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A CONVERGENCE THEOREM FOR MANN ITERATION IN THE CLASS OF ZAMFIRESCU OPERATORS

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Dedicated to Professor Mihail Megan on the occasion of his sixtieth anniversary

Abstract. A convergence theorem of Rhoades regarding the approximation of fixed points of some quasi-contractive operators in uniformly convex Banach spaces using the Mann iterative procedure, is extended to arbitrary Banach spaces. The conditions on the parameters $\{\alpha_n\}$ that define the Mann iteration are also weakened. Our result extends many other fixed point theorems in literature.

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1. Introduction

In the last four decades, numerous papers were published on the iterative approximation of fixed points of self and nonself contractive type operators in metric spaces, Hilbert spaces or several classes of Banach spaces, see for example the recent monograph [1] and the references therein. While for strict contractive type operators, the Picard iteration is usually used to approximate the (unique) fixed point, see e.g. [1], [15], [23], [24], for operators satisfying slightly weaker contractive type conditions, instead of Picard iteration, which does not generally converge, it was necessary to consider other fixed point iteration procedures. The Krasnoselskij iteration [16], [6], [13], [14], the Mann iteration [17], [9], [18] and the Ishikawa iteration [11] are certainly the most studied of these fixed point iteration procedures, see [1].

Let E be a normed linear space and $T : E \to E$ a given operator. Let $x_0 \in E$ be arbitrary and $\{\alpha_n\} \subset [0, 1]$ a sequence or real numbers.

The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n\,, \quad n = 0,1,2,\dots \tag{1}$$

is called the *Mann iteration* or *Mann iterative procedure*, in light of [17].

The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by

$$\begin{cases} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n\,, \quad n = 0,1,2,\dots \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n\,, \quad n = 0,1,2,\dots\,, \end{cases} \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0,1]$, and $x_0 \in E$ arbitrary, is called the *Ishikawa iteration* or *Ishikawa iterative procedure*, due to [11].

Remark 1. For $\alpha_n = \lambda$ (constant), the iteration (1) reduces to the so called *Krasnoselskij iteration*, while for $\alpha_n \equiv 1$ we obtain the *Picard iteration* or method of successive approximations, as it is commonly known, see [1]. Obviously, for $\beta_n \equiv 0$ the Ishikawa iteration (2) reduces to (1).

The classical Banach's contraction principle is one of the most useful results in fixed point theory. In a metric space setting it can be briefly stated as follows.

Theorem 1.1. *Let (X,d) be a complete metric space and $T : X \longrightarrow X$ a strict contraction, i.e., a map satisfying*

$$d(Tx,Ty) \le a\, d(x,y)\,, \tag{3}$$

for all $x,y \in X$, where $0 \le a < 1$ is constant. Then:

$(p1)$ T has a unique fixed point p in X;

$(p2)$ The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n\,, \quad n = 0,1,2,\dots \tag{4}$$

converges to p, for any $x_0 \in X$.

Note. A map satisfying $(p1)$ and $(p2)$ is said to be a *Picard operator*, see Rus [22].

Theorem 1.1, together with its direct generalizations have many applications in solving nonlinear equations, but suffer from one drawback - the contractive condition (3) forces T be continuous on X.

It is then natural to ask if there exist contractive conditions which do not imply the continuity of T. This was answered in the affirmative by R. Kannan [12] in 1968, who proved a fixed point theorem which extends Theorem B to mappings that need not be continuous, by considering instead of (3) the next condition: there exists $b \in \left(0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq b\left[d(x, Tx) + d(y, Ty)\right], \text{ for all } x, y \in X. \tag{5}$$

Example 1.1. Let $X = \mathbf{R}$, the set of real numbers with the usual norm, and $T : \mathbf{R} \to \mathbf{R}$, given by $Tx = 0$, if $x \in (-\infty, 2]$ and $Tx = -\frac{1}{2}$, if $x > 2$. Then T is not continuous on $\mathbf{R}$ and satisfies condition (5) with $b = \frac{1}{5}$.

Following Kannan's theorem, a lot of papers were devoted to obtaining fixed point theorems for various classes of contractive type conditions that do not require the continuity of T, see for example, Rus [22], and references therein. One of them, actually a sort of dual of Kannan fixed point theorem, due to Chatterjea [7], is based on a condition similar to (5): there exists $c \in \left(0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big], \quad \text{for all } x, y \in X. \tag{6}$$

It is known, see [20], that (3) and (5), (3) and (6), respectively, are independent contractive conditions.

In 1972, Zamfirescu [25] obtained a very interesting fixed point theorem, by combining (3), (5) and (6).

Theorem 1.2. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a map for which there exist the real numbers a, b and c satisfying $0 \leq a < 1$, $0 \leq b,\ c < 1/2$ such that for each pair x, y in X, at least one of the following is true*:

(z_1) $d(Tx, Ty) \leq a\, d(x, y)$;

(z_2) $d(Tx, Ty) \leq b\big[d(x, Tx) + d(y, Ty)\big]$;

(z_3) $d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big]$.

Then T is a Picard operator.

One of the most general contraction condition for which the map satisfying it is still a Picard operator, has been obtained by Ciric [8] in 1974: there exists $0 \leq h < 1$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq h \cdot \max\big\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big\}. \tag{7}$$

Remarks. A mapping satisfying (7) is commonly called quasi-contraction.

It is obvious that each of the conditions (5), (6) and (z_1)-(z_3) implies (7);

There exist many other fixed point theorems based on contractive conditions of this type, see for example Rhoades [20], [21] and the monographs Berinde [1], Rus [24].

An operator T which satisfies the contractive conditions in Theorem 1.3 will be called a *Zamfirescu operator* (alternatively, we shall say that T satisfies condition Z, see Rhoades [18]). Obviously, a Zamfirescu operator is generally not continuous, see Example 1.1.

The class of Zamfirescu operators T is one of the most studied class of quasi-contractive type operators, for which all important fixed point iteration procedures, i.e., Picard [25], Mann [18] and Ishikawa [19] iterations, are known to converge to the unique fixed point of T . Zamfirescu showed in [25] that an operator satisfying condition Z has a unique fixed point that can be approximated using the Picard iteration. Later, Rhoades [18], [19] proved that the Mann and Ishikawa iterations can also be used to approximate fixed points of Zamfirescu operators.

The class of operators satisfying condition Z is independent of the class of strictly (strongly) pseudocontractive operators, extensively studied by several authors in the last years, see Rhoades [18]. For the case of pseudocontractive type operators, the pioneering convergence theorems, due to Browder [5] and Browder and Petryshyn [6], established in Hilbert spaces, were successively extended to more general Banach spaces and to weaker conditions on the parameters that define the fixed point iteration procedures, as well as to several classes of weaker contractive type operators.

For a recent survey and a comprehensive bibliography, we refer to the author's monograph [1].

The following result was obtained by Rhoades ([18], Theorem 4).

Theorem 1.3. *Let E be a uniformly convex Banach space, K a closed convex subset of E and $T : K \to K$ a Zamfirescu operator. Let $\{x_n\}_{n=0}^{\infty}$ be defined by* (1) *and $x_0 \in K$, with $\{\alpha_n\}$ satisfying*

(i) $\alpha_0 = 1$; *(ii)* $0 < \alpha_n < 1$ *for* $n \geq 1$; *(iii)* $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$.

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.

The proof of Theorem 1.3 in [18] is based on a Lemma in Groetsch [9].

The main aim of this paper is to extend Theorem 1.3 from uniformly convex Banach spaces to arbitrary Banach spaces and also to Mann iterations defined by weaker assumptions on the sequence $\{\alpha_n\}$.

2. Main result

Theorem 2.1. *Let E be an arbitrary Banach space, K a closed convex subset of E, and $T : K \to K$ an operator satisfying condition Z. Let $\{x_n\}_{n=0}^{\infty}$ be defined by* (1) *and $x_0 \in K$, with $\{\alpha_n\} \subset [0,1]$ satisfying*

$$\text{(iv)} \qquad \sum_{n=0}^{\infty} \alpha_n = \infty\,.$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.

Proof. By Theorem 1.2, we known that T has a unique fixed point in K. Call it p and consider $x, y \in K$.

At least one of (z_1), (z_2) and (z_3) is satisfied. If (z_2) holds, then

$$\begin{aligned} \|Tx - Ty\| &\le b\big[\|x - Tx\| + \|y - Ty\|\big] \le \\ &\le b\Big\{\|x - Tx\| + \big[\|y - x\| + \|x - Tx\| + \|Tx - Ty\|\big]\Big\}. \end{aligned}$$

So

$$(1-b)\|Tx - Ty\| \le b\|x - y\| + 2b\|x - Tx\|\,,$$

which yields (since $0 \le b < 1$)

$$\|Tx - Ty\| \le \frac{b}{1-b}\,\|x - y\| + \frac{2b}{1-b}\,\|x - Tx\|\,. \tag{8}$$

If (z_3) holds, then similarly we get

$$\|Tx - Ty\| \le \frac{c}{1-c}\,\|x - y\| + \frac{2c}{1-c}\,\|x - Tx\|\,. \tag{9}$$

If we denote

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}, \tag{10}$$

then we have $0 \le \delta < 1$ and, in view of (z_1), (8) and (9) it follows that the inequality

$$\|Tx - Ty\| \le \delta \|x - y\| + 2\delta \|x - Tx\| \tag{11}$$

holds, for all $x, y \in K$.

Let $\{x_n\}_{n=0}^{\infty}$ be the Mann iteration (1), with $x_0 \in K$ arbitrary. Then

$$\begin{aligned} \|x_{n+1} - p\| &= \big\|(1-\alpha_n)x_n + \alpha_n T x_n - (1 - \alpha_n + \alpha_n)p\big\| = \\ &= \big\|(1-\alpha_n)(x_n - p) + \alpha_n(Tx_n - p)\big\| \le \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n \|Tx_n - p\| . \end{aligned} \tag{12}$$

Take $x := p$ and $y := x_n$ in (11) to obtain

$$\|Tx_n - p\| \le \delta \cdot \|x_n - p\|,$$

which together with (12) yields

$$\|x_{n+1} - p\| \le \big[1 - (1-\delta)\alpha_n\big] \|x_n - p\| , \quad n = 0, 1, 2, \ldots . \tag{13}$$

Inductively we get

$$\|x_{n+1} - p\| \le \prod_{k=0}^{n} \big[1 - (1-\delta)\alpha_k\big] \cdot \|x_0 - p\| , \quad n = 0, 1, 2, \ldots . \tag{14}$$

As $\delta < 1$, $\alpha_k \in [0, 1]$ and $\sum\limits_{k=0}^{\infty} \alpha_k = \infty$, it results that

$$\lim_{n\to\infty} \prod_{k=0}^{n} \big[1 - (1-\delta)\alpha_k\big] = 0 ,$$

which by (14) implies

$$\lim_{n\to\infty} \|x_{n+1} - p\| = 0 ,$$

i.e., $\{x_n\}_{n=0}^{\infty}$ converges strongly to p. □

Remarks.

1) Condition (iv) in our Theorem 2.1 is more relaxed than conditions (i) - (iii) in Theorem 1.3. Indeed, in view of

$$0 < \alpha_k(1 - \alpha_k) < \alpha_k,$$

valid for all α_k satisfying (i) - (ii), condition (iii) implies (iv).
There also exist values of $\{\alpha_n\}$, e.g., $\alpha_n \equiv 1$, such that (iv) is satisfied but (iii) is not;

2) Since the contractive condition of Kannan (5) is a special case of that of Zamfirescu, Theorems 2 and 3 of Kannan [13] are special cases of Theorem 2.1 or Theorem 1.3 in this paper, with $\alpha_n = 1/2$. Theorem 3 of Kannan [14] is the special case of Theorem 2.1 or Theorem 1.3 with $\alpha_n = \lambda$, $0 < \lambda < 1$. However, note that all the results of Kannan [13], [14] are obtained in uniformly Banach spaces, like Theorem 1.3;

3) Because of the more restrictive assumptions (i) - (ii), the convergence of Picard iteration cannot be obtained as a particular case of Theorem 1.3, but it can be obtained by our theorem Theorem 2.1, taking $\alpha_n = 1$, due to the more natural assumption (iv);

4) By Theorem 2.1 we can obtain, as a particular case, a convergence theorem for Mann iteration in the class of operators that satisfy a contractive condition of the form (6);

5) The stability of the Mann iteration for Zamfirescu operators was studied in [10].

Conclusions. Our Theorem 2.1 improves Theorem 4 in Rhoades [18] by extending it from uniformly convex Banach spaces to arbitrary Banach spaces and simultaneously by weakening the assumptions on the sequence $\{\alpha_n\}$ that defines the Mann iteration. Moreover, many other results in literature are also extended in this way, e.g.:

1) The convergence theorems of two mean value fixed point iteration procedures for Kannan operators [13], [14] are extended to the larger class of Zamfirescu operators and simultaneously from uniformly convex Banach spaces to arbitrary Banach spaces;

2) The fixed point theorem of Chatterjea is extended from the Picard iteration to the Mann iteration, which also contains, as a particular case, the corresponding convergence theorem for Krasnoselskij iteration;

3) While the convergence of Picard iteration in the class of Zamfirescu operators cannot be deduced by Rhoades' Theorem 1.3, our main result also include, as a particular case, the convergence of both Picard and Krasnoselskij iterations.

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