algorithm 5 is bounded, there exists a subsequence $\{t_{2n_r}\}$ of $\{t_{2n}\}$ such that $t_{2n_r} \rightarrow t^*$ as $r \rightarrow \infty$. Consequently, from (3.97), (3.98), (3.101), (3.102), (3.103), (3.104) and the assignment of u_{2n_r} in Algorithm 5, we see that $t_{2n_r+1} \rightarrow t^*$ and $t_{2n_r+2} \rightarrow t^*$ as $r \rightarrow \infty$.

Combining (3.77) and (3.97), we have

$$\lim_{r \rightarrow \infty} \|z_{2n_r} - h_{2n_r}\| = 0. \quad (3.105)$$

In connection with the assignment of θ_{2n} and (3.76), we get

$$\begin{aligned} \theta_{2n_r}^2 \|f_{2n_r}\|^2 &= \theta_{2n_r} \left(\langle z_{2n_r} - h_{2n_r}, f_{2n_r} \rangle + 2\tau \lambda_{2n_r} g(h_{2n_r}) \right) \\ &\geq \theta_{2n_r} \left((1 - \pi_{2n_r}) \|z_{2n_r} - h_{2n_r}\|^2 + 2\tau \lambda_{2n_r} g(h_{2n_r}) \right). \end{aligned} \quad (3.106)$$



Combining (3.97), (3.105) and (3.106), we obtain

$$\lim_{r \rightarrow \infty} g(h_{2n_r}) = \lim_{r \rightarrow \infty} \frac{1}{2} \|(I - J_{\zeta}^{F_2})Bh_{2n_r}\|^2 = 0. \quad (3.107)$$

Then the remaining part of the proof follows from the proof of Theorem 3.6. $\blacksquare$

Next, we present the modified version of Algorithm 5. In the following modified version, we computed Steps 4 differently from Algorithm 5 and a new adaptive step length strategy similar to the one used in Algorithm 4 that is different from the one defined in (3.72) is employed. Similarly, Algorithm 6 improves Algorithm 4 from the point of view of complexity and computational cost, by reducing to a single resolvent operation $J_{\zeta}^{F_1}$ per iteration.

Algorithm 6

Initialization: Take $\lambda_1, \zeta, \varepsilon, \sigma, \tau, \{\delta_n\}, \{\alpha_n\}, \{\mu_n\}, \{\xi_n^{(1)}\}$, and $\{\beta_n\}$ such that (B3) and Assumption 3.2 hold. Select $\varepsilon_2 \in (0, \frac{4}{\tau})$ and $\xi_n^{(2)} \in (-\infty, +\infty)$. Choose $t_0, t_1 \in \mathcal{H}_1, \gamma_0 = -\nabla g(t_0)$ and $\{w_n\} \subset \mathcal{H}_1$. Set $n = 1$.

Step 1. Compute u_n by (3.1) and

$$z_n = u_n + \xi_n^{(1)} \gamma_{n-1}.$$

Step 2. Compute $h_n = J_{\zeta}^{F_1}(z_n - \tau \lambda_n \nabla g(z_n))$. If $h_n = z_n$, stop the iteration and $h_n \in \Omega_{SVIP}$. Else, move to step 3.

Step 3. Compute f_n by (3.70). If $f_n = 0$, stop the iteration and $h_n \in \Omega_{SVIP}$. Else, compute $m_n = z_n - \sigma \theta_n f_n$, where θ_n is defined by (3.71).

Step 4. Compute $t_{n+1} = \alpha_n f(h_n) + \delta_n h_n + \mu_n m_n$, update γ_n by (3.73) and λ_{n+1} by

$$\lambda_{n+1} = \begin{cases} \min \left\{ \frac{\varepsilon \|z_n - h_n\|}{\|\check{A}_n\|}, \frac{\varepsilon_2 g(z_n)}{\|\nabla g(z_n)\|^2}, \lambda_n \right\} & \text{if } \check{A}_n \neq 0 \text{ and } \nabla g(z_n) \neq 0, \\ \min \left\{ \frac{\varepsilon \|z_n - h_n\|}{\|\check{A}_n\|}, \lambda_n \right\} & \text{if } \check{A}_n \neq 0 \text{ and } \nabla g(z_n) = 0, \\ \min \left\{ \frac{\varepsilon_2 g(z_n)}{\|\nabla g(z_n)\|^2}, \lambda_n \right\} & \text{if } \check{A}_n = 0 \text{ and } \nabla g(z_n) \neq 0, \\ \lambda_n, & \text{otherwise,} \end{cases} \quad (3.108)$$

where $\check{A}_n = \nabla g(z_n) - \nabla g(h_n)$. Set $n := n + 1$ and go back to Step 1.

We first prove the following lemma.

Lemma 3.13. Suppose that $\{\lambda_n\}$ is a sequence of step lengths generated by (3.108). Then it is bounded away from zero and there exists $\lambda > 0$ such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda$.

Proof. For the proof of this lemma, refer to the proof of Lemma 3.7. $\blacksquare$

Next, we show that an even subsequence $\{t_{2n}\}$ of $\{t_n\}$ generated by Algorithm 6 is bounded.

Lemma 3.14. Let $\{t_n\}$ be the sequence generated by Algorithm 6. Then, for any point $z \in \Omega$, the even subsequence $\{\|t_{2n} - z\|\}$ of $\{\|t_n - z\|\}$ is bounded.

Proof. Similarly, proceeding as in the proof of Lemma 3.11, the results in (3.76)-(3.82) can easily be derived. Moreover, in view of (3.80), the assignment of h_n in Algorithm 6



and the same arguments in the derivations of (3.61) and (3.63), one sees that

$$\|h_n - z\|^2 \leq \|z_n - z\|^2 - \tau\lambda_n\sigma_n g(z_n) \quad (3.109)$$

and

$$\|h_n - z\| \leq \|z_n - z\| \leq \|u_n - z\| + \xi_n^{(1)}\|\gamma_{n-1}\|, \forall n \geq \mathcal{R}. \quad (3.110)$$

Similarly, taking $\check{c}_n := \frac{1}{1-\alpha_n}(\delta_n h_n + \mu_n m_n)$, $\forall n \geq 1$, we obtain from (3.82), (3.110) and the fact that $\alpha_n + \delta_n + \mu_n = 1$, $\forall n \geq 1$ that

$$\begin{aligned} \|\check{c}_n - z\| &\leq \frac{\delta_n}{1-\alpha_n}\|h_n - z\| + \frac{\mu_n}{1-\alpha_n}\|m_n - z\| \\ &\leq \|u_n - z\| + \xi_n^{(1)}\|\gamma_{n-1}\|, \forall n \geq \mathcal{N}. \end{aligned} \quad (3.111)$$

From (3.110), (3.111), the assignment of Algorithm 6 and similar arguments used in (3.65), we have

$$\begin{aligned} \|t_{n+1} - z\| &\leq (1 - \alpha_n(1 - c))\|u_n - z\| + \alpha_n\|f(z) - z\| \\ &\quad + (1 - \alpha_n(1 - c))\xi_n^{(1)}\|\gamma_{n-1}\|, \forall n \geq \mathcal{N}. \end{aligned} \quad (3.112)$$

Similarly, the rest of the proof can be derived as in the proof of Lemma 3.11. This completes the proof. ■

Next, we prove the following as the convergence theorem of Algorithm 6.

Theorem 3.15. *Suppose that Assumptions 3.1 and 3.2 hold and $\{t_n\}$ is a sequence produced by Algorithm 6. Then it strongly converges to a point $z = P_{\Omega_{SVIP}} f(z)$.*

Proof. From (3.8), (3.80), (3.88), (3.109) and the fact that $\alpha_n + \delta_n + \mu_n = 1$, we deduce

$$\begin{aligned} \|\check{c}_n - z\|^2 &\leq \|u_n - z\|^2 + \zeta_n^{(1)}(\zeta_n^{(1)} + 2\|h_n - z\|) - \frac{\delta_n\mu_n}{(1-\alpha_n)^2}\|u_n - m_n\|^2 \\ &\quad - \frac{\mu_n}{1-\alpha_n}\left(\frac{2-\sigma}{\sigma}\right)\sigma^2\theta_n^2\|f_n\|^2 - \frac{\tau\delta_n\lambda_n\sigma_n}{1-\alpha_n}g(z_n). \end{aligned} \quad (3.113)$$

Moreover, from the fact that $\alpha_n, c \in (0, 1)$, $\forall n \geq 1$, (3.8), (3.65), (3.80), (3.88), (3.109), (3.113) and the assignment of t_{n+1} in Algorithm 6, we deduce

$$\begin{aligned} \|t_{n+1} - z\|^2 &\leq \alpha_n^2 c^2 \|h_n - z\|^2 + \alpha_n^2 \|f(z) - z\|^2 + (1 - \alpha_n)^2 \|\check{c}_n - z\|^2 + \Lambda_n \\ &\leq \alpha_n c \|u_n - z\|^2 + \alpha_n^2 \|f(z) - z\|^2 - \delta_n \mu_n \|z_n - m_n\|^2 + \Lambda_n \\ &\quad + (1 - \alpha_n) \|u_n - z\|^2 + \left((1 - \alpha_n)^2 + \alpha_n^2 c^2\right) \zeta_n^{(1)} (\zeta_n^{(1)} + 2\|u_n - z\|) \\ &\quad - \mu_n (1 - \alpha_n) \left(\frac{2-\sigma}{\sigma}\right) \sigma^2 \theta_n^2 \|f_n\|^2 - \tau \lambda_n \sigma_n (\alpha_n^2 c^2 + (1 - \alpha_n) \delta_n) g(z_n) \\ &= (1 - \alpha_n(1 - c)) \|u_n - z\|^2 + \check{\Phi}_n + \Lambda_n - \check{\Psi}_n. \end{aligned} \quad (3.114)$$

where

$$\Lambda_n := 2\alpha_n c \|h_n - z\| (\alpha_n \|f(z) - z\| + (1 - \alpha_n) \|\check{c}_n - z\|) + 2\alpha_n (1 - \alpha_n) \|f(z) - z\| \|\check{c}_n - z\|$$

and

$$\begin{aligned} \check{\Psi}_n &:= \mu_n (1 - \alpha_n) \left(\frac{2-\sigma}{\sigma}\right) \sigma^2 \theta_n^2 \|f_n\|^2 + \delta_n \mu_n \|z_n - m_n\|^2 \\ &\quad + \tau \lambda_n \sigma_n (\alpha_n^2 c^2 + (1 - \alpha_n) \delta_n) g(z_n). \end{aligned}$$



In the same fashion, proceeding as in the proof of Theorem 3.12, the updated versions of (3.91) and (3.92) can easily be obtained. Similarly, from (3.8), (3.110), (3.111), the assignment of t_{n+1} in Algorithm 6 and the c -contractiveness of f , we get

$$\begin{aligned} \|t_{n+1} - z\|^2 &\leq (1 - \alpha_n)\|\check{c}_n - z\|^2 + \alpha_n\|f(h_n) - f(z)\|^2 + 2\alpha_n\langle f(z) - z, t_{n+1} - z \rangle \\ &\leq (1 - (1 - c)\alpha_n)\|u_n - z\|^2 + ((1 - \alpha_n) + \alpha_n c^2)\zeta_n^{(1)}(\zeta_n^{(1)} + 2\|u_n - z\|) \\ &\quad + 2\alpha_n\langle f(z) - z, t_{n+1} - z \rangle, \quad \forall n \geq \mathcal{N}. \end{aligned} \quad (3.115)$$

Furthermore, the updated versions of (3.94)-(3.96) can be easily deduced. Consequently, the rest of the proof follows from the proof of Theorem 3.12. This completes the proof. ■

4. APPLICATIONS

In this section, we use our proposed algorithms (that is, Algorithms 3-6) to solve some particular cases of the problem (1.1), including split minimization and split feasibility problems.

4.1. SPLIT FEASIBILITY PROBLEMS

Recall that one of the special cases of the SVIP (1.1) is the split feasibility problem (SFP):

$$\text{find } t \in C \text{ such that } Bt \in Q, \quad (4.1)$$

where C and Q are nonempty, closed and convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The SFP (4.1) was introduced by Censor and Elfving [12] to approach many significant inverse problems that arise in real-life applications, including the problems in intensity modulated radiation therapy (IMRT); see [33], machine learning; see [2, 8] and Jointly constrained Nash equilibrium problems; see [9, 17]. In this work, we denote the set of solutions to SFP (4.1) by Ω_{SFP} and it is assumed to be nonempty. However, following the detailed explanation provided in ([41], Subsection 4.1), we obtain the following corollary from Algorithm 3 and Theorem 3.6.

Corollary 4.1. *Suppose that Assumptions 3.1 and 3.2 hold and $\{t_n\}$ is a sequence produced by Algorithm 3 with $J_\zeta^{F_1} = P_C$ and $J_\zeta^{F_2} = P_Q$. Then it strongly converges to $z = P_{\Omega_{SFP}} f(z)$.*

Additionally, if $J_\zeta^{F_1} = P_C$ and $J_\zeta^{F_2} = P_Q$ in Algorithms 4-6, one can obtain three more sub-results on the split feasibility problems. It is further noted that the algorithm in Corollary 4.1 and in the three sub-results of Algorithms 4-6, which are introduced for the first time in this work, improve several existing related algorithms on the split feasibility problems; see [2, 4] and the references therein.

4.2. SPLIT MINIMIZATION PROBLEMS

Recall that another special case of the SVIP (1.1) is the split minimization problem (SMP):

$$\text{find } t^* \in \mathcal{H}_1 \text{ such that } t^* \in \operatorname{argmin}_{t \in \mathcal{H}_1} p(t) \text{ and } Bt^* \in \operatorname{argmin}_{s \in \mathcal{H}_2} q(s), \quad (4.2)$$

where $p : \mathcal{H}_1 \rightarrow \mathbb{R}$ and $q : \mathcal{H}_2 \rightarrow \mathbb{R}$ are proper and lower semicontinuous. We similarly represent by Ω_{SMP} the set of solutions of the SMP (4.2) and assume that $\Omega_{SMP} \neq \emptyset$.



Similarly, following the detailed explanation provided in ([41], Subsection 4.2), we obtain the following corollary from Algorithm 5 and Theorem 3.12.

Corollary 4.2. *Suppose that Assumptions 3.1 and 3.2 hold and $\{t_n\}$ is a sequence generated by Algorithm 5 with $F_1 = \partial p$ and $F_2 = \partial q$. Then it strongly converges to $z = P_{\Omega_{SMF}} f(z)$.*

Similarly, if $F_1 = \partial p$ and $F_2 = \partial q$ in Algorithms 3, 4 and 6, we can obtain three more sub-results on the split minimization problems. It is also observed that the algorithm in Corollary 4.2 and the three sub-results of Algorithms 3, 4 and 6, which are introduced for the first time in this work, improve several existing related algorithms on split minimization problems.

5. NUMERICAL EXPERIMENTS

This section is devoted to investigating the performance of the proposed algorithms (namely, Algorithms 3–6) and three recently developed related algorithms, including Algorithm 1 (Tan et al. [39], Algorithm 3.1), Algorithm 2 (Zhang and Ma [43], Algorithm 1), and Algorithm 3.3 of Thong et al. [42], abbreviated as TQY, ZM, and TDC, respectively. The implementations of the TQY, ZM, and TDC algorithms follow the original formulations in [39, 42, 43]. To maintain consistent notation throughout this paper, we replace the original variables and parameters with our notation, while leaving all algorithmic steps and parameter choices unchanged. For performance comparisons in all numerical experiments, we employ some varying tolerance levels $\epsilon > 0$ and the following residuals: $E_n = \|h_n - u_n\| \leq \epsilon$ for Algorithms 3, 4, TQY, and TDC, $E_n = \|h_n - z_n\| \leq \epsilon$ for Algorithms 5 and 6, and $E_n = \|h_n - u_n\| + \|\bar{\nu}_n \gamma_n - \alpha_n \lambda_n u_n\| \leq \epsilon$ for ZM.

All numerical experiments were carried out in the MATLAB R2023b environment on a Lenovo laptop equipped with a 12th Gen Intel(R) Core(TM) i5-1240P processor running at 1.70 GHz and 16.0 GB of RAM. For pairwise comparisons, each algorithm was executed over 30 independent runs, where all random matrices and initial points were generated using the MATLAB random number generator with different seeds, namely, `rng(run)` for $\text{run} = 1, 2, \dots, 30$. The performance of each algorithm was evaluated using the following metrics: the average number of iterations (Iter.), the average execution time (Time (s)), and the average and standard deviation of the final residual error (E_n (Aver. $\pm$ Std.)), where Aver. and Std. denote the average and standard deviation computed over the 30 independent runs, respectively.

Example 5.1. Consider three matrices $B, G_1, G_2 : \mathbb{R}^m \rightarrow \mathbb{R}^m$ generated from a normal distribution with mean zero and unit variance. Define the operators $F_1, F_2 : \mathbb{R}^m \rightarrow \mathbb{R}^m$ by $F_1(t) = G_1^T G_1 t$ and $F_2(s) = G_2^T G_2 s$, respectively. We seek a point $x = (x_1, \dots, x_m)^T \in \mathbb{R}^m$ such that $F_1(x) = (0, \dots, 0) \in \mathbb{R}^m$ and $F_2(Bx) = (0, \dots, 0) \in \mathbb{R}^m$. Clearly, the unique solution is $x = (0, \dots, 0) \in \mathbb{R}^m$. Since $F_i, i = 1, 2$, are linear monotone operators, the resolvent $J_\zeta^{F_i} = (I + \zeta F_i)^{-1}, i = 1, 2$, is well defined and single-valued for every $\zeta > 0$. In the numerical implementation, $J_\zeta^{F_i}(x), i = 1, 2$, is obtained by solving the linear system $(I + \zeta G_i^T G_i)z = x, i = 1, 2$, and setting $J_\zeta^{F_i}(x) = z$. Equivalently, the resolvent is computed numerically using the MATLAB backslash operator, $J_\zeta^{F_i}(x) = (I + \zeta G_i^T G_i) \backslash x$. The experiments were conducted for two problem sizes, $m = 500$



and $m = 1000$, using four tolerance levels, $\epsilon = 10^{-6}$, 10^{-8} , 10^{-10} , and 10^{-12} . For the implementation of all algorithms, we choose $f(t) = 0.99t$, set $t_0 = \text{randn}(m, 1)$, $t_1 = \text{randn}(m, 1)$, and use 1000 as the maximum number of iterations. The parameters are selected as follows:

(i) We set $\lambda_1 = 0.007$, $\zeta = 0.5$, $\varepsilon = 0.96$, $\tau = \frac{0.8}{\varepsilon+0.1}$, $\sigma = 1.9999$, $\eta_1 = 0.003$, $\eta_2 = 3$, $\eta_3 = 0.0001$, $\alpha_n = \frac{1}{10^{1.2n}}$, $\mu_n = 0.4$, $\delta_n = 1 - (\alpha_n + \mu_n)$, $\varsigma_n = \zeta_n^{(1)} = \zeta_n^{(2)} = \alpha_n^2$, and generate β_n , $\xi_n^{(1)}$, and $\xi_n^{(2)}$ according to (3.6) and (3.7) for Algorithms 3–6. We also choose $\{w_n\}$ to be the sequence $\{h_n\}$ generated by each proposed algorithm. In addition, we set $\varepsilon_2 = \frac{4}{\tau+0.1}$ for Algorithms 4 and 6.

(ii) For TDC, we choose $\beta = 0.5$, $\theta = 0.003$, $\lambda_n = \frac{0.4}{\|B\|^2}$, $\beta_n = \frac{1}{10^{1.2n}}$, and $\tau_n = \beta_n^2$.

(iii) For ZM, we set $\lambda_1 = 0.007$, $\zeta = 0.5$, $\varepsilon = 0.96$, $\eta_1 = 0.003$, $\alpha_n = \frac{1}{10^{1.2n}}$, and $\varsigma_n = \alpha_n^2$, while the remaining parameters are adopted from [43, Example 5.1].

(iv) For TQY, we set $\zeta = 0.5$, $\varepsilon = 0.96$, $\sigma = 1.9$, $\eta_1 = 0.003$, $\alpha_n = \frac{1}{10^{1.2n}}$, and $\varsigma_n = \alpha_n^2$, while the remaining parameters are adopted from [39, Numerical Experiments].

The numerical results for Example 5.1 obtained by all algorithms are presented in Table 1, while the residual error trajectories corresponding to the run with the highest number of iterations for all algorithms are depicted in Figures 1–2.

TABLE 1. Comparison of the performance of all algorithms for Example 5.1.

ϵ	Algorithms	m = 500			m = 1000		
		Iter.	Time(s)	E_n (Aver. $\pm$ Std)	Iter.	Time(s)	E_n (Aver. $\pm$ Std)
10^{-6}	Alg 3	34.5	0.6555	6.57E-07 $\pm$ 2.26E-07	35.57	3.4451	6.15E-07 $\pm$ 1.83E-07
	Alg 4	39.03	0.9955	8.54E-07 $\pm$ 7.97E-08	41.87	4.856	8.56E-07 $\pm$ 9.80E-08
	Alg 5	34.83	0.4831	6.67E-07 $\pm$ 2.15E-07	35.9	2.4034	6.79E-07 $\pm$ 1.98E-07
	Alg 6	39.83	0.6704	8.09E-07 $\pm$ 7.87E-08	42.53	3.402	8.30E-07 $\pm$ 7.75E-08
	TDC	121.5	1.3001	9.47E-07 $\pm$ 3.20E-08	118.67	5.9553	9.58E-07 $\pm$ 2.71E-08
	ZM	264.07	4.1745	9.51E-07 $\pm$ 5.06E-08	335.3	16.8694	9.70E-07 $\pm$ 1.82E-08
	TQY	209.6	33.0528	9.62E-07 $\pm$ 2.46E-08	226.83	205.677	9.62E-07 $\pm$ 2.01E-08
10^{-8}	Alg 3	42.9	0.698	7.56E-09 $\pm$ 1.97E-09	43.37	6.9699	7.32E-09 $\pm$ 1.83E-09
	Alg 4	51.77	1.03	8.16E-09 $\pm$ 7.52E-10	54.83	14.767	8.63E-09 $\pm$ 8.91E-10
	Alg 5	43.67	0.8862	6.57E-09 $\pm$ 1.72E-09	44.53	5.4406	6.78E-09 $\pm$ 2.05E-09
	Alg 6	52.7	0.7236	8.23E-09 $\pm$ 8.80E-10	55.8	8.8819	8.31E-09 $\pm$ 8.25E-10
	TDC	169.33	1.5882	9.46E-09 $\pm$ 2.63E-10	164.9	14.0375	9.52E-09 $\pm$ 3.06E-10
	ZM	335.03	4.182	9.56E-09 $\pm$ 3.49E-10	429.53	43.5204	9.64E-09 $\pm$ 2.66E-10
	TQY	272.03	44.5273	9.55E-09 $\pm$ 4.73E-10	289.83	665.6153	9.56E-09 $\pm$ 3.41E-10
10^{-10}	Alg 3	52.03	0.9686	6.97E-11 $\pm$ 1.69E-11	51.87	8.6904	6.88E-11 $\pm$ 1.79E-11
	Alg 4	64.53	1.5187	8.38E-11 $\pm$ 8.95E-12	68.1	7.4035	8.43E-11 $\pm$ 8.42E-12
	Alg 5	52.5	0.7392	6.11E-11 $\pm$ 1.76E-11	53	5.8909	6.04E-11 $\pm$ 1.94E-11
	Alg 6	65.63	1.1256	8.50E-11 $\pm$ 8.93E-12	69.1	10.39	8.51E-11 $\pm$ 8.61E-12
	TDC	217.5	2.7925	9.63E-11 $\pm$ 3.10E-12	211.7	19.402	9.52E-11 $\pm$ 2.94E-12
	ZM	405.93	10.322	9.62E-11 $\pm$ 2.56E-12	523.17	51.2855	9.66E-11 $\pm$ 2.60E-12
	TQY	333.7	65.2479	9.44E-11 $\pm$ 3.76E-12	353.33	326.7711	9.64E-11 $\pm$ 2.43E-12
10^{-12}	Alg 3	61.4	1.4122	6.72E-13 $\pm$ 1.94E-13	60.9	18.2208	6.75E-13 $\pm$ 2.06E-13
	Alg 4	77.53	2.1516	8.29E-13 $\pm$ 8.69E-14	81.43	23.0255	8.42E-13 $\pm$ 7.91E-14
	Alg 5	61.2	0.8293	6.48E-13 $\pm$ 1.79E-13	61.4	9.7165	7.13E-13 $\pm$ 1.88E-13
	Alg 6	78.77	1.4603	8.36E-13 $\pm$ 9.03E-14	82.5	22.9302	8.67E-13 $\pm$ 8.21E-14
	TDC	266.17	3.3944	9.63E-13 $\pm$ 2.20E-14	259	40.0376	9.46E-13 $\pm$ 2.77E-14
	ZM	472.63	5.5047	9.59E-13 $\pm$ 3.17E-14	615.87	80.0011	9.67E-13 $\pm$ 1.96E-14
	TQY	394.87	59.2837	9.56E-13 $\pm$ 3.49E-14	416.9	440.5447	9.72E-13 $\pm$ 2.48E-14



Figure 1. Error trajectories corresponding to the run requiring the largest number of iterations for all algorithms with $m = 500$ and different tolerance levels in Example 5.1.

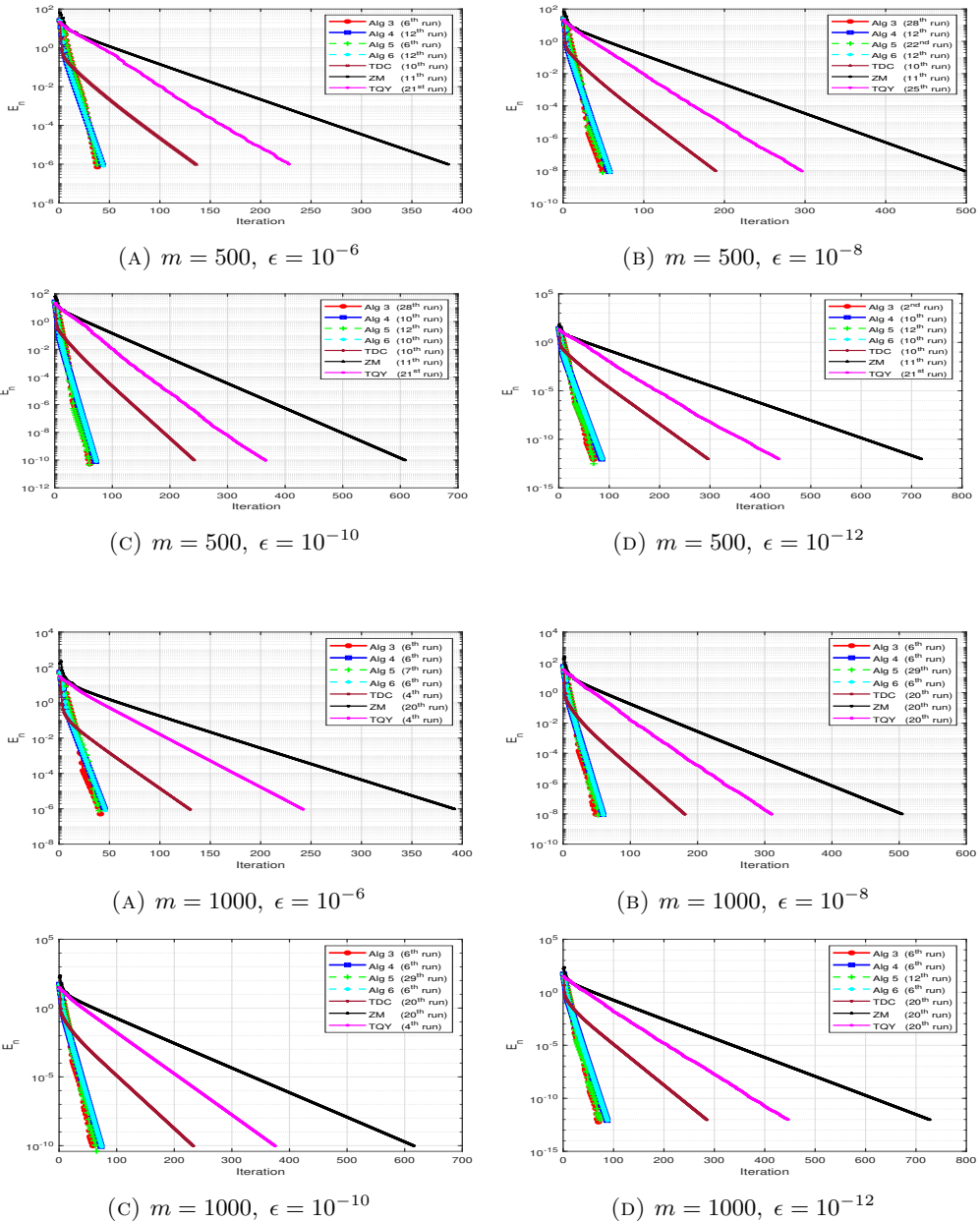


Figure 2. Error trajectories corresponding to the run requiring the largest number of iterations for all algorithms with $m = 1000$ and different tolerance levels in Example 5.1.



Remark 5.2. Although all algorithms successfully converge for every assigned tolerance level, the algorithms differ significantly in computational efficiency across all three metrics (number of iterations, CPU time, and residual error) as highlighted from Table 1 and Figures 1-2.

(i) The numerical results clearly demonstrate that the proposed algorithms (i.e., Algorithms 3-6) consistently outperform the competing methods (TDC, ZM, and TQY) in terms of faster convergence. In particular, they achieve extremely smaller averages of the final residual errors, with significantly fewer iterations and substantially reduced execution times for both problem sizes ($m = 500$ and $m = 1000$), across all tolerance levels, illustrating their efficiency and stability in solving the problem in Example 5.1. This advantage becomes more pronounced as the tolerance becomes tighter (from 10^{-6} to 10^{-12}), where the gap in performance increases substantially in favor of the proposed algorithms, except in only one scenario ($m = 1000$, $\epsilon = 10^{-8}$), in which TDC has slightly smaller execution time than Algorithm 4.

(ii) The reduction to a single resolvent evaluation $J_{\zeta}^{F_1}$ per iteration plays a crucial role in improving computational performance. This simplification significantly lowers the per-iteration cost, which explains the consistently shorter execution times observed for Algorithms 5 and 6 compared to their counterparts Algorithms 3 and 4, respectively. This effect is evident for almost all problem sizes ($m = 500$ and $m = 1000$) and across all the tolerance levels. The only minor exceptions occur at ($m = 500$, $\epsilon = 10^{-8}$) and ($m = 1000$, $\epsilon = 10^{-10}$), where Algorithms 3 and 4 produce slightly smaller execution times compared to Algorithms 5 and 6, respectively.

(iii) It is also observed that Algorithms 4 and 6 generally require slightly more iterations and execution time than their respective variants Algorithms 3 and 5. These may be attributed to the slight changes in Step 4 and new adaptive step length strategies incorporated to Algorithms 4 and 6, which, while enhancing stability and robustness, introduce additional computational overhead per iteration and impose stricter update conditions. Consequently, a modest increase in iteration count and runtime is expected depending on the problems.

(iv) Furthermore, among the competing methods, ZM and TQY exhibit significantly higher iteration counts and execution times, especially as the problem size increases from $m = 500$ to $m = 1000$. This indicates poorer scalability compared to both the proposed algorithms and TDC. In contrast, the proposed methods maintain relatively stable growth in computational cost, confirming their efficiency and robustness for larger-scale problems.

(v) Overall, the results indicate that the proposed algorithms strike a favorable balance between accuracy, stability and efficiency, making them more suitable for many practical implementations, especially in large-scale settings and under stringent accuracy requirements.

Example 5.3. In this example, we consider the SMP (4.2) by setting $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}^m$, $h(t) = \frac{1}{2}d_C^2(t)$ and $q(s) = \frac{1}{2}\|s\|^2$, where $C \subset \mathbb{R}^m$ is a unit ball, $d_C(t)$ represents the distance of a point t onto C and $B \in \mathbb{R}^{m \times m}$ is a Gaussian random matrix whose entries are independent and identically distributed according to $N(0, 1)$. The projection onto C is explicitly computed by $P_C(t) = \frac{t}{\max\{1, \|t\|\}}$. Thus, the subdifferentials of h and q are as follows:



$$\partial h(t) = \begin{cases} 0, & \text{if } \|t\| \leq 1, \\ t - \frac{t}{\|t\|}, & \text{if } \|t\| > 1 \end{cases} \quad \text{and} \quad \partial q(s) = s.$$

Using these subdifferentials and proximal optimality condition, we explicitly compute the proximal mappings associated to h and q as follows.

$$J_{\zeta}^{\partial h}(t) = \text{Prox}_{\zeta h}(t) = \frac{t + P_C(t)}{1 + \zeta} \quad \text{and} \quad J_{\zeta}^{\partial q}(s) = \text{Prox}_{\zeta q}(s) = \frac{s}{1 + \zeta}.$$

In these experiments, we consider four varying tolerance levels $\epsilon = 10^{-3}$, 10^{-4} , 10^{-6} and 10^{-8} , and three problem sizes $m = 500$, $m = 1000$ and $m = 2000$. We randomly generated the initial points t_0 and t_1 and applied the same mapping f as in Example 5.1 and set 100,000 as the maximum iteration counts. We further set $\zeta = \beta = 6.3$, $\alpha_n = \beta_n = \frac{1}{10n}$ and the remaining of the ones as selected for all algorithms in Example 5.1, except $\tau = \frac{1}{10^9(\epsilon+0.05)}$ for Algorithms 3-6 and $\varepsilon_2 = \frac{4}{\tau+0.1} - 2.62$ for Algorithms 4 and 6.

Similarly, the numerical results for Example 5.3 obtained by all algorithms are presented in Table 2, while the residual error trajectories corresponding to the run with the highest number of iterations for all algorithms are depicted in Figures 3-5.

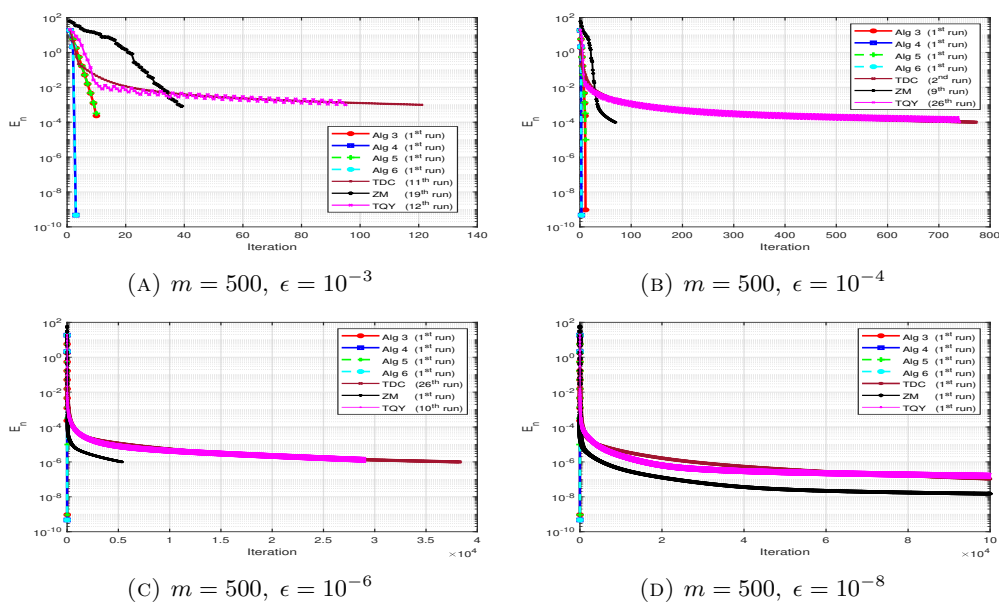


Figure 3. Error trajectories corresponding to the run requiring the largest number of iterations for all algorithms with $m = 500$ and different tolerance levels in Example 5.3.



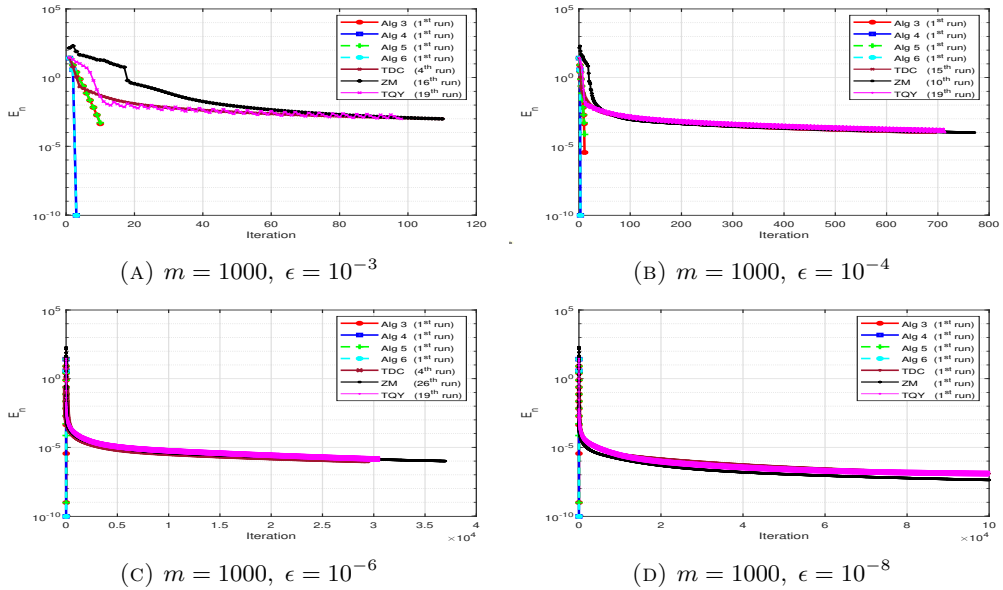


Figure 4. Error trajectories corresponding to the run requiring the largest number of iterations for all algorithms with $m = 1000$ and different tolerance levels in Example 5.3.

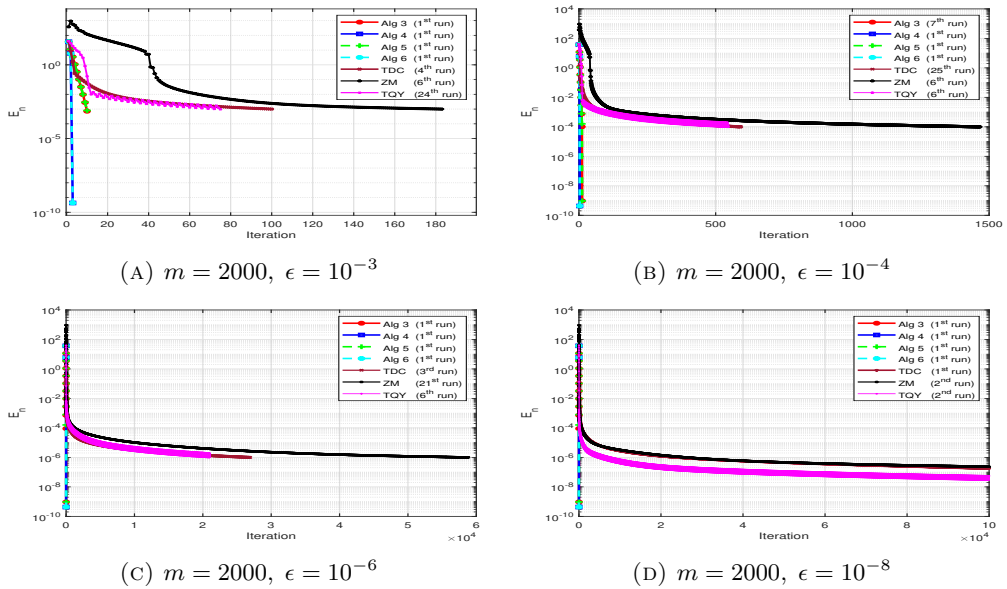


Figure 5. Error trajectories corresponding to the run requiring the largest number of iterations for all algorithms with $m = 2000$ and different tolerance levels in Example 5.3.



TABLE 2. Comparison of the performance of all algorithms for Example 5.3

ϵ	Algorithms	m = 500			m = 1000			m = 2000		
		Iter.	Time(s)	E_n (Aver. $\pm$ Std)	Iter.	Time(s)	E_n (Aver. $\pm$ Std)	Iter.	Time(s)	E_n (Aver. $\pm$ Std)
10^{-3}	Alg 3	10	0.003	2.35E-04 $\pm$ 1.05E-05	10	0.0064	4.37E-04 $\pm$ 1.85E-05	10	0.035	7.23E-04 $\pm$ 1.59E-05
	Alg 4	3	0.0019	4.65E-10 $\pm$ 1.95E-11	3	0.0036	8.68E-11 $\pm$ 3.45E-11	3	0.0105	4.54E-10 $\pm$ 2.95E-11
	Alg 5	10	0.0029	3.21E-04 $\pm$ 1.05E-05	10	0.0062	5.24E-04 $\pm$ 1.84E-05	10	0.0356	8.14E-04 $\pm$ 1.59E-05
	Alg 6	3	0.0021	4.65E-10 $\pm$ 1.95E-11	3	0.0032	8.67E-11 $\pm$ 3.45E-11	3	0.0111	4.54E-10 $\pm$ 2.95E-11
	TDC	110.87	0.0267	9.93E-04 $\pm$ 3.54E-06	102.67	0.0516	9.93E-04 $\pm$ 3.82E-06	92.57	0.2763	9.92E-04 $\pm$ 4.79E-06
	ZM	35.13	0.0095	6.99E-04 $\pm$ 2.30E-04	51.5	0.0288	9.21E-04 $\pm$ 1.14E-04	77.9	0.233	9.64E-04 $\pm$ 4.39E-05
	TQY	47.33	0.0721	9.25E-04 $\pm$ 1.27E-04	37.2	0.1356	8.23E-04 $\pm$ 1.72E-04	26.2	0.6185	8.03E-04 $\pm$ 2.14E-04
10^{-4}	Alg 3	11	0.0032	9.50E-10 $\pm$ 3.35E-14	11	0.0075	5.19E-06 $\pm$ 4.77E-06	11.17	0.0367	7.74E-05 $\pm$ 3.54E-05
	Alg 4	3	0.0016	4.65E-10 $\pm$ 1.95E-11	3	0.003	8.68E-11 $\pm$ 3.45E-11	3	0.0102	4.54E-10 $\pm$ 2.95E-11
	Alg 5	11	0.0038	1.10E-05 $\pm$ 3.26E-06	11	0.0074	7.37E-05 $\pm$ 5.70E-06	12	0.0389	9.50E-10 $\pm$ 1.94E-15
	Alg 6	3	0.0017	4.65E-10 $\pm$ 1.95E-11	3	0.0029	8.67E-11 $\pm$ 3.45E-11	3	0.0098	4.54E-10 $\pm$ 2.95E-11
	TDC	665.03	0.1395	9.99E-05 $\pm$ 6.28E-08	620.3	0.4555	9.99E-05 $\pm$ 6.48E-08	542.93	1.6602	9.99E-05 $\pm$ 6.87E-08
	ZM	46.63	0.0123	7.88E-05 $\pm$ 2.89E-05	260.77	0.1337	9.69E-05 $\pm$ 9.35E-06	538.47	1.6054	9.92E-05 $\pm$ 2.27E-06
	TQY	342.73	0.5498	9.84E-05 $\pm$ 2.26E-06	237.3	1.363	9.75E-05 $\pm$ 2.41E-06	137.43	3.2466	9.54E-05 $\pm$ 4.13E-06
10^{-6}	Alg 3	11	0.0037	9.50E-10 $\pm$ 3.35E-14	11.73	0.0087	2.43E-08 $\pm$ 1.28E-07	12	0.0413	9.50E-10 $\pm$ 4.51E-14
	Alg 4	3	0.0017	4.65E-10 $\pm$ 1.95E-11	3	0.0035	8.68E-11 $\pm$ 3.45E-11	3	0.0116	4.54E-10 $\pm$ 2.95E-11
	Alg 5	12	0.0041	9.50E-10 $\pm$ 4.36E-14	12	0.0093	9.50E-10 $\pm$ 3.42E-15	12	0.0394	9.50E-10 $\pm$ 1.94E-15
	Alg 6	3	0.0017	4.65E-10 $\pm$ 1.95E-11	3	0.0032	8.67E-11 $\pm$ 3.45E-11	3	0.011	4.54E-10 $\pm$ 2.95E-11
	TDC	25480.73	9.5543	1.00E-06 $\pm$ 2.10E-11	24819.7	17.649	1.00E-06 $\pm$ 1.60E-11	21731.9	191.8961	1.00E-06 $\pm$ 1.65E-11
	ZM	1842.27	0.4578	9.53E-07 $\pm$ 1.68E-07	12832.37	9.1458	1.00E-06 $\pm$ 2.57E-10	24862.13	170.674	1.00E-06 $\pm$ 5.72E-10
	TQY	13202	31.7767	9.99E-07 $\pm$ 1.78E-09	9452.17	259.527	9.98E-07 $\pm$ 8.30E-09	5201.8	263.308	9.96E-07 $\pm$ 1.09E-08
10^{-8}	Alg 3	11	0.0038	9.50E-10 $\pm$ 3.35E-14	11.77	0.008	9.50E-10 $\pm$ 3.17E-13	12	0.0356	9.50E-10 $\pm$ 4.51E-14
	Alg 4	3	0.0017	4.65E-10 $\pm$ 1.95E-11	3	0.003	8.68E-11 $\pm$ 3.45E-11	3	0.0094	4.54E-10 $\pm$ 2.95E-11
	Alg 5	12	0.0033	9.50E-10 $\pm$ 4.36E-14	12	0.0117	9.50E-10 $\pm$ 3.42E-15	12	0.0386	9.50E-10 $\pm$ 1.94E-15
	Alg 6	3	0.0015	4.65E-10 $\pm$ 1.95E-11	3	0.0034	8.67E-11 $\pm$ 3.45E-11	3	0.0089	4.54E-10 $\pm$ 2.95E-11
	TDC	100000*	32.7369	1.79E-07 $\pm$ 7.12E-08	100000*	192.7932	1.77E-07 $\pm$ 4.51E-08	100000*	876.2602	1.47E-07 $\pm$ 3.08E-08
	ZM	52744.3	31.2788	1.25E-08 $\pm$ 4.71E-09	95317.93	160.175	7.20E-08 $\pm$ 6.87E-08	87173.33	462.0057	1.91E-07 $\pm$ 1.60E-07
	TQY	95420.83	367.9924	9.72E-08 $\pm$ 7.44E-08	896652.2	882.5459	7.64E-08 $\pm$ 7.89E-08	83100.97	2894.286	4.09E-08 $\pm$ 4.27E-08

Remark 5.4. Similarly, from Table 2 and Figures 3-5, we have the following observations.

- (i) The numerical results clearly indicate that the proposed algorithms (i.e., Algorithms 3-6) consistently outperform the competing algorithms (TDC, ZM, and TQY) in terms of convergence efficiency. They achieve significantly smaller averages of the final residual errors with far fewer iterations and substantially reduced execution times for all problem sizes $m = 500, 1000$ and 2000 , across all tolerance levels. Additionally, the proposed algorithms achieve comparably attractive standard deviations of the final residual errors in most experiments. These demonstrate their efficiency, robustness, and stability in solving Example 5.3. These superior performance remain efficient stable as the problem size increases, highlighting the robustness and good scalability properties of the proposed methods.
- (ii) It is particularly noteworthy that Algorithms 4 and 6 require the least number of iterations in all cases, typically converging within 3 iterations, regardless of the tolerance level and problem size. These algorithms also exhibit extremely low execution times while maintaining high accuracy with very small average final residual errors over 30 independent runs across all problem sizes and tolerance levels, and more stability than Algorithms 3 and 5, especially in the smaller tolerance levels (10^{-3} and 10^{-4}), whereas the proposed Algorithms 3 and 5 achieve extremely smaller standard deviation of the final errors for the tighter tolerance levels (10^{-6} and 10^{-8}). This behavior strongly suggests the effectiveness and stability of the slight modifications in the fourth step and the new adaptive step length strategies incorporated to Algorithms 4 and 6, which significantly accelerate

* Maximum iteration limit ($N_{\max} = 100,000$) reached before satisfying the stopping criterion.



their convergence without sacrificing efficiency and stability in solving Example 5.3.

(iii) As the tolerance level becomes more stringent, the difference in performance becomes increasingly pronounced. The competing algorithms (TDC, ZM, and TQY) often require a dramatically larger number of iterations. In particular, it is noted from Table 2 and Figures 3-5(D) that the competing algorithms (ZM, and TQY) and TDC reach the prescribed maximum iteration limit (that is, 100,000 iterations) for the tightest tolerance level 10^{-8} without converging in some runs and in all runs, respectively. This is accompanied by a substantial increase in execution time. In contrast, the proposed algorithms maintain efficient and stable convergence behavior, requiring only a few iterations and modest computational time while still achieving the desired accuracy and stability.

(iv) Furthermore, the performance gap widens as the problem dimension increases. While the computational cost of TDC, ZM, and TQY grows rapidly with m , the proposed algorithms exhibit only a mild increase in execution time, confirming their superior scalability. In particular, the consistent performance of Algorithms 4 and 6 across all tested scenarios demonstrates their suitability for large-scale problems.

(v) Overall, these results confirm that the proposed algorithms not only provide faster convergence but also maintain high numerical accuracy, stability and computational efficiency under increasingly challenging conditions, making them more reliable and effective for practical large-scale applications.

6. CONCLUSION

This work introduces four proximal viscosity-type and contraction algorithms for finding a unique viscosity-selected solution of a split variational inclusion problem in infinite-dimensional real Hilbert spaces. Each of the proposed algorithms incorporates an alternating inertial technique and a three-term conjugate gradient-like direction, both of which significantly improve its convergence properties. In contrast to several existing and related algorithms, the proposed schemes incorporate relaxation effects together with simple adaptive step-length strategies that are bounded away from zero, enabling them to operate effectively without requiring prior knowledge of the norm of the bounded linear operator or the implementation of a costly line-search procedure at each iteration. Notably, the adaptive step-length strategy employed in the second and fourth algorithms (namely, Algorithms 4 and 6) is introduced for the first time in this work and differs from that used in the first and third algorithms (namely, Algorithms 3 and 5). Another distinctive feature of the proposed algorithms based on the three-term conjugate gradient-like direction (1.11) is the incorporation of simple adaptive techniques for generating the conjugate gradient-like parameters at each iteration, thereby avoiding the restrictive boundedness assumption commonly imposed on sequences generated by existing algorithms based on the conjugate gradient-like direction (1.12); see [43] and the references therein. Under mild conditions, the strong convergence of each of the proposed algorithms to a unique viscosity-selected solution of the split variational inclusion problem is established. Moreover, applications of the proposed algorithms to significant problems, including split feasibility and split minimization problems, are explored, thereby providing a unified framework for transforming and solving important inverse problems arising in practical applications such as image and signal recovery, machine learning, and many other related areas. Furthermore, numerical experiments demonstrate the robustness and efficiency of the proposed algorithms, consistently outperforming the comparison algorithms in terms of convergence speed, computational time, and accuracy across all the



examples considered.

Acknowledgments This research was funded by King Mongkut's University of Technology North Bangkok, Contract No. KMUTNB-69-KNOW-60.

DECLARATIONS

Competing interests

The authors declare no conflict of interest.

Consent for publication

The authors agreed to publish this article in the Bangmod International Journal of Mathematical & Computational Science.

Code availability

Available upon reasonable request.

Authors' contributions

All authors contributed equally in this article.

REFERENCES

- [1] J.A. Abuchu, A.E. Ofem, G.C. Ugwunnadi, O.K. Narain, A. Hussain, Hybrid alternated inertial projection and contraction algorithm for solving bilevel variational inequality problems, *Journal of Mathematics* 2023 (2023) 3185746. Available from: <https://doi.org/10.1155/2023/3185746>.
- [2] A. Ahmad, P. Kumam, Strong convergence accelerated alternated inertial relaxed algorithm for split feasibilities with applications in breast cancer detection, *Bangmod Journal of Mathematical and Computational Sciences* 11 (2025) 109–134. Available from: <https://doi.org/10.58715/bangmodjmcs.2025.11.6>.
- [3] A. Ahmad, P. Kumam, V. Berinde, T. Seangwattana, Viscosity-type algorithms with acceleration techniques for variational inequalities and demicontractive mappings with applications in cybersecurity investments and classification problems, *Communications in Nonlinear Science and Numerical Simulation* 135 (2025) 109549. Available from: <https://doi.org/10.1016/j.cnsns.2025.109549>.
- [4] A. Ahmad, P. Kumam, Y.J. Cho, K. Sitthithakerngkiet, Halpern-type relaxed algorithms with alternated and multi-step inertia for split feasibility problems with applications in classification problems, *Constructive Mathematical Analysis* 8 (2025) 50–80. Available from: <https://doi.org/10.33205/cma.1563173>.
- [5] A. Ahmad, P. Kumam, M.H. Harbau, Convergence theorems for common solutions of nonlinear problems and applications, *Carpathian Journal of Mathematics* 40 (2024) 207–241. Available from: <https://doi.org/10.37193/CJM.2024.02.01>.
- [6] A. Ahmad, P. Kumam, M.H. Harbau, K. Sitthithakerngkiet, Inertial hybrid algorithm for generalized mixed equilibrium problems, zero problems, and fixed points of some nonlinear mappings in the intermediate sense, *Mathematical Methods in the Applied Sciences* 47 (2024) 8527–8550. Available from: <https://doi.org/10.1002/mma.10030>.



- [7] A. Ahmad, P. Kumam, T. Seangwattana, Strong convergence multi-step inertial golden ratio-based algorithms for split feasibility problems with applications, *Mathematical Modelling and Numerical Simulation with Applications* 5 (2025) 307–347. Available from: <https://doi.org/10.53391/mmnsa.1595657>.
- [8] A. Ahmad, P. Kumam, M.M. Yahaya, K. Sitthithakerngkiet, Alternated and multi-step inertial algorithm with three-term conjugate gradient-like direction for split feasibilities with applications in classification problems and elastic net, *Computational and Applied Mathematics* 44 (2025) 325. Available from: <https://doi.org/10.1007/s40314-025-03276-x>.
- [9] A. Ahmad, P. Kumam, M.M. Yahaya, K. Sitthithakerngkiet, W. Chalamjiak, Two-step alternated inertial algorithm for split feasibility problems with some applications, *Journal of Nonlinear and Convex Analysis* 25 (2024) 3165–3191.
- [10] A. Beck, M. Teboulle, A fast iterative shrinkage-thresholding algorithm for linear inverse problems, *SIAM Journal on Imaging Sciences* 2 (2009) 183–202. Available from: <https://doi.org/10.1137/080716542>.
- [11] C. Byrne, Y. Censor, A. Gibali, S. Reich, The split common null point problem, *Journal of Nonlinear and Convex Analysis* 13 (2012) 759–775.
- [12] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, *Numerical Algorithms* 8 (1994) 221–239. Available from: <https://doi.org/10.1007/BF02142692>.
- [13] Y. Censor, A. Gibali, S. Reich, Algorithms for the split variational inequality problem, *Numerical Algorithms* 59 (2012) 301–323. Available from: <https://doi.org/10.1007/s11075-011-9490-5>.
- [14] P. Chen, J. Huang, X. Zhang, A primal-dual fixed point algorithm for convex separable minimization with applications to image restoration, *Inverse Problems* 29 (2013) 025011. Available from: <https://doi.org/10.1088/0266-5611/29/2/025011>.
- [15] C.S. Chuang, Strong convergence theorems for the split variational inclusion problem in Hilbert spaces, *Fixed Point Theory and Applications* 2013 (2013) 350. Available from: <https://doi.org/10.1186/1687-1812-2013-350>.
- [16] J. Deepho, P. Thounthong, P. Kumam, S. Phiangsungnoen, A new general iterative scheme for split variational inclusion and fixed point problems of k -strict pseudo-contraction mappings with convergence analysis, *Journal of Computational and Applied Mathematics* 318 (2017) 293–306. Available from: <https://doi.org/10.1016/j.cam.2016.09.009>.
- [17] Q.L. Dong, X.-H. Li, T.M. Rassias, Two projection algorithms for a class of split feasibility problems with jointly constrained Nash equilibrium models, *Optimization* 70 (2021) 871–897. Available from: <https://doi.org/10.1080/02331934.2020.1753741>.
- [18] Q.L. Dong, H.-B. Yuan, Accelerated Mann and CQ algorithms for finding a fixed point of a nonexpansive mapping, *Fixed Point Theory and Applications* 2015 (2015) 125. Available from: <https://doi.org/10.1186/s13663-015-0374-6>.
- [19] R. Fadhil, S. Salman, Fixed-point acceleration algorithm of total asymptotically nonexpansive mappings, *F1000Research* 14 (2025) 1430. Available from: <https://doi.org/10.12688/f1000research.172972.1>.
- [20] K. Goebel, S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, *Monographs and Textbooks in Pure and Applied Mathematics*, vol. 83, Marcel Dekker, New York, 1984.



- [21] B. Halpern, Fixed points of nonexpanding maps, *Bulletin of the American Mathematical Society* 73 (1967) 957–961. Available from: <https://doi.org/10.1090/S0002-9904-1967-11864-0>.
- [22] M.H. Harbau, A. Ahmad, B. Ali, G.C. Ugwunnadi, Inertial residual algorithm for fixed points of finite family of strictly pseudocontractive mappings in Banach spaces, *International Journal of Nonlinear Analysis and Applications* 13 (2022) 2257–2269. Available from: <https://doi.org/10.22075/ijnaa.2021.23301.2517>.
- [23] M.H. Harbau, G.C. Ugwunnadi, L.O. Jolaoso, A. Abdulwahab, Inertial accelerated algorithm for fixed point of asymptotically nonexpansive mapping in real uniformly convex Banach spaces, *Axioms* 10 (2021) 147. Available from: <https://doi.org/10.3390/axioms10030147>.
- [24] S. He, C. Yang, Solving the variational inequality problem defined on intersection of finite level sets, *Abstract and Applied Analysis* 2013 (2013) 942315. Available from: <https://doi.org/10.1155/2013/942315>.
- [25] Z. He, The split equilibrium problem and its convergence algorithms, *Journal of Inequalities and Applications* 2012 (2012) 162. Available from: <https://doi.org/10.1186/1029-242X-2012-162>.
- [26] H. Iiduka, Fixed point optimization algorithms for distributed optimization in networked systems, *SIAM Journal on Optimization* 23 (2013) 1–26. Available from: <https://doi.org/10.1137/120866877>.
- [27] H. Iiduka, Acceleration method for convex optimization over the fixed point set of a nonexpansive mapping, *Mathematical Programming* 149 (2015) 131–165. Available from: <https://doi.org/10.1007/s10107-013-0741-1>.
- [28] D.A. Lorenz, T. Pock, An inertial forward-backward algorithm for monotone inclusions, *Journal of Mathematical Imaging and Vision* 51 (2015) 311–325. Available from: <https://doi.org/10.1007/s10851-014-0523-2>.
- [29] G. Marino, H.-K. Xu, Convergence of generalized proximal point algorithms, *Communications on Pure and Applied Analysis* 3 (2004) 791–808. Available from: <https://doi.org/10.3934/cpaa.2004.3.791>.
- [30] A. Moudafi, Viscosity approximation methods for fixed point problems, *Journal of Mathematical Analysis and Applications* 241 (2000) 46–55. Available from: <https://doi.org/10.1006/jmaa.1999.6615>.
- [31] A. Moudafi, Split monotone variational inclusions, *Journal of Optimization Theory and Applications* 150 (2011) 275–283. Available from: <https://doi.org/10.1007/s10957-011-9814-6>.
- [32] Z. Mu, Y. Peng, A note on the inertial proximal point method, *Statistics, Optimization and Information Computing* 3 (2015) 241–248. Available from: <https://doi.org/10.19139/soic.v3i3.124>.
- [33] S. Penfold, R. Zalas, M. Casiraghi, M. Brooke, Y. Censor, R. Schulte, Sparsity constrained split feasibility for dose-volume constraints in inverse planning of intensity-modulated photon or proton therapy, *Physics in Medicine and Biology* 62 (2017) 3599–3618. Available from: <https://doi.org/10.1088/1361-6560/aa602b>.
- [34] B.T. Polyak, Some methods of speeding up the convergence of iteration methods, *USSR Computational Mathematics and Mathematical Physics* 4 (1964) 1–17. Available from: [https://doi.org/10.1016/0041-5553\(64\)90137-5](https://doi.org/10.1016/0041-5553(64)90137-5).
- [35] R.T. Rockafellar, On the maximal monotonicity of subdifferential mappings, *Pacific Journal of Mathematics* 33 (1970) 209–216. Available from: <https://doi.org/10.2140/>



- pjm.1970.33.209.
- [36] K. Sitthithakerngkiet, J. Deepho, P. Kumam, A hybrid viscosity algorithm via modified hybrid steepest descent method for solving split variational inclusion in image reconstruction and fixed point problems, *Applied Mathematics and Computation* 250 (2015) 986–1001. Available from: <https://doi.org/10.1016/j.amc.2014.10.130>.
 - [37] K. Sitthithakerngkiet, J. Deepho, J. Martínez-Moreno, P. Kumam, Convergence analysis of a general iterative algorithm for finding a common solution of split variational inclusion and optimization problems, *Numerical Algorithms* 79 (2018) 801–824. Available from: <https://doi.org/10.1007/s11075-017-0462-2>.
 - [38] S. Suantai, P. Chalamjiak, P. Inkrong, S. Kesornprom, Modified iterative schemes with two inertia and linesearch rule for split variational inclusion and applications to image deblurring and diabetes prediction, *Carpathian Journal of Mathematics* 40 (2024) 459–476. Available from: <https://doi.org/10.37193/CJM.2024.02.15>.
 - [39] B. Tan, X. Qin, J.C. Yao, Strong convergence of self-adaptive inertial algorithms for solving split variational inclusion problems with applications, *Journal of Scientific Computing* 87 (2021) 20. Available from: <https://doi.org/10.1007/s10915-021-01428-9>.
 - [40] Y. Tang, A. Gibali, New self-adaptive step size algorithms for solving split variational inclusion problems and its applications, *Numerical Algorithms* 83 (2020) 305–331. Available from: <https://doi.org/10.1007/s11075-019-00683-0>.
 - [41] Y. Tang, Z. Ji, Perturbation resilience of self-adaptive step-size algorithms for solving split variational inclusion problems and their applications, *Numerical Functional Analysis and Optimization* 44 (2023) 1343–1370. Available from: <https://doi.org/10.1080/01630563.2023.2247615>.
 - [42] D.V. Thong, V.T. Dung, Y.J. Cho, A new strong convergence theorem for solving split variational inclusion problems, *Numerical Algorithms* 86 (2021) 565–591. Available from: <https://doi.org/10.1007/s11075-020-00901-0>.
 - [43] Y. Zhang, X. Ma, An accelerated conjugate method for split variational inclusion problems with applications, *AIMS Mathematics* 10 (2025) 11465–11487. Available from: <https://doi.org/10.3934/math.2025522>.

