



Accelerated Averaged Iterative Methods for the Multiple-Set Split Common Fixed Point Problem of Demicontractive Mappings

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Abstract

In this paper, we propose accelerated averaged iterative algorithms for solving the multiple-set split common fixed point problem involving demicontractive mappings. The proposed schemes are obtained by incorporating averaged terms into existing algorithms originally developed for quasi-nonexpansive mappings. This approach allows us to address the multiple-set split common fixed point problem within the broader class of demicontractive mappings, which strictly contains the class of quasi-nonexpansive mappings. For the proposed algorithms, we establish weak convergence theorems in the framework of real Hilbert spaces. In addition, numerical experiments are provided to illustrate the convergence behavior of the proposed iterative schemes.

1 Introduction

In recent years, there has been increasing interest in designing convergent iterative algorithms to solve split feasibility problems [10, 13], and split common fixed point problems, see [15, 41], and their various generalizations.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, $C \subseteq \mathcal{H}_1$ and $Q \subseteq \mathcal{H}_2$ be two nonempty closed convex subsets, and $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ a bounded linear operator.

The *split feasibility problem* (SFP) requires finding

$$x \in C \text{ such that } Ax \in Q. \quad (1)$$

Key Words: Hilbert space, demicontractive mapping, multiple-set split common fixed point problem, iterative schemes.

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Let $F : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ and $G : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ be two mappings, $\mathcal{D} \subset \mathcal{H}_1$ and

$$\text{Fix}(F) := \{u \in \mathcal{D} : Fu = u\}$$

denote the set of fixed points of F . If $C = \text{Fix}(F)$ and $Q = \text{Fix}(G)$, then Problem (1) becomes

$$x \in \text{Fix}(F) \text{ such that } Ax \in \text{Fix}(G), \quad (2)$$

which is called *the split common fixed point problem* (SCFPP).

Problem (1) has been further generalized to the *multiple-sets split feasibility problem* (MSSFP), see [14] or [38], which requires finding

$$x \in \bigcap_{i=1}^m C_i \text{ such that } Ax \in Q, \quad (3)$$

where C_i , $i = 1, 2, \dots, m$, are nonempty closed convex subsets of $\mathcal{H}_1$.

Similarly, given integers $m, n \geq 1$, the MSSFP can be formulated in a more general form:

$$x \in \bigcap_{i=1}^m C_i \text{ such that } Ax \in \bigcap_{j=1}^n Q_j, \quad (4)$$

where Q_j , $j = 1, 2, \dots, n$, are nonempty closed convex subsets of $\mathcal{H}_2$.

If the sets C_i and Q_j in (4) are replaced by $\text{Fix}(F_i)$ and $\text{Fix}(G_j)$, then the MSSFP reduces to the *multiple-set split common fixed point problem* (MSSCFP), which requires finding

$$x \in \bigcap_{i=1}^m \text{Fix}(F_i) \text{ such that } Ax \in \bigcap_{j=1}^n \text{Fix}(G_j), \quad (5)$$

where, for $1 \leq i \leq m$ and $1 \leq j \leq n$, $\text{Fix}(F_i)$ and $\text{Fix}(G_j)$ are the sets of fixed points of $F_i : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ and $G_j : \mathcal{H}_2 \rightarrow \mathcal{H}_2$, respectively.

In particular, when $m = n = 1$, the MSSFP and MSSCFP reduces to the SFP and SCFP, respectively.

Multiple-set split feasibility problems (MSSFP) are important in real applications, such as signal processing and image reconstruction, CT image reconstruction of medical imaging and signal processing, see, for instance [10, 13, 45].

Several iterative schemes have been proposed to solve the above or related problems. By exploiting several properties of the demicontractive mappings in connection with the directed mappings, Cui [16] introduced and analyzed three iterative methods for solving the split common fixed point problem with multiple output sets for demicontractive mappings in Hilbert spaces, for which he established the weak convergence under mild conditions, see also Wang [35].

Eslamian and Kamandi [20] studied a variational inequality problem defined over the solution of multiple-set split common fixed point problem for a finite family of generalized demimetric mappings (which are closely related to demicontractive mappings) and presented an accelerated steepest descent algorithm for solving this problem. In the same context, Eslamian [19] studied the split common fixed point problem for demimetric mappings and Bregman relatively nonexpansive mappings.

Zhao et al. [45] introduced some algorithms to solve the MSSCFP in the class of quasi-nonexpansive mappings. There is a significant interest in solving MSSCFPs, as shown by the next selected list of recent works [1], [3], [9], [11], [12], [23], [27], [28], [29], [30], [31], [32], [33], [37], [35], [36], [34], [40], [42], [43] and [44], see also [2], [17], [18], [21], [22], [26], [46].

In the present paper, by using the possibility of embedding demicontractive mappings in the class of quasi-nonexpansive mappings, see Berinde [4, 5] and Berinde and Saleh [6, 7, 8], we propose several averaged self-adaptive inertial parallel iterative algorithms aimed at solving the MSSCFP in a more general class of demicontractive mappings.

We prove the weak convergence of the proposed algorithms to the solution of the MSSCFP and also provide some appropriate numerical examples to illustrate the convergence of our algorithms and their superiority over other related methods.

2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{D} \subset \mathcal{H}$ be a closed convex set and consider the self-mapping $F : \mathcal{D} \rightarrow \mathcal{D}$.

The mapping F is said to be *Lipschitz* if there is a real number $L \geq 0$ such that

$$\|Fu - Fv\| \leq L\|u - v\|, \quad \forall u, v \in \mathcal{D}.$$

If $L < 1$, then F is called *Banach contraction* or *strict contraction* and if $L = 1$, then F is called *nonexpansive*. Furthermore, F is called

- (a) *strictly pseudo-contractive* if there exists $\beta \in (0, 1)$ such that

$$\|Fu - Fv\|^2 \leq \|u - v\|^2 + \beta\|(I - F)u - (I - F)v\|^2, \quad \forall u, v \in \mathcal{D}. \quad (6)$$

- (b) *quasi-nonexpansive* if $\text{Fix}(F) \neq \emptyset$ and

$$\|Fu - v\| \leq \|u - v\|, \quad \text{for all } u \in \mathcal{D} \text{ and } v \in \text{Fix}(F); \quad (7)$$

- (c) β -*demicontractive* if $\text{Fix}(F) \neq \emptyset$ and there exists a positive number $\beta < 1$ such that

$$\|Fu - v\|^2 \leq \|u - v\|^2 + \beta\|u - Fu\|^2, \quad \text{for all } u \in \mathcal{D} \text{ and } v \in \text{Fix}(F); \quad (8)$$

By the previous definitions, it is obvious that any nonexpansive mapping is strictly pseudocontractive, any nonexpansive mapping F with $\text{Fix}(F) \neq \emptyset$ is quasi-nonexpansive and that any quasi-nonexpansive mapping is demicontractive, too, but the reverses of all these assertions are no more true, as illustrated by the next two examples. For a complete diagram of the relationships between the four classes of nonexpansive type mappings defined before, we refer to [5].

Example 1. [8] Let $\mathcal{H}$ be the real line with the usual norm and $\mathcal{D} = [0, 1]$. Define F on $\mathcal{D}$ as

$$F(u) = \begin{cases} \frac{2}{3}, & \text{if } 0 \leq u \leq \frac{2}{3} \\ u, & \text{if } \frac{2}{3} < u < \frac{3}{4} \\ \frac{3}{4}, & \text{if } \frac{3}{4} \leq u < \frac{9}{10} \\ \frac{1}{3}, & \text{if } \frac{9}{10} \leq u \leq 1. \end{cases} \quad (9)$$

- (a) F is β -demicontractive with $\beta = \frac{7}{17}$;
- (b) F is neither quasi-nonexpansive nor nonexpansive;
- (c) F is not strictly pseudocontractive (see Berinde and Saleh [8] for details).

The following lemma is due to Berinde [5].

Lemma 1. [5] Let $\mathcal{D}$ be a closed convex subset of the Hilbert space $\mathcal{H}$. If $F : \mathcal{D} \rightarrow \mathcal{D}$ is a β -demicontractive mapping, then for any $\mu \in (0, 1 - \beta)$ the mapping

$$F_\mu = (1 - \mu)I + \mu F$$

is quasi-nonexpansive.

Let us recall the following two straightforward identities in a real Hilbert space (see for example Zhao et al. [45]):

$$2\langle u, v \rangle = \|u\|^2 + \|v\|^2 - \|u - v\|^2 = \|u + v\|^2 - \|u\|^2 - \|v\|^2$$

and

$$\|\alpha u + (1 - \alpha)v\|^2 = \alpha\|u\|^2 + (1 - \alpha)\|v\|^2 - \alpha(1 - \alpha)\|u - v\|^2$$

for all $u, v \in \mathcal{H}$ and $\alpha \in (0, 1)$. It is easy to see that the following inequality also holds in Hilbert spaces:

$$\|\alpha_0 u_0 + \alpha_1 u_1 + \cdots + \alpha_r u_r\|^2 \leq \sum_{i=0}^r \alpha_i \|u_i\|^2 - \alpha_s \alpha_t \|u_s - u_t\|^2, \quad (10)$$

where $s, t \in \{0, 1, \dots, r\}$, $u_i \in \mathcal{H}$ ($i = 0, 1, \dots, r$) and $0 \leq \alpha_i \leq 1$ with $\sum_{i=0}^r \alpha_i = 1$.

Lemma 2. [25] *Let $\mathcal{H}$ be a real Hilbert space and let $F : \mathcal{H} \rightarrow \mathcal{H}$ be a quasi-nonexpansive mapping. Set $F_\alpha = (1 - \alpha)I + \alpha F$ for $\alpha \in (0, 1)$. For all $u \in \mathcal{H}$, $q \in \text{Fix}(F)$, the following assertions hold:*

- (i) $\langle u - Fu, u - q \rangle \geq \frac{1}{2} \|u - Fu\|^2$ and $\langle u - Fu, q - Fu \rangle \geq \frac{1}{2} \|u - Fu\|^2$;
- (ii) $\|F_\alpha u - q\|^2 \leq \|u - q\|^2 - \alpha(1 - \alpha) \|Fu - u\|^2$;
- (iii) $\langle u - F_\alpha u, u - q \rangle \geq \frac{\alpha}{2} \|Fu - u\|^2$.

It follows from (iii) of Lemma 2 that the following result is easily obtained.

Proposition 1. [45] *Let F be a quasi-nonexpansive mapping and $\alpha \in (0, 1)$. If*

$$F_\alpha = (1 - \alpha)I + \alpha F,$$

then

$$\|(I - F_\alpha)u\|^2 \leq 2\alpha \langle u - q, (I - F_\alpha)u \rangle$$

for all $u \in \mathcal{H}$ and $q \in \text{Fix}(F)$.

Remark 1. *Let $F_\alpha = (1 - \alpha)I + \alpha F$, where $F : \mathcal{H} \rightarrow \mathcal{H}$ is a quasi-nonexpansive mapping and $\alpha \in (0, 1)$. We have $\text{Fix}(F_\alpha) = \text{Fix}(F)$ and*

$$\|F_\alpha - u\|^2 = \alpha^2 \|Fu - u\|^2.$$

It follows from (ii) of Lemma 2 that

$$\|F_\alpha u - q\|^2 \leq \|u - q\|^2 - \frac{1 - \alpha}{\alpha} \|F_\alpha u - u\|^2,$$

which implies that F_α is firmly quasi-nonexpansive provided that $\alpha = \frac{1}{2}$. On the other hand, if $\hat{F}$ is a firmly quasi-nonexpansive mapping, we can easily obtain $\hat{F} = \frac{1}{2}I + \frac{1}{2}F$, where F is a quasi-nonexpansive mapping.

Lemma 3. [24] *Let $\{\phi_p\} \subset [0, \infty)$, $\{\theta_p\} \subset [0, \infty)$, and $\{\delta_k\} \subset [0, \infty)$ be sequences such that*

- (i) $\phi_{k+1} - \phi_p \leq \theta_p(\phi_p - \phi_{k-1}) + \delta_k$;
- (ii) $\sum_{k=1}^{\infty} \delta_k < \infty$;
- (iii) $\{\delta_k\} \subset [0, \theta]$, where $\theta \in [0, 1)$.

Then $\{\phi_p\}$ is a convergent sequence and $\sum_{k=1}^{\infty} [\phi_{k+1} - \phi_p]_+ < \infty$, where $[t]_+ = \max\{t, 0\}$ for $t \in \mathbb{R}$.

Lemma 4. [39] Let $\{u_p\}$ be a bounded sequence that satisfies:

- (i) every weak limit point of $\{u_p\}$ lies in $\mathcal{D}$, a convex and closed subset of $\mathcal{H}$;
- (ii) $\lim_{p \rightarrow \infty} \|u_p - u\|$ exists for every $u \in \mathcal{D}$.

Then $\{u_p\}$ converges weakly to a point in $\mathcal{D}$.

3 Main Results

In this paper, if no contrarily specified, we shall make use of the following assumptions:

- (A1) $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ are real Hilbert spaces, $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a bounded and linear operator with $A \neq 0$;
- (A2) $F_i : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ ($1 \leq i \leq m$) and $G_j : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ ($1 \leq j \leq n$) are β_i -demicontractive and β_j -demicontractive mappings, respectively;

$$F_{\nu_i} = (1 - \nu_i)I + \nu_i F_i, \quad G_{\nu_j} = (1 - \nu_j)I + \nu_j G_j,$$

$$\nu_i \in (0, 1 - \beta_i) \text{ and } \nu_j \in (0, 1 - \beta_j).$$

- (A3) For $1 \leq i \leq m$ and $1 \leq j \leq n$, $\{\delta_p^i\}$ and $\{\xi_p^j\}$ are sequences in $[0, 1]$ such that $\sum_{i=1}^m \delta_p^i = \sum_{j=1}^n \xi_p^j = 1$ for all $p \geq 1$, $\{a_p^i\}$ and $\{b_p^j\}$ are sequences in $(0, 1)$,

$$\mathcal{F}_p^j = (1 - a_p^i)I + a_p^i F_{\nu_i} \text{ and } \mathcal{G}_p^j = (1 - b_p^j)I + b_p^j G_{\nu_j},$$

$$\bar{a} = \sup_{p \geq 1, 1 \leq i \leq m} \{a_p^i\}, \quad \underline{a} = \inf_{p \geq 1, 1 \leq i \leq m} \{a_p^i\},$$

$$\bar{b} = \sup_{p \geq 1, 1 \leq j \leq n} \{b_p^j\}, \quad \underline{b} = \inf_{p \geq 1, 1 \leq j \leq n} \{b_p^j\},$$

- (A4) The set of solution of the (MSSCFP) of demicontractive mappings Ω is assumed to be nonempty.

Algorithm 1.

Step 1 : Choose two sequences $\{\vartheta_p\} \subset (0, \infty)$ and $\{\varphi_p\} \subset (0, \infty)$ with $\sum_{k=1}^{\infty} \varphi_p < \infty$. Select initial values $u_0, u_1, y_0 \in \mathcal{H}_1, \kappa \in [0, 1], c \in (0, 1], \eta > 0$ and set

$y_1 = y_0$.

Step 2 : Given u_{p-1}, u_p and y_p , $p \geq 1$, choose α_p such that $0 \leq \alpha_p \leq \bar{\alpha}_p$, where

$$\bar{\alpha}_p = \begin{cases} \min \left\{ \kappa, \frac{\varphi_p}{\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2} \right\}, & \text{if } u_p \neq u_{p-1} \text{ or } y_{p-1} \neq 0, \\ \kappa, & \text{otherwise.} \end{cases} \quad (11)$$

Compute

$$\begin{cases} v_p = u_p + \alpha_p(u_p - u_{p-1}), \\ z_p = v_p - \eta_p \sum_{j=1}^n \xi_p^j A^* b_p^j (I - G_{\nu_j}) A v_p, \\ y_{p+1} = (I - \sum_{i=1}^m \delta_p^i [(1 - a_p^i)I + a_p^i F_{\nu_i}]) (z_p + (1 - c)y_p), \\ u_{p+1} = z_p - c y_{p+1}, \end{cases}$$

where the step size η_p is chosen in such a way that

$$\eta_p = \begin{cases} \frac{\vartheta_p \sum_{j=1}^n \xi_p^j \|(I - \mathcal{G}_p^j) A v_p\|^2}{\sum_{j=1}^n \xi_p^j \|A^*(I - \mathcal{G}_p^j) A v_p\|^2}, & \sum_{j=1}^n \xi_p^j \|(I - \mathcal{G}_p^j) A v_p\| \neq 0 \\ \eta, & \sum_{j=1}^n \xi_p^j \|(I - \mathcal{G}_p^j) A v_p\| = 0. \end{cases} \quad (12)$$

Step 3 : Update $p := p + 1$ and go to Step 2.

Remark 2. If $\varphi_p \equiv 0$ in Algorithm 1, we can take $\alpha_p \equiv 0$. The proposed algorithm is reduced to the original algorithm without inertial extrapolation.

Remark 3. From (11), we have

$$\alpha_p(\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2) \leq \bar{\alpha}_p(\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2) \leq \varphi_p,$$

so

$$\sum_{p=1}^{\infty} \alpha_p(\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2) < \infty. \quad (13)$$

Theorem 1. Let $\{(y_p, u_p)\}$ be the sequence generated by Algorithm 1. Assume the following conditions hold

- (i) $I - F_i$ and $I - G_j$ are demiclosed at zero for $1 \leq i \leq m$ and $1 \leq j \leq n$;
- (ii) $0 < \underline{a} \leq a_p^i \leq \bar{a} \leq \frac{1}{2}$ and $0 < \underline{b} \leq b_p^i \leq \bar{b} \leq 1$

(iii) $0 < \liminf_{p \rightarrow \infty} \delta_p^i \leq \delta_p^i \leq 1$ and $0 < \liminf_{p \rightarrow \infty} \xi_p^i \leq \xi_p^i \leq 1$

(iv) $0 < \liminf_{p \rightarrow \infty} \vartheta_p \leq \limsup_{p \rightarrow \infty} \vartheta_p < \frac{1}{b}$.

Then the sequences $\{u_p\}$ and $\{y_p\}$ converge weakly to $u^* \in \Omega$ and 0, respectively.

Proof. First we show that $\lim_{p \rightarrow \infty} \|u_p - u\|$ exists for all $u \in \Omega$. Let $u \in \Omega$. It follows that $u \in \cap_{i=1}^m \text{Fix}(F_i)$ and $Au \in \cap_{j=1}^p \text{Fix}(G_j)$. From the definitions of $\mathcal{F}_p^j$ and $\mathcal{G}_p^j$, and Remark 2 we have $u \in \cap_{i=1}^m \text{Fix}(\mathcal{F}_p^i)$ and $Au \in \cap_{j=1}^p \text{Fix}(\mathcal{G}_p^j)$. From Algorithm 1 and Proposition 1,

$$\begin{aligned} \|y_{p+1}\|^2 &= \left\| \sum_{i=1}^m \delta_p^i (I - \mathcal{F}_p^i)(z_p + (1-c)y_p) \right\|^2 \\ &\leq \sum_{i=1}^m \delta_p^i \|(I - \mathcal{F}_p^i)(z_p + (1-c)y_p)\|^2 \\ &\leq \sum_{i=1}^m 2\delta_p^i a_p^i \langle (I - \mathcal{F}_p^i)(z_p + (1-c)y_p), z_p - u + (1-c)y_p \rangle \\ &\leq 2\bar{a} \langle y_{p+1}, z_p - u + (1-c)y_p \rangle \end{aligned}$$

and

$$\|u_{p+1} - u\|^2 = \|z_p - u\|^2 - 2c \langle z_p - u, y_{p+1} \rangle + c^2 \|y_{p+1}\|^2. \quad (14)$$

One can see that

$$\begin{aligned} \|u_{p+1} - u\|^2 + c \|y_{p+1}\|^2 &= \|z_p - u\|^2 - 2c \langle z_p - u, y_{p+1} \rangle + \frac{c}{a} \|y_{p+1}\|^2 - c \left(\frac{1}{a} - c - 1 \right) \|y_{p+1}\|^2 \\ &\leq \|z_p - u\|^2 + 2c(1-c) \langle y_{p+1}, y_p \rangle - c \left(\frac{1}{a} - c - 1 \right) \|y_{p+1}\|^2 \\ &\leq \|z_p - u\|^2 + c(1-c)(\|y_p\|^2 - \|y_{p+1} - y_p\|^2) - c \left(\frac{1}{a} - 2 \right) \|y_{p+1}\|^2. \end{aligned} \quad (15)$$

Since

$$\langle v_p - u, A^*(I - \mathcal{G}_p^j)Av_p \rangle \geq \frac{1}{2b} \|(I - \mathcal{G}_p^j)Av_p\|^2,$$

it follows that

$$\begin{aligned}
\|z_p - u\|^2 &= \|v_p - u\|^2 - 2\eta_p \left\langle v_p - u, \sum_{j=1}^n \xi_p^j A^* (I - \mathcal{G}_p^j) A v_p \right\rangle + \eta_p^2 \left\| \sum_{j=1}^n \xi_p^j A^* (I - \mathcal{G}_p^j) A v_p \right\|^2 \\
&\leq \|v_p - u\|^2 - 2\eta_p \sum_{j=1}^n \left\langle v_p - u, \xi_p^j A^* (I - \mathcal{G}_p^j) A v_p \right\rangle + \eta_p^2 \left\| \sum_{j=1}^n \xi_p^j A^* (I - \mathcal{G}_p^j) A v_p \right\|^2 \\
&\leq \|v_p - u\|^2 - \eta_p \left(\frac{1}{b} \sum_{j=1}^n \xi_p^j \left\| (1 - \mathcal{G}_p^j) A v_p \right\|^2 - \eta_p \sum_{j=1}^n \xi_p^j \left\| A^* (I - \mathcal{G}_p^j) A v_p \right\|^2 \right).
\end{aligned}$$

It can be seen that

$$\|v_p - u\|^2 = (1 + \alpha_p) \|u_p - u\|^2 - \alpha_p \|u_{p-1} - u\|^2 + \alpha_p (1 + \alpha_p) \|u_p - u_{p-1}\|^2, \quad (16)$$

and

$$\begin{aligned}
\|z_p - u\|^2 &\leq (1 + \alpha_p) \|u_p - u\|^2 - \alpha_p \|u_{p-1} - u\|^2 + \alpha_p (1 + \alpha_p) \|u_p - u_{p-1}\|^2 \\
&\quad - \eta_p \left(\frac{1}{b} \sum_{j=1}^n \xi_p^j \left\| (1 - \mathcal{G}_p^j) A v_p \right\|^2 - \eta_p \sum_{j=1}^n \xi_p^j \left\| A^* (I - \mathcal{G}_p^j) A v_p \right\|^2 \right).
\end{aligned}$$

The conditions in $\bar{\alpha}$ and (15) yield

$$\begin{aligned}
&\|u_{p+1} - u\|^2 + c \|y_{p+1}\|^2 \\
&\leq \|u_p - u\|^2 + c \|y_p\|^2 + \alpha_p (\|u_p - u\|^2 - \|u_{p-1} - u\|^2) + \alpha_p (1 + \alpha_p) \|u_p - u_{p-1}\|^2 \\
&\quad - c^2 \|y_p\|^2 - \eta_p \left(\frac{1}{b} \sum_{j=1}^n \xi_p^j \left\| (1 - \mathcal{G}_p^j) A v_p \right\|^2 - \eta_p \sum_{j=1}^n \xi_p^j \left\| A^* (I - \mathcal{G}_p^j) A v_p \right\|^2 \right).
\end{aligned}$$

Letting $\omega_p = \|u_p - u\|^2 + c \|y_p\|^2$ gives

$$\begin{aligned}
\omega_{p+1} &\leq \omega_p + \alpha_p (\|u_p - u\|^2 - \|u_{p-1} - u\|^2) + \alpha_p (1 + \alpha_p) \|u_p - u_{p-1}\|^2 \\
&\quad - c^2 \|y_p\|^2 - \eta_p \left(\frac{1}{b} \sum_{j=1}^n \xi_p^j \left\| (1 - \mathcal{G}_p^j) A v_p \right\|^2 - \eta_p \sum_{j=1}^n \xi_p^j \left\| A^* (I - \mathcal{G}_p^j) A v_p \right\|^2 \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\omega_{p+1} - \omega_p &\leq \alpha_p (\omega_p - \omega_{p-1}) + 2\alpha_p (\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2) \\
&\quad - c^2 \|y_p\|^2 - \eta_p \left(\frac{1}{b} \sum_{j=1}^n \xi_p^j \left\| (1 - \mathcal{G}_p^j) A v_p \right\|^2 - \eta_p \sum_{j=1}^n \xi_p^j \left\| A^* (I - \mathcal{G}_p^j) A v_p \right\|^2 \right).
\end{aligned} \tag{17}$$

If $\sum_{j=1}^n \xi_p^j \|(I - \mathcal{G}_p^j)Av_p\| = 0$, then

$$\begin{aligned} \omega_{p+1} - \omega_p &\leq \alpha_p(\omega_p - \omega_{p-1}) + 2\alpha_p(\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2) - c^2\|y_p\|^2 \\ &\leq \alpha_p(\omega_p - \omega_{p-1}) + 2\alpha_p(\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2). \end{aligned} \quad (18)$$

Otherwise, it follows from (12) and (17) that

$$\begin{aligned} \omega_{p+1} - \omega_p &\leq \alpha_p(\omega_p - \omega_{p-1}) + 2\alpha_p(\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2) \\ &\quad - c^2\|y_p\|^2 - \vartheta_p \left(\frac{1}{b} - \vartheta_p \right) \frac{\left(\sum_{j=1}^n \xi_p^j \|(I - \mathcal{G}_p^j)Av_p\|^2 \right)^2}{\sum_{j=1}^n \xi_p^j \|A^*(I - \mathcal{G}_p^j)Av_p\|^2}. \end{aligned} \quad (19)$$

The conditions in ϑ_p and c imply that

$$\omega_{p+1} - \omega_p \leq \alpha_p(\omega_p - \omega_{p-1}) + 2\alpha_p(\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2). \quad (20)$$

Let $\phi_p = \omega_p$, $\theta_p = \alpha_p$ and $\delta_p = 2\alpha_p(\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2)$. From (13),

$$\sum_{p=1}^{\infty} 2\alpha_p(\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2) < \infty.$$

Applying Lemma 3 to (20), we conclude that $\lim_{p \rightarrow \infty} \omega_p$ exists. This implies that $\{\omega_p\}$ is bounded, as are $\{u_p\}$, $\{v_p\}$ and $\{z_p\}$. From (18)-(19), it follows that

$$c^2\|y_p\|^2 \leq \alpha_p(\omega_p - \omega_{p-1}) - (\omega_{p+1} - \omega_p) + 2\alpha_p(\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2).$$

This indicates that $\lim_{p \rightarrow \infty} \|y_p\| = 0$. Therefore,

$$\lim_{p \rightarrow \infty} \|u_p - u\|^2 = \lim_{p \rightarrow \infty} (\omega_p - c\|y_p\|^2) = \lim_{p \rightarrow \infty} \omega_p \quad \text{exists.}$$

Secondly, we show that $\lim_{p \rightarrow \infty} \|(I - \mathcal{G}_p^j)Av_p\| = 0$ for $1 \leq j \leq n$.

If $\sum_{j=1}^n \xi_p^j \|(I - \mathcal{G}_p^j)Av_p\| = 0$, then $v_p - z_p = \eta_p \sum_{j=1}^n \xi_p^j A^*(I - \mathcal{G}_p^j)Av_p = 0$. Otherwise, (19) yields

$$\begin{aligned} &\vartheta_p \left(\frac{1}{b} \right) \frac{\left(\sum_{j=1}^n \xi_p^j \|(I - \mathcal{G}_p^j)Av_p\|^2 \right)^2}{\sum_{j=1}^n \xi_p^j \|A^*(I - \mathcal{G}_p^j)Av_p\|^2} \\ &\leq \alpha_p(\omega_p - \omega_{p-1}) - (\omega_{p+1} - \omega_p) + 2\alpha_p(\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2) - c^2\|y_p\|^2, \end{aligned}$$

and hence,

$$\lim_{p \rightarrow \infty} \frac{\left(\sum_{j=1}^n \xi_p^j \|(I - \mathcal{G}_p^j)Av_p\|^2 \right)^2}{\sum_{j=1}^n \xi_p^j \|A^*(I - \mathcal{G}_p^j)Av_p\|^2} = 0$$

Moreover,

$$\lim_{p \rightarrow \infty} \frac{\sum_{j=1}^n \xi_p^j \|(I - \mathcal{G}_p^j)Av_p\|^2}{\sqrt{\sum_{j=1}^n \xi_p^j \|A^*(I - \mathcal{G}_p^j)Av_p\|^2}} = 0 \quad (21)$$

As A is a bounded linear operator, the following inequality holds.

$$\|A^*(I - \mathcal{G}_p^j)Av_p\| \leq \|A\| \cdot \|(I - \mathcal{G}_p^j)Av_p\|.$$

Thus,

$$\begin{aligned} \frac{1}{\|A\|} \sqrt{\sum_{j=1}^n \xi_p^j \|(I - \mathcal{G}_p^j)Av_p\|^2} &= \frac{\sum_{j=1}^n \xi_p^j \|(I - \mathcal{G}_p^j)Av_p\|^2}{\|A\| \sqrt{\sum_{j=1}^n \xi_p^j \|(I - \mathcal{G}_p^j)Av_p\|^2}} \\ &\leq \frac{\sum_{j=1}^n \xi_p^j \|(I - \mathcal{G}_p^j)Av_p\|^2}{\sqrt{\sum_{j=1}^n \xi_p^j \|A^*(I - \mathcal{G}_p^j)Av_p\|^2}} \end{aligned} \quad (22)$$

in view of the fact that $A \neq 0$. From (21) and (22), it follows that

$$\lim_{p \rightarrow \infty} \sum_{j=1}^n \xi_p^j \|(I - \mathcal{G}_p^j)Av_p\|^2 = 0.$$

Taking into account the condition

$$0 < \lim_{p \rightarrow \infty} \xi_p^j \leq \xi_p^j \leq 1$$

this implies that

$$\lim_{p \rightarrow \infty} \|(I - \mathcal{G}_p^j)Av_p\| = 0.$$

One can see that

$$\left\| \sum_{j=1}^n \xi_p^j A^*(I - \mathcal{G}_p^j)Av_p \right\| \leq \sqrt{\sum_{j=1}^n \xi_p^j \|A^*(I - \mathcal{G}_p^j)Av_p\|^2}$$

which yields

$$\frac{\left\| \sum_{j=1}^n \xi_p^j A^*(I - \mathcal{G}_p^j)Av_p \right\|}{\sqrt{\sum_{j=1}^n \xi_p^j \|A^*(I - \mathcal{G}_p^j)Av_p\|^2}} \leq 1. \quad (23)$$

In fact,

$$\begin{aligned}
\|v_p - z_p\| &= \frac{\vartheta_p \sum_{j=1}^n \xi_p^j \|(I - \mathcal{G}_p^j)Av_p\|^2}{\sum_{j=1}^n \xi_p^j \|A^*(I - \mathcal{G}_p^j)Av_p\|^2} \left\| \sum_{j=1}^n \xi_p^j A^*(I - \mathcal{G}_p^j)Av_p \right\| \\
&= \frac{\vartheta_p \sum_{j=1}^n \xi_p^j \|(I - \mathcal{G}_p^j)Av_p\|^2}{\sqrt{\sum_{j=1}^n \xi_p^j \|A^*(I - \mathcal{G}_p^j)Av_p\|^2}} \frac{\left\| \sum_{j=1}^n \xi_p^j A^*(I - \mathcal{G}_p^j)Av_p \right\|}{\sqrt{\sum_{j=1}^n \xi_p^j \|A^*(I - \mathcal{G}_p^j)Av_p\|^2}} \\
&\leq \frac{\vartheta_p \sum_{j=1}^n \xi_p^j \|(I - \mathcal{G}_p^j)Av_p\|^2}{\sqrt{\sum_{j=1}^n \xi_p^j \|A^*(I - \mathcal{G}_p^j)Av_p\|^2}}
\end{aligned}$$

It follows from (21) and (23) that $\lim_{p \rightarrow \infty} \|v_p - z_p\| = 0$. Now we can conclude that

$$\lim_{p \rightarrow \infty} \|(I - \mathcal{G}_p^j)Av_p\| = \lim_{p \rightarrow \infty} \|v_p - z_p\| = 0 \quad (24)$$

for $1 \leq j \leq n$. Since $u_{p+1} = z_p - cy_{p+1}$, we obtain

$$\lim_{p \rightarrow \infty} \|u_{p+1} - z_p\| = \lim_{p \rightarrow \infty} c\|y_{p+1}\| = 0. \quad (25)$$

It follows from (13) that $\lim_{p \rightarrow \infty} a_p(\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2) = 0$. Thus, $\lim_{p \rightarrow \infty} a_p\|u_p - u_{p-1}\|^2 = 0$. Since $a_p^2\|u_p - u_{p-1}\|^2 \leq \alpha_p\|u_p - u_{p-1}\|^2$, we have $\lim_{p \rightarrow \infty} a_p\|u_p - u_{p-1}\| = 0$. Thus,

$$\lim_{p \rightarrow \infty} \|v_p - u_p\| = \lim_{p \rightarrow \infty} a_p\|u_p - u_{p-1}\| = 0. \quad (26)$$

From (24) and (26), it follows that $\lim_{p \rightarrow \infty} \|u_p - z_p\| = 0$. Using (25), we obtain $\lim_{p \rightarrow \infty} \|u_{p+1} - u_p\| = 0$. We conclude that $\{u_p\}$ is asymptotically regular. Hence, $\{Au_p\}$ is also asymptotically regular, i.e., $\lim_{p \rightarrow \infty} \|Au_{p+1} - Au_p\| = 0$.

Next we show that

$$\lim_{p \rightarrow \infty} \|(I - \mathcal{F}_p^j)(z_p + (1 - c)y_p)\| = 0$$

for $1 \leq i \leq m$. It can be seen that

$$\|u_{p+1} - u\|^2 = \|z_p - u\|^2 - 2c\langle y_{p+1}, z_p - u \rangle + c^2\|y_{p+1}\|^2,$$

which yields

$$\|z_p - u\|^2 = \|u_{p+1} - u\|^2 + 2c\langle y_{p+1}, z_p - u \rangle - c^2\|y_{p+1}\|^2.$$

The boundedness of $\{z_p\}$ implies that $\lim_{p \rightarrow \infty} \|z_p - u\| = \lim_{p \rightarrow \infty} \|u_{p+1} - u\|$ exists. Similarly, since

$$\|z_p - cy_p - u\|^2 = \|z_p - u\|^2 - 2c\langle y_p, z_p - u \rangle + c^2\|y_p\|^2$$

and

$$\|z_p + (1 - c)y_p - u\|^2 = \|z_p - u\|^2 + 2(1 - c)\langle y_p, z_p - u \rangle + (1 - c)^2\|y_p\|^2,$$

it follows that

$$\lim_{p \rightarrow \infty} \|z_p - cy_p - u\| = \lim_{p \rightarrow \infty} \|z_p + (1 - c)y_p - u\| = \lim_{p \rightarrow \infty} \|z_p - u\| = \lim_{p \rightarrow \infty} \|u_{p+1} - u\|. \quad (27)$$

From Algorithm 1,

$$\begin{aligned} u_{p+1} &= (1 - c)z_p - c(1 - c)y_p + c \sum_{i=1}^m \delta_p^i \mathcal{F}_p^j(z_p + (1 - c)y_p) \\ &= (1 - c)(z_p - cy_p) + c \sum_{i=1}^m \delta_p^i \mathcal{F}_p^j(z_p + (1 - c)y_p) \end{aligned} \quad (28)$$

Using (10), (28) and Remark 2, we obtain

$$\begin{aligned} \|u_{p+1} - u\|^2 &\leq (1 - c)\|z_p - cy_p - u\|^2 + c \sum_{i=1}^m \delta_p^i \|\mathcal{F}_p^j(z_p + (1 - c)y_p) - u\|^2 \\ &\leq (1 - c)\|z_p - cy_p - u\|^2 + c\|z_p + (1 - c)y_p - u\|^2 \\ &\quad - c \frac{1 - \bar{\alpha}}{\bar{\alpha}} \sum_{i=1}^m \delta_p^i \|(I - \mathcal{F}_p^j)(z_p + (1 - c)y_p)\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} &c \frac{1 - \bar{\alpha}}{\bar{\alpha}} \sum_{i=1}^m \delta_p^i \|(I - \mathcal{F}_p^j)(z_p + (1 - c)y_p)\|^2 \\ &\leq (1 - c)\|z_p - cy_p - u\|^2 + c\|z_p + (1 - c)y_p - u\|^2 - \|u_{p+1} - u\|^2. \end{aligned}$$

Applying (27) and the conditions in $\{\delta_p^i\}, 1 \leq i \leq m$, we obtain

$$\lim_{p \rightarrow \infty} \|(I - \mathcal{F}_p^j)(z_p + (1 - c)y_p)\| = 0, \text{ for } 1 \leq i \leq m.$$

Finally, we show that $y_w(u_p) \subset \Omega$. Let $\hat{u} \in y_w(u_p)$, i.e., there exists a subsequence $\{u_{p_l}\}$ of $\{u_p\}$ such that $u_{p_l} \rightharpoonup \hat{u}$ as $l \rightarrow \infty$. At the same time, it follows from (26) that $y_{p_l} \rightharpoonup \hat{u}, Ay_{p_l} \rightharpoonup A\hat{u}$ as $l \rightarrow \infty$. It can be seen that $z_{p_l} + (1 - c)y_{p_l} \rightharpoonup \hat{u}$ as $l \rightarrow \infty$. Since $I - F_{\nu_i}$ and $I - G_{\nu_j}$ are demiclosed at 0, from

$$I - \mathcal{F}_p^i = (1 - \alpha_p^i)(I - F_{\nu_i}) \text{ and } I - \mathcal{G}_p^j = (1 - \beta_p^j)(I - G_{\nu_j}),$$

we see that $I - \mathcal{F}_p^i$ and $I - \mathcal{G}_p^j$ are demiclosed at 0. Thus, it follows from (24) and Remark 2 that

$$\hat{u} \in \bigcap_{i=1}^m \text{Fix}(\mathcal{F}_p^i) = \bigcap_{i=1}^m \text{Fix}(F_{\nu_i}) = \bigcap_{i=1}^m \text{Fix}(F_i)$$

and

$$A\hat{u} \in \bigcap_{j=1}^n \text{Fix}(\mathcal{G}_p^j) = \bigcap_{j=1}^n \text{Fix}(G_{\nu_j}) = \bigcap_{j=1}^n \text{Fix}(G_j),$$

which yield $\hat{u} \in \Omega$, so $\omega_w(u_p) \subset \Omega$. By Lemma 4, $u_p \rightharpoonup u^*$ as $p \rightarrow \infty$, where u^* is a solution of the MSSCFP. Since $y_p \rightarrow 0$, it follows that $(y_p, u_p) \rightharpoonup (0, u^*)$ as $p \rightarrow \infty$. $\square$

Corollary 1. *Let $0 < c \leq 1$ and $0 \leq \liminf_{p \rightarrow \infty} \vartheta_p \leq \limsup_{p \rightarrow \infty} \vartheta_p < 2$. Choose $u_0, u_1, y_0 \in \mathcal{H}_1$, set $y_1 = y_0$, and let $\{(y_p, u_p)\}$ be the sequences generated by*

$$\begin{cases} v_p = u_p + \alpha_p(u_p - u_{p-1}), \\ z_p = v_p - \eta_p \sum_{j=1}^n \xi_p^j A^*(I - P_{Q_j})Av_p, \\ y_{p+1} = (I - \sum_{i=1}^m \delta_p^i P_{C_i})(z_p + (1-c)y_p), \\ u_{p+1} = z_p - cy_{p+1}, \end{cases} \quad (29)$$

where $0 \leq \alpha_p \leq \bar{a}_k$, $\bar{a}_k$ is chosen by (11) with $\kappa \in [0, 1)$ and $\sum_{k=1}^{\infty} \varphi_p < \infty$, and η_p is chosen by the following way that

$$\eta_p = \begin{cases} \frac{\vartheta_p \sum_{j=1}^n \xi_p^j \|(I - P_{Q_j})Av_p\|^2}{\sum_{j=1}^n \xi_p^j \|A^*(I - P_{Q_j})Av_p\|^2}, & \sum_{j=1}^n \xi_p^j \|(I - P_{Q_j})Av_p\| \neq 0 \\ \eta, & \sum_{j=1}^n \xi_p^j \|(I - P_{Q_j})Av_p\| = 0. \end{cases} \quad (30)$$

with $0 < \eta < 1$. Then $\{u_p\}$ converges weakly to a point u^* , where u^* is a solution of the MSFP, and $\{(y_p, u_p)\}$ converges weakly to $(0, u^*)$.

In addition to assumptions (A1)-(A3), we consider the following assumption:

(A4) for all $p \geq 1$, $[p]_1 = p(\bmod m) + 1$ and $[p]_2 = p(\bmod n) + 1$. Let $\{\theta_p\} \subset (0, 1)$, $\{\mu_p\} \subset (0, 1)$,

$$\hat{F}_{\nu_p} = (1 - \theta_p)I + \theta_p F_{\nu_{[p]_1}}, \quad \hat{G}_{\nu_p} = (1 - \mu_p)I + \mu_p G_{\nu_{[p]_2}},$$

$$\bar{\theta} = \sup_{p \geq 1} \{\theta_p\}, \quad \underline{\theta} = \inf_{p \geq 1} \{\theta_p\}, \quad \bar{\mu} = \sup_{p \geq 1} \{\mu_p\}, \quad \text{and} \quad \underline{\mu} = \inf_{p \geq 1} \{\mu_p\}.$$

Algorithm 2.

Step 1 : Choose two sequences $\{\vartheta_p\} \subset [0, \infty)$ and $\{\varphi_p\} \subset [0, \infty)$ with $\sum_{k=1}^{\infty} \varphi_p < \infty$. Select initials $u_0, u_1, y_0 \in \mathcal{H}_1, \kappa \in [0, 1], c \in (0, 1], \eta > 0$ and set $y_1 = y_0$.

Step 2 : Given the iterates u_{p-1}, u_p and y_p , for $p \geq 1$, choose a_k such that $0 \leq \alpha_p \leq \bar{a}_k$, where

$$\begin{cases} \min \left\{ \kappa, \frac{\varphi_p}{\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2} \right\}, & \text{if } u_p \neq u_{p-1} \text{ or } y_{p-1} \neq 0, \\ \kappa, & \text{otherwise.} \end{cases}$$

Compute

$$\begin{cases} v_p = u_p + \alpha_p(u_p - u_{p-1}), \\ z_p = v_p - \eta_p A^*(I - \hat{G}_{\nu_p})Av_p, \\ y_{p+1} = (I - \sum_{i=1}^m \delta_p^i [(1 - a_p^i)I + a_p^i F_{\nu_i}]) (z_p + (1 - c)y_p), \\ u_{p+1} = z_p - cy_{p+1}, \end{cases}$$

where the step-size η_p is chosen in such a way that

$$\eta_p \begin{cases} \frac{\vartheta_p \|(I - \mathcal{G}_p^j)Av_p\|^2}{\|A^*(I - \mathcal{G}_p^j)Av_p\|^2}, & (I - \mathcal{G}_p^j)Av_p \neq 0, \\ \eta, & (I - \mathcal{G}_p^j)Av_p = 0. \end{cases} \quad (31)$$

Step 3 : Update $p := p + 1$ and go to Step 2.

Theorem 2. Let $\{(y_p, u_p)\}$ be the sequence generated by Algorithm 2. Assume the following conditions hold

- (i) $I - F_i$ and $I - G_j$ are demiclosed at zero for $1 \leq i \leq m$ and $1 \leq j \leq n$;
- (ii) $0 < \underline{\alpha} \leq \alpha_p^i \leq \bar{\alpha} \leq \frac{1}{2}$ and $0 < \underline{\mu} \leq \mu_p^i \leq \bar{\mu} \leq 1$
- (iii) $0 < \liminf_{p \rightarrow \infty} \vartheta_p \leq \limsup_{p \rightarrow \infty} \vartheta_p < \frac{1}{\bar{\mu}}$.

Then the sequences $\{u_p\}$ and $\{y_p\}$ converge weakly to $u^* \in \Omega$ and 0, respectively.

Proof. Firstly, we prove that $\lim_{p \rightarrow \infty} \|u_p - u\|$ exists for any $u \in \Omega$. Let $u \in \Omega$, we have $u \in \cap_{i=1}^m \text{Fix}(F_i)$ and $Au \in \cap_{j=1}^n \text{Fix}(G_j)$, and Remark 1 that $u \in$

$\cap_{i=1}^m \text{Fix}(\mathcal{F}_p^j)$ and $Au \in \cap_{j=1}^n \text{Fix}(\mathcal{G}_p^j)$. Similar to the proof of Theorem 1, we obtain (15) and (16). Since

$$\langle v_p - u, A^*(I - \hat{G}_{\nu_p})Av_p \rangle \geq \frac{1}{2\bar{\mu}} \|(I - \hat{G}_{\nu_p})Av_p\|^2,$$

it follows that

$$\begin{aligned} & \|z_p - u\|^2 \|v_p - u\|^2 - 2\eta_p \langle v_p - u, A^*((I - \hat{G}_{\nu_p})Av_p) \rangle + \eta_p^2 \|A^*(I - \hat{G}_{\nu_p})Av_p\|^2 \\ & \leq \|v_p - u\|^2 - \eta_p \left(\frac{1}{\bar{\mu}} \|(I - \hat{G}_{\nu_p})Av_p\|^2 - \eta_p \|A^*(I - \hat{G}_{\nu_p})Av_p\|^2 \right). \end{aligned}$$

Adding (16) into this relation yields

$$\begin{aligned} \|z_p - u\|^2 & \leq (1 + \alpha_p) \|u_p - u\|^2 - \alpha_p \|u_p - u_{p-1}\|^2 + \alpha_p (1 - \alpha_p) \|u_p - u_{p-1}\|^2 \\ & \quad - \eta_p \left(\frac{1}{\bar{\mu}} \|(I - \hat{G}_{\nu_p})Av_p\|^2 - \eta_p \|A^*(I - \hat{G}_{\nu_p})Av_p\|^2 \right) \end{aligned}$$

Similar to the first step in Theorem 1, we have

$$\begin{aligned} \omega_{p+1} - \omega_p & \leq \alpha_p (\omega_p - \omega_{p-1} + 2\alpha_p (\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2) \\ & \quad - c^2 \|y_p\|^2 - \eta_p \left(\frac{1}{\bar{\mu}} \|(I - \hat{G}_{\nu_p})Av_p\|^2 - \eta_p \|A^*(I - \hat{G}_{\nu_p})Av_p\|^2 \right) \end{aligned} \quad (32)$$

where $\omega_p = \|u_p - u\|^2 + c\|y_p\|^2$. For the case $(I - \hat{G}_{\nu_p})Av_p = 0$,

$$\begin{aligned} \omega_{p+1} - \omega_p & \leq \alpha_p (\omega_p - \omega_{p-1} + 2\alpha_p (\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2) - c^2 \|y_p\|^2) \\ & \leq \alpha_p (\omega_p - \omega_{p-1} + 2\alpha_p (\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2)) \end{aligned}$$

Otherwise, we deduce from (31) and (32) that

$$\begin{aligned} \omega_{p+1} - \omega_p & \leq \alpha_p (\omega_p - \omega_{p-1} + 2\alpha_p (\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2) \\ & \quad - c^2 \|y_p\|^2 - \vartheta_p \left(\frac{1}{\bar{\mu}} - \vartheta_p \right) \frac{\|(I - \hat{G}_{\nu_p})Av_p\|^4}{\|A^*(I - \hat{G}_{\nu_p})Av_p\|^2} \end{aligned} \quad (33)$$

By Theorem 1, we also have $\lim_{p \rightarrow \infty} \|y_p\| = 0$ and $\lim_{p \rightarrow \infty} \|u_p - u\|^2 = \lim_{p \rightarrow \infty} \omega_p$ exists. Thus, $\{u_p\}$ is bounded and $\{z_p\}$ is bounded.

Secondly, we prove that $\lim_{p \rightarrow \infty} \|(I - \hat{G}_{\nu_p})Av_p\| = 0$ and $\lim_{p \rightarrow \infty} \|(I - \mathcal{F}_p^j)(z_p + (1 - c)y_p)\| = 0$ for $1 \leq i \leq m$. If $(I - \hat{G}_{\nu_p})Av_p = 0$, then $v_p - z_p = \eta_p A^*(I - \hat{G}_{\nu_p})Av_p = 0$. Otherwise, it follows from (33) that

$$\begin{aligned} \vartheta_p \left(\frac{1}{\bar{\mu}} - \vartheta_p \right) \frac{\|(I - \hat{G}_{\nu_p})Av_p\|^4}{\|A^*(I - \hat{G}_{\nu_p})Av_p\|^2} & \leq \alpha_p (\omega_p - \omega_{p-1}) - (\omega_{p+1} - \omega_p) \\ & \quad + 2\alpha_p (\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2) - c^2 \|y_p\|^2, \end{aligned}$$

which implies that

$$\frac{\|(I - \hat{G}_{\nu_p})Av_p\|^2}{\|A^*(I - \hat{G}_{\nu_p})Av_p\|} = 0. \quad (34)$$

Since $A \neq 0$ and

$$\frac{1}{\|A\|} \|(I - \hat{G}_{\nu_p})Av_p\| = \frac{\|(I - \hat{G}_{\nu_p})Av_p\|^2}{\|A\| \|(I - \hat{G}_{\nu_p})Av_p\|} \leq \frac{\|(I - \hat{G}_{\nu_p})Av_p\|^2}{\|A^*(I - \hat{G}_{\nu_p})Av_p\|},$$

we have $\lim_{p \rightarrow \infty} \|(I - \hat{G}_{\nu_p})Av_p\| = 0$. It follows from (34) that

$$\lim_{p \rightarrow \infty} \|v_p - z_p\| = \lim_{p \rightarrow \infty} \|\eta_p A^*(I - \hat{G}_{\nu_p})Av_p\| = \lim_{p \rightarrow \infty} \vartheta_p \frac{\|(I - \hat{G}_{\nu_p})Av_p\|^2}{\|A^*(I - \hat{G}_{\nu_p})Av_p\|} = 0.$$

From the two cases discussed above, we obtain

$$\lim_{p \rightarrow \infty} \|(I - \hat{G}_{\nu_p})Av_p\| = \lim_{p \rightarrow \infty} \|v_p - z_p\| = 0.$$

Theorem 1 implies that

$$\lim_{p \rightarrow \infty} \|(I - \mathcal{F}_p^i)(z_p + (1 - c)y_p)\| = 0$$

for all $1 \leq i \leq m$, and

$$\lim_{p \rightarrow \infty} \|v_p - u_p\| = \lim_{p \rightarrow \infty} \|u_p - u_p\| = \lim_{p \rightarrow \infty} \|u_{p+1} - u_p\| = 0.$$

Finally, we prove that $\omega_w(u_p) \subset \Omega$. Let $\hat{u} \in \omega_w(u_p)$, i.e., there exists a subsequence $\{u_{p_l}\}$ of $\{u_p\}$ such that $u_{p_l} \rightharpoonup \hat{u}$ as $l \rightarrow \infty$. Similar to Theorem 1 we have $y_{p_l} \rightharpoonup \hat{u}$, $Ay_{p_l} \rightharpoonup A\hat{u}$ and $v_{p_l} + (1 - c)\omega_{p_l} \rightharpoonup \hat{u}$ as $l \rightarrow \infty$. Since $\{Au_p\}$ is asymptotically regular, for any $1 \leq j \leq n$, we can choose a subsequence $\{p_{j_k}\} \subset \{p\}$ such that $Ay_{p_{j_k}} \rightharpoonup A\hat{u}$ as $n \rightarrow \infty$ and $[p_{j_k}]_2 = j$ for all n . It turns out that

$$\begin{aligned} \lim_{p \rightarrow \infty} \|(I - G_{\nu_j})Ay_{p_{j_k}}\| &= \lim_{p \rightarrow \infty} \|(I - G_{\nu_{[p_{j_k}]_2}})Ay_{p_{j_k}}\| \\ &= \lim_{p \rightarrow \infty} \frac{1}{\mu_{p_{j_k}}} \|(I - \hat{G}_{\nu_{p_{j_k}}})Ay_{p_{j_k}}\| = 0 \end{aligned} \quad (35)$$

By the same proof in Theorem 1, we have $\hat{u} \in \cap_{i=1}^m \text{Fix}(F_{\nu_i}) = \cap_{i=1}^m \text{Fix}(F_i)$. Since $I - G_{\nu_j}$ ($1 \leq j \leq n$) are demiclosed at zero, it follows from (35) that $A\hat{u} \in \cap_{j=1}^n \text{Fix}(G_{\nu_j}) = \cap_{j=1}^n \text{Fix}(G_j)$, so $\hat{u} \in \Omega$. Using Lemma 4, we have $u_p \rightharpoonup u^*$ as $p \rightarrow \infty$, where u^* is a solution to the MSSCFP. Thus, $y_p \rightharpoonup 0$ and $u_p \rightharpoonup u^*$ as $p \rightarrow \infty$. $\square$

Algorithm 3.

Step 1: Choose two sequences $\{\vartheta_p\} \subset [0, \infty)$ and $\{\varphi_p\} \subset [0, \infty)$ with $\sum_{k=1}^{\infty} \varphi_p < \infty$. Select initials $u_0, u_1, y_0 \in \mathcal{H}_1, \kappa \in [0, 1], c \in (0, 1], \eta > 0$ and set $y_1 = y_0$.

Step 2: Given the iterates u_{p-1}, u_p and y_p , for $p \geq 1$, choose α_p such that $0 \leq \alpha_p \leq \bar{\alpha}_p$, where

$$\bar{\alpha}_p = \begin{cases} \min \left\{ \kappa, \frac{\varphi_p}{\|u_p - u_{p-1}\|^2 + \|y_{p-1}\|^2} \right\}, & \text{if } u_p \neq u_{p-1} \text{ or } y_{p-1} \neq 0, \\ \kappa, & \text{otherwise.} \end{cases}$$

Compute

$$\begin{cases} v_p = u_p + \alpha_p(u_p - u_{p-1}), \\ z_p = v_p - \eta_p \sum_{j=1}^n \xi_p^j A^{*j} b_p^j (I - G_{\nu_j}) A v_p, \\ y_{p+1} = (I - \hat{F}_{\nu_p})(z_p + (1 - c)y_p), \\ u_{p+1} = z_p - c y_{p+1}, \end{cases}$$

where the step-size η_p is chosen as in (12).

Step 3: Update $p := p + 1$ and go to Step 2.

Theorem 3. Let $\{(y_p, u_p)\}$ be the sequence generated by Algorithm 3. Assume the following conditions hold

- (i) $I - F_i$ and $I - G_j$ are demiclosed at zero for $1 \leq i \leq m$ and $1 \leq j \leq n$;
- (ii) $0 < \underline{\theta} \leq \theta_p \leq \bar{\theta} \leq \frac{1}{2}$ and $0 < \underline{\beta} \leq \beta_k \leq \bar{b} \leq 1$
- (iii) $0 < \liminf_{p \rightarrow \infty} \xi_p^i \leq \xi_p^i \leq 1$
- (iv) $0 < \liminf_{p \rightarrow \infty} \vartheta_p \leq \limsup_{p \rightarrow \infty} \vartheta_p < \frac{1}{\bar{b}}$.

Then the sequences $\{u_p\}$ and $\{y_p\}$ converge weakly to $u^* \in \Omega$ and 0, respectively.

Proof. Fix $u \in \Omega$. We have $u \in \cap_{i=1}^m \text{Fix}(F_i)$ and $Au \in \cap_{j=1}^n \text{Fix}(G_j)$. For $1 \leq i \leq m$ and $1 \leq j \leq n$, it follows from Remark 1 that $u \in \cap_{p=1}^{\infty} \text{Fix}(\hat{F}_{\nu_p})$ and $Au \in \cap_{p=1}^{\infty} \text{Fix}(\mathcal{G}_p^j)$. Hence,

$$\begin{aligned} \|y_{p+1}\|^2 &= \|(I - \hat{F}_{\nu_p})(z_p + (1 - c)y_p) - (I - \hat{F}_{\nu_p})u\|^2 \\ &\leq 2\bar{\theta}\langle y_{p+1}, z_p - u + (1 - c)y_p \rangle. \end{aligned}$$

The rest of the proof follows from Theorem 1 or Theorem 2. $\square$

Remark 4. Note that in Theorems 1, 2, and 3 the sequences $\{u_p\}$ and $\{Au_p\}$ are shown to be asymptotically regular.

Corollary 2. Let $0 < c \leq 1$ and

$$0 \leq \liminf_{p \rightarrow \infty} \vartheta_p \leq \liminf_{p \rightarrow \infty} \vartheta_p \leq \limsup_{p \rightarrow \infty} \vartheta_p < 2.$$

Choose $u_0, u_1, y_0 \in \mathcal{H}_1$, set $y_1 = y_0$, and let $\{(y_p, u_p)\}$ be the sequences generated by

$$\begin{cases} v_p = u_p + \alpha_p(u_p - u_{p-1}), \\ z_p = v_p - \eta_p A^*(I - P_{Q_{[p]_2}})Av_p, \\ y_{p+1} = (I - \sum_{i=1}^m \delta_p^i P_{C_i})(z_p + (1-c)y_p), \\ u_{p+1} = z_p - cy_{p+1}, \end{cases} \quad (36)$$

where $0 \leq \alpha_p \leq \bar{a}_k$, $\bar{a}_k$ is chosen by (11) with $\kappa \in [0, 1)$ and $\sum_{p=1}^{\infty} \varphi_p < \infty$, and η_p is chosen by the following way:

$$\eta_p = \begin{cases} \frac{\vartheta_p \|(I - P_{Q_{[p]_2}})Av_p\|^2}{\|A^*(I - P_{Q_{[p]_2}})Av_p\|^2}, & (I - P_{Q_{[p]_2}})Av_p \neq 0 \\ \eta, & (I - P_{Q_{[p]_2}})Av_p = 0. \end{cases} \quad (37)$$

with $0 < \eta < 1$. Then $\{u_p\}$ converges weakly to a point u^* , where u^* is a solution of the MSFP, and $\{(y_p, u_p)\}$ converges weakly to $(0, u^*)$.

4 Numerical illustrations

In this section, we consider a numerical example to illustrate the numerical behavior of the iterations in algorithms (1), (2) and (3) to approximate the solution of the multiple-set split common fixed point problem.

Example 2. Let $\mathcal{H}$ be the real line with the usual norm and $\mathcal{D} = [0, 1]$. Define the following functions on $\mathcal{D}$.

$$F_1(u) = \begin{cases} \frac{1}{2}u + \frac{3}{8}, & \text{if } 0 \leq u \leq \frac{3}{4} \\ u, & \text{if } \frac{3}{4} < u < \frac{4}{5} \\ \frac{4}{5}, & \text{if } \frac{4}{5} \leq u < \frac{9}{10} \\ \frac{1}{2}, & \text{if } \frac{9}{10} \leq u \leq 1, \end{cases} \quad (38)$$

$$F_2(u) = \begin{cases} \frac{4}{5}, & \text{if } 0 \leq u < 1 \\ \frac{1}{5}, & \text{if } u = 1, \end{cases} \quad (39)$$

$$G_1(u) = \begin{cases} \frac{5}{6}, & \text{if } 0 \leq u \leq \frac{5}{6} \\ u, & \text{if } \frac{5}{6} < u < \frac{7}{8} \\ \frac{7}{8}, & \text{if } \frac{7}{8} \leq u < 1 \\ \frac{1}{8}, & \text{if } u = 1, \end{cases} \quad (40)$$

and

$$G_2(u) = \begin{cases} \frac{7}{8}, & \text{if } 0 \leq u < 1 \\ \frac{1}{8}, & \text{if } u = 1. \end{cases} \quad (41)$$

The above functions (F_1, F_2, G_1, G_2) are demicontractive but neither quasi-nonexpansive nor nonexpansive, see Berinde and Saleh [6, 8]. We set the following sequences and constants:

$$\varphi_p = \frac{1}{p^2}, \quad \vartheta_p = \frac{3}{p+2}, \quad \xi_p^1 = \frac{1+2p}{3p}, \quad \xi_p^2 = 1 - \xi_p^1, \quad \kappa_p^1 = \frac{1+p}{3p}, \quad \kappa_p^2 = 1 - \kappa_p^1,$$

$$c = 0.5, \quad \eta = 0.5, \quad \kappa = 0.2.$$

All assumptions in Theorems 1, 2, and 3 are satisfied. Hence, the sequences $\{u_p\}$ and $\{y_p\}$ generated by Algorithms (1), (2) and (3) converge to the solution of the MSSCFP, namely is $u = 0.8 \in \Omega$ and 0, respectively.

Several numerical experiments were conducted in **Python** for various choices of the parameters. The numerical results corresponding to different initial values, obtained using Algorithms (1), (2), and (3), are reported in Tables 1, 2, and 3, respectively.

We observe that for the initial value $(u_0, y_0) = (1, 0.3)$, the solution of the MSSCFP is obtained after 80, 116 and 81 iterations using Algorithms 1, 2 and 3, respectively. For the initial value $(u_0, y_0) = (0.2, 0.6)$, the same solution is reached after 88, 127 and 82 iterations for Algorithms 1, 2 and 3, respectively. Overall, the three algorithms converge at comparable rates, although Algorithm 2 converges slightly more slowly than Algorithms 1 and 3.

5 Conclusions

We establish weak convergence theorems for several averaged iterative methods designed to approximate solutions of the multiple-set split common fixed point

Table 1: Numerical results for Example 2 with different initial values using Algorithm 1

Iteration (p)	u_p	y_p	u_p	y_p
0	1	0.3	0.2	0.6
1	0.80246667	0.07506667	0.30677968	-0.03374984
2	0.76434598	0.00056163	0.43840056	-0.09130827
3	0.76323949	-0.0081066	0.54247468	-0.07893457
4	0.76881632	-0.0082679	0.6158808	-0.06004483
5	0.77472902	-0.00712048	0.66638754	-0.04449937
...	...	...	...	...
10	0.79169215	-0.00254265	0.76483163	-0.01085815
...	...	...	...	...
50	0.79999796	-0.00000066	0.79999139	-0.00000278
...	...	...	...	...
75	0.79999999	0.0	0.79999994	-0.00000002
...	...	...	...	...
80	0.8	0.0	0.79999998	-0.00000001
...	...	...	...	...
82	0.8	0.0	0.79999999	0.0
88	0.8	0.0	0.8	0.0

problem in Hilbert spaces involving demicontractive mappings. These results are obtained by exploiting known weak convergence theorems for the class of quasi-nonexpansive operators.

Our analysis relies on an embedding technique based on average mappings: if F is β -demicontractive, then for any $\mu \in (0, 1 - \beta)$, the average operator $F_\mu = (1 - \mu)I + \mu F$ is quasi-nonexpansive.

As demonstrated by the numerical experiments presented in Section 4, the theoretical results are effectively applicable to the multiple-set split common fixed point problem within the broader class of demicontractive mappings.

Competing interests

The authors declare that they have no conflict of interest.

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Table 2: Numerical results for Example 2 with different initial values using Algorithm 2.

Iteration (p)	u_p	y_p	u_p	y_p
0	1	0.3	0.2	0.6
1	0.8058	0.0684	0.32772190	-0.07563429
2	0.76823506	0.00042349	0.42917651	-0.02988672
3	0.76937237	-0.01335331	0.54817956	-0.11711582
4	0.77096784	0.0	0.60223339	-0.01719371
5	0.77742399	-0.00991142	0.66068916	-0.06493509
...	...	...	...	...
10	0.78960016	0.0	0.75107995	-0.00094223
...	...	...	...	...
50	0.79996758	0.0	0.79985026	0.0
...	...	...	...	...
100	0.79999996	0.0	0.79999982	0.0
...	...	...	...	...
116	0.8	0.0	0.79999998	0.0
...	...	...	...	...
120	0.8	0.0	0.79999999	0.0
...	...	...	...	...
127	0.8	0.0	0.8	0.0

Table 3: Numerical results for Example 2 with different initial values using Algorithm 3

Iteration (p)	u_p	y_p	u_p	y_p
0	1	0.3	0.2	0.6
1	0.6406	-0.0012	0.42194667	-0.01189333
2	0.61385893	-0.04391352	0.53425992	-0.06491133
3	0.66925123	-0.03875765	0.63466786	-0.05079473
4	0.70214601	-0.03058063	0.68220772	-0.0377717
5	0.73347528	-0.02133576	0.72131988	-0.02579493
...	...	...	...	...
10	0.78248105	-0.00536745	0.77985152	-0.00617449
...	...	...	...	...
50	0.79999724	-0.00000089	0.79999682	-0.00000103
...	...	...	...	...
76	0.79999999	0.0	0.79999998	0.0
...	...	...	...	...
80	0.79999999	0.0	0.79999999	0.0
81	0.8	0.0	0.79999999	0.0
82	0.8	0.0	0.8	0.0

Data Availability

Not applicable.

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