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A COMMON FIXED POINT THEOREM FOR QUASI CONTRACTIVE TYPE MAPPINGS

By

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1. Introduction

The well known Banach's fixed point theorem (also named contraction mapping principle) is one of the most useful results in fixed point theory. In a metric space setting it can be briefly stated as follows.

THEOREM B. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a strict contraction, i.e., a map satisfying*

$$d(Tx, Ty) \le a\, d(x, y)\,, \quad \textit{for all} \quad x, y \in X, \tag{1.1}$$

where $0 < a < 1$ is constant. Then T has a unique fixed point in X.

Theorem B, together with its direct generalizations and local variants, has many applications in solving nonlinear functional equations, but suffers from one drawback - the contractive condition (1.1) forces that T be continuous throughout X. In order to remove this drawback, in 1968 Kannan [9] obtained a fixed point theorem for mappings T that need not be continuous.

THEOREM K. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a mapping for which there exists $a \in \left(0, \frac{1}{2}\right)$ such that*

$$d(Tx, Ty) \le a\left[d(x, Tx) + d(y, Ty)\right], \quad \textit{for all} \quad x, y \in X. \tag{1.2}$$

Then T has a unique fixed point in X.

EXAMPLE 1. Let $X = \mathbb{R}$ be the set of real numbers with the usual metric and $T : \mathbb{R} \longrightarrow \mathbb{R}$, given by $Tx = 0$, if $x \in (-\infty, 2]$ and $Tx = -\frac{1}{2}$, if $x \in (2, \infty)$.

Then T satisfies (1.2) with $a = \frac{1}{5}$, T is not continuous and $F_T = \{0\}$.

Following Kannan's theorem, a lot of papers were devoted to obtaining fixed point theorems for various classes of contractive type conditions that do not require the continuity of T, see for example Rus [13]. In this context, a very interesting theorem which extends both Banach's and Kannan's fixed point theorems, alongside many other similar results of this kind, was obtained in 1972 by Zamfirescu [14].

THEOREM Z. *Let* (X,d) *be a complete metric space and* $T : X \longrightarrow X$ *a mapping for which there exist the real numbers* α, β *and* γ *satisfying* $0 < \alpha < 1$, $0 < \beta < 1/2$ *and* $0 < \gamma < 1/2$ *such that, for each* $x, y \in X$, *at least one of the following is true:*

(z_1) $\quad d(Tx, Ty) \le \alpha\, d(x, y)$;

(z_2) $\quad d(Tx, Ty) \le \beta \left[d(x, Tx) + d(y, Ty)\right]$;

(z_3) $\quad d(Tx, Ty) \le \gamma \left[d(x, Ty) + d(y, Tx)\right]$.

Then T *has a unique fixed point in* X.

One of the most general contraction conditions obtained in this way, for which the Picard iteration still converge to the unique fixed point, was given by Ciric [7] in 1974.

THEOREM C. *Let* (X,d) *be a complete metric space and* $T : X \longrightarrow X$ *a mapping that satisfies*

(1.3)
$$d(Tx, Ty) \le h \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for all $x, y \in X$ *and some constant* $0 < h < 1$.
Then T *has a unique fixed point in* X.

REMARK. It is easy to see that if T is an operator that satisfies the assumptions in any of the Theorems B, K and Z, then T also satisfies the assumptions of Theorem C.

The set $0_T(x) = \{x, Tx, T^2x, \ldots\}$ is called *the orbit* of T relative to x. It is shown in [15] that condition (1.3) does in fact assure that the orbits of T are bounded.

There exist many extensions and generalizations of these results. One of them was given in [1], for the class of the so called generalized φ-contractions, as a unifying fixed point theorem of many results of the same kind.

A mapping $T : X \longrightarrow X$ is said to be a *generalized φ-contraction* if there exists a function $\varphi : \mathbb{R}_+^5 \longrightarrow \mathbb{R}_+$ (called *comparison function* and satisfying certain appropriate conditions) such that for all $x, y \in X$

$$d(Tx, Ty) \leq \varphi\left(d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right). \tag{1.4}$$

EXAMPLE 2. The functions

1) $\varphi_1(t) = \alpha t_1$, for all $t = (t_1, t_2, t_3, t_4, t_5) \in \mathbb{R}_+^5$ $(0 \leq \alpha < 1)$;

2) $\varphi_2(t) = a(t_2 + t_3)$, for all $t = (t_1, t_2, t_3, t_4, t_5) \in \mathbb{R}_+^5$, $0 \leq a < \frac{1}{2}$;

3) $\varphi_3(t) \in \{\alpha t_1, \beta(t_2+t_3), \gamma(t_4+t_5)\}$, for all $t = (t_1, t_2, t_3, t_4, t_5) \in$ $\in \mathbb{R}_+^5$, $0 \leq \alpha < 1$; $0 \leq \beta < \frac{1}{2}$; $0 \leq \gamma < \frac{1}{2}$;

4) $\varphi_4(t) = h \cdot \max\{t_1, t_2, t_3, t_4, t_5\}$, for all $t = (t_1, t_2, t_3, t_4, t_5) \in$ $\in \mathbb{R}_+^5$, $0 < h < 1$

are all comparison functions. (Recall that a map satisfying (1.4) with $\varphi \equiv \varphi_4$ is usually called quasi contraction).

In a slightly corriged version, see Berinde [2], the main result in [1] can be briefly restated as follows.

THEOREM G. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a generalized φ-contraction with φ such that $\psi(t) = \varphi(t,t,t,t,t)$ is a continuous comparison function and $h(t) = t - \psi(t)$ is an increasing bijection. Then*

(i) T has a unique fixed point p in X;

(ii) The Picard iteration $\{x_n\}_{n=0}^{\infty}$, given by $x_{n+1} = Tx_n$, $n \geq 0$ and $x_0 \in X$, converges to p;

(iii) $d(x_n, p) \leq \psi^n\left(h^{-1}(d(x_0, x_1))\right)$, $n \geq 1$.

It is the main purpose of the present paper to extend Theorem G, and hence all fixed point theorems contained by it as particular cases, to a common fixed point theorem.

2. A common fixed point theorem

The important result given by Theorem C has been also extended in many directions: to nonself mappings, Ciric ([8], Theorem 2.1) by using Rothe's boundary condition, to generalized orbitally complete metric spaces with the metric satisfying a quadrilateral inequality instead of the usual triangle inequality, see Lahiri and Das [10], as well as to a common fixed point for nonself mappings, see Rakocevic [11] and Berinde [4], and also to orbitally complete metric spaces, see Ciric [6].

In this section we state and prove a general common fixed point theorem for self operators satisfying a generalized condition of quasi-contractive type.

To this end we need some appropriate notions and results related to mappings with contracting orbital diameters.

REMARKS.

1) A mapping satisfying a contractive condition of the form (1.4) is generally not continuous throughout X. However, as shown by Rhoades ([12], Theorem 2), a contractive mapping satisfying (1.3) is continuous *at the fixed point*. The argument is easily extendable to mappings satisfying (1.4) with φ an appropriate comparison function.

2) One of the first authors who considered conditions of the form (1.4) with $\varphi(t) \equiv \varphi(t_1)$, $t = (t_1, t_2, t_3, t_4, t_5) \in \mathbb{R}_+^5$, was Browder [5].
A scalar function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ involved in such a fixed point theorem is also called *comparison function* and is supposed to satisfy at least the following two conditions:

(i_φ) φ is monotonically increasing, i.e., $t_1 < t_2 \Rightarrow \varphi(t_1) \leq \varphi(t_2)$;

(ii_φ) The sequence $\{\varphi^n(t)\}_{n=0}^{\infty}$ converges to zero for each $t \in \mathbb{R}_+$, where φ^n stands for the n^{th} iterate of φ.

A prototype for the scalar comparison functions is $\varphi(t) = a \cdot t$, $t \in \mathbb{R}_+$, with $0 \leq a < 1$.
Considering $\varphi_1(t) = \frac{t}{1+t}$, $t \in \mathbb{R}_+$ and $\varphi_2(t) = \frac{1}{2}t$, if $0 \leq t < 1$ and $\varphi_2(t) = t - \frac{1}{3}$, if $t \geq 1$, it is easy to check that comparison functions need not be neither linear, nor continuous.

To prove our main result we shall use the following Lemma.

LEMMA 1. *Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfy (i_φ) and (ii_φ) and suppose*

$$t \leq \varphi(t), \tag{2.1}$$

for a certain $t \in \mathbb{R}_+$. Then $t = 0$.

PROOF. Assume the contrary, i.e., there exists $t > 0$ such that (2.1) is satisfied. Then, by (i_φ) we inductively get

$$t \leq \varphi^n(t), \quad n \geq 1$$

and so, in view of (ii_φ), this implies

$$0 \le t \le \lim_{n\to\infty} \varphi^n(t) = 0\,,$$

a contradiction. ∎

The main result of this paper is given by the next theorem.

THEOREM 1. *Let (X,d) be a complete metric space and $S,\ T : X \longrightarrow X$ two mappings with bounded orbits. Suppose T is continuous and*

$$d(Sx,Sy) \le \varphi\left(M(x,y)\right), \quad \text{for all} \quad x,y \in X, \tag{2.2}$$

where

$$M(x,y) = \max\left\{d(Tx,Ty),\, d(Tx,Sx),\, d(Ty,Sy),\, d(Tx,Sy), d(Ty,Sx)\right\}, \tag{2.3}$$

with $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a continuous scalar comparison function. Suppose

$$S(X) \subset T(X) \tag{2.4}$$

and also suppose T and S are weakly commutative, i.e.,

$$d(TSx,STx) \le d(Tx,Sx)\,, \quad \text{for every} \quad x \in X\,. \tag{2.5}$$

Then T and S have a unique common fixed point.

PROOF. Let $x_0 \in X$ be arbitrary. Then by (2.4) $Sx_0 \in T(X)$, which shows that there exists $x_1 \in X$ such that

$$Tx_1 = Sx_0\,.$$

Consider now Sx_1. Since $Sx_1 \in T(X)$, there exists $x_2 \in X$ such that

$$Tx_2 = Sx_1\,.$$

By induction, we construct a sequence $\{x_n\}_{n=0}^{\infty}$ of points in X such that

$$Tx_{n+1} = Sx_n\,, \quad n = 0,1,2,\dots\,.$$

We shall prove that $\{Tx_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

To this end, consider

$$B(n,k) = \{Tx_j, Sx_j : n \le j \le n+k\}\,;\ b(n,k) = \operatorname{diam}\left(B(n,k)\right);$$

$$B(n) = \{Tx_j, Sx_j : n \le j\}\,;\ b(n) = \operatorname{diam}\left(B(n)\right).$$

It easy to see that $b(n,k) \uparrow b(n)$ as $k \to \infty$ and that $\{b(n)\}_{n=0}^{\infty}$ is a decreasing sequence of positive terms, hence

$$b = \lim_{n\to\infty} b(n)$$

does exist.

To prove that $\{Tx_n\}_{n=0}^{\infty}$ is a Cauchy sequence we must show that $b = 0$.

We claim that

$$b(n,k) \le \varphi\left(b(n-2,k-2)\right), \quad n,k \ge 2, \tag{2.6}$$

and discuss the following three cases.

Case 1. $b(n,k) = d(Tx_i, Sx_j)$ with $n \le i, j \le n+k$:
Then $Tx_i = Sx_{i-1}$ and, by (2.2), we get

$$b(n,k) = d(Sx_{i-1}, Sx_j) \le \varphi\left(M(x_{i-1}, x_j)\right) \le \varphi\left(b(n-2,k+2)\right),$$

since φ is monotonically increasing. The remaining cases:

Case 2. $b(n,k) = d(Sx_i, Sx_j)$ with $n \le i, j \le n+k$
and

Case 3. $b(n,k) = d(Tx_i, Tx_j)$ with $n \le i, j \le n+k$
can be easily reduced to Case 1.

Therefore (2.6) is true. Now, if we let $k \to \infty$ in (2.6) and use the continuity of φ we obtain

$$b(n) \le \varphi\left(b(n-2)\right), \quad n \ge 1. \tag{2.7}$$

By (ii_φ) and continuity of φ, letting $n \to \infty$ in (2.7) we get

$$b \le \varphi(b)$$

which by Lemma 1 implies $b = 0$.

This shows that both $\{Tx_n\}_{n=1}^{\infty}$ and $\{Sx_n\}_{n=0}^{\infty}$ are Cauchy sequences. Since (X,d) is a complete metric space, we conclude that

$$\lim_{n\to\infty} Tx_n = p \in X,$$

and hence $\lim\limits_{n\to\infty} Sx_n = p$, too.

Since T is continuous, we obtain

$$\lim_{n\to\infty} T(Sx_n) = T\left(\lim_{n\to\infty} Sx_n\right) = Tp$$

which, in view of the weak commutativity condition (2.4), yields

$$\begin{aligned} & d(STx_n, Tp) \le d(STx_n, TSx_n) + d(TSx_n, Tp) \le \\ & \le d(Tx_n, Sx_n) + d(TSx_n, Tp) \longrightarrow 0, \quad \text{as} \quad n \to \infty. \end{aligned} \tag{2.8}$$

This shows that

$$\lim_{n\to\infty} (ST)(x_n) = Tp, \tag{2.9}$$

and therefore, by (2.8) and (2.9), we have

$$\begin{aligned} M(Tx_n, p) = \max\{&d(TTx_n, Tp), d(TTx_n, Sp), d(Tp, Sp), d(TTx_n, Sp), \\ &d(Tp, Sx_n)\} \longrightarrow \max\{d(Tp, Tp), d(Tp, Sp), d(Tp, Sp), d(Tp, Sp), \\ &d(Tp, Sp)\} = d(Tp, Sp), \quad \text{as} \quad n \to \infty. \end{aligned}$$

So by (2.3)

$$d(STx_n, Sp) \le \varphi\left(M(Tx_n, p)\right),$$

which by letting $n \to \infty$, yields

$$d(Tp, Sp) \le \varphi\left(d(Tp, Sp)\right)$$

and which by Lemma 1 implies $d(Tp, Sp) = 0$, i.e.,

$$Tp = Sp. \tag{2.10}$$

To show that Sp is a common fixed point of S and T it suffices to show that Sp is a fixed point of S. Indeed, by (2.10) and (2.5) it results that

$$TSp = STp = SSp. \tag{2.11}$$

Now, by (2.2), (2.10) and (2.11), we have

$$d(SSp, Sp) \le \varphi\left(M(Sp, p)\right) = \varphi\left(d(SSp, Sp)\right),$$

which again by Lemma 1 implies $SSp = Sp$. From (2.11) it results that Sp is a fixed point of T, too. The uniqueness follows by (2.2). ∎

REMARKS.

1) For $T = 1_X$, the identity map, by Theorem 1 we obtain a fixed point theorem similar to Theorem G;

2) For $\varphi(t) = h \cdot t$, $t \in \mathbb{R}_+$, $0 < h < 1$, from Theorem 1 we obtain a common fixed point theorem that contains Ciric's fixed point theorem as a particular case;

3) Note that if we denote for all $x, y \in X$,

$$D(x, y) = \left(d(x, y),\, d(x, Sx),\, d(y, Sy),\, d(x, Sy),\, d(y, Sx)\right),$$

then

$$\varphi_i\left(D(x, y)\right) \le \varphi\left(M(x, y)\right),$$

for all functions φ_1, φ_2 and φ_3 in Example 2.

This shows that, in the particular case $T = 1_X$, Theorem 1 provides extensions of Banach's, Kannan's, Zamfirescu's and Ciric's fixed point theorems.

THEOREM 2. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a generalized φ-contraction, i.e., a mapping satisfying*

$$d(Tx, Ty) \leq \varphi\big(C(x, y)\big), \quad \text{for all} \quad x, y \in X,$$

where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous comparison function and

$$C(x, y) = \max\{d(x, y),\, d(x, Tx),\, d(y, Ty),\, d(x, Ty),\, d(y, Tx)\}.$$

If T has bounded orbits, then it has a unique fixed point.

PROOF. Take $T = 1_X$ and $S := T$ in Theorem 1. ∎

The continuity of T in Theorem 1 can be weakened to obtain a more general result, similar to Theorem 3 in Rakocevic [11] and Berinde [4]. Actually all the results given in Rakocevic [11] can be similarly adapted for self mappings, but we restrict to the result corresponding to Theorem 3 in [11].

THEOREM 3. *Let (X, d) be a complete metric space and $S, T : X \to X$ two mappings with bounded orbits. Suppose that T^m is continuous for some fixed positive integer m, that S and T satisfy* (2.2), (2.4) *and are commutative, that is,*

$$TSx = STx, \quad \text{for each} \quad x \in K.$$

Then S and T have a unique common fixed point in K.

PROOF. Let $\{x_n\}$ be constructed as in the proof of Theorem 1.
Hence

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = p \in X.$$

For each $n \geq 1$,

$$d\big(T^m Sx_n, ST^{m-1}p\big) = d\big(ST^m x_n, ST^{m-1}p\big) \leq$$

$$\leq \varphi\left(M\big(T^m x_n, T^{m-1}p\big)\right) =$$

$$= \varphi\Big(\max\Big\{d\big(T^m Tx_n, T^m p\big),\, d\big(T^m Tx_n, T^m Sx_n\big),\, d\big(T^m p, ST^{m-1}p\big),$$

$$\big(T^m Tx_n, ST^{m-1}p\big),\, d\big(T^m p, T^m Sx_n\big)\Big\}\Big).$$

Then by the continuity of T^m and letting $n \to \infty$ we get

$$d\big(T^m p, ST^{m-1}p\big) \leq \varphi\left(d\big(T^m p, ST^{m-1}p\big)\right),$$

which by Lemma 1 shows that $T^m p = ST^{m-1}p$.
In order to prove that $T^m p$ is a fixed point of S, i.e.,

$$ST^m p = T^m p,$$

in view of $T^m p = ST^{m-1}p$, it suffices to show that

$$ST^m p = ST^{m-1}p. \tag{2.12}$$

Since

$$M\left(T^m p, T^{m-1}p\right) = \max\Big\{d\left(T^{m+1}p, T^m p\right), d\left(T^{m+1}p, ST^m p\right),$$
$$d\left(T^m p, ST^{m-1}p\right), d\left(T^{m+1}p, ST^{m-1}p\right), d\left(T^m p, ST^m p\right)\Big\},$$

in view of $T^m p = ST^{m-1}p$ and $T^{m+1}p = T\left(ST^{m-1}p\right) = ST^m p$,
we obtain

$$M(T^m p, T^{m-1}p) =$$
$$\max\Big\{d\left(ST^m p, ST^{m-1}p\right), 0, 0, d\left(ST^m p, ST^{m-1}p\right), d\left(ST^m p, ST^{m-1}p\right)\Big\}.$$

Now by (2.3) we have

$$d\left(ST^m p, ST^{m-1}p\right) \le \varphi\left(d\left(T^m p, T^{m-1}p\right)\right) = \varphi\left(d\left(ST^m p, ST^{m-1}p\right)\right)$$

which by Lemma 1 gives

$$d\left(ST^m p, ST^{m-1}p\right) = 0.$$

This proves (2.12) and hence $T^m p$ is a fixed point of S. Now

$$TT^m p = T^{m+1}p = ST^m p = T^m p,$$

which shows that $T^m p$ is a fixed point of T as well.

The uniqueness follows similarly, by the contraction condition (2.2). ∎

REMARKS.

1) Note that the results for nonself mappings in Rakocevic [11] and Berinde [4] are proven in a Banach space setting, while the results in this paper are obtained in the general setting of a complete metric space.

If we impose additional conditions on the comparison function φ, it is possible to obtain an error estimate for the method of successive approximations, like in Theorem G.

2) It is known, see Lemma 4.3.1 in [13] that if T is a generalized strict φ-contraction, i.e., T satisfies (1.4), with

$$t - \varphi(t, t, t, t, t) \to \infty, \text{as } t \to \infty,$$

then T has bounded orbits.

It is however an open question whether or not two mappings S and T satisfying (2.2) or the mapping T in Theorem 2, with φ an arbitrary comparison function, have bounded orbits.

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