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Within the world of demicontractive mappings. In Memoriam Professor Ştefan Măruşter (1937-2017)

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Abstract.
Professor Ştefan Măruşter, an outstanding Romanian mathematician and computer scientist, passed away on December 24th 2017. The aim of this note is to present some relevant information on his life and impressive professional activity. Most of the material included here is taken from the conference presentations [Berinde, V., *In honour and celebration of the 80th birthday of Professor Ştefan Măruşter*, Plenary lecture, Ceremony in honour of Professor Ştefan Măruşter's 80th birthday, SYNASC 2017, September 23rd, 2017] and [Berinde, V., Păcurar, M., *On some contributions of Professor Ştefan Măruşter to the study of demicontractive type mappings*, Workshop "Iterative Approximation of Fixed Points", SYNASC 2017, September 22nd, 2017].

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1 Brief biography

Ştefan Măruşter was born on September 5th, 1937 in Ineu, Arad county, Romania, and studied in his native village during elementary school (1944-1951), and then in Arad during high school (1951-1956). He then studied

Mathematics and Physics at West University of Timişoara (1956-1960) and, after graduation, he was a high school teacher of mathematics in Baia Mare for five years (1960-1965). In the period 1965-1972 he was a researcher at the Computing Centre of Politehnica University of Timişoara. Since 1972 Ştefan Măruşter has been affiliated at West University of Timişoara, where he held all academic positions: Assistant Professor (1970-1981); Associate Professor (1981-1991); Full Professor & Professor Emeritus (1991-2017). He passed away on December 24th, 2017, after a long and harsh disease, but he was active, collaborative and very enthusiastic about some joint papers we had been working on for a few weeks before his death.

He has been celebrated on the occasion of his 80th birthday, on September 23rd, 2017, by an impressive ceremony organized in the framework of SYNASC 2017 (19th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing, 21-24 September 2017, West University of Timişoara, Romania), by his former students, collaborators and friends.

Directly related to his research work, Ştefan Măruşter was also a dedicated supervisor. What is really impressive is that Professor Măruşter has mixed Mathematics and Computer Science not only in his research activity but also as a PhD supervisor in a very original and productive way, as shown by the list of the theses of his students: Dana Petcu, *Numerical Solution of Stiff Differential Equations*, 1994; Octavian Cira, *Numerical Methods for Algebraic Equations*, 1998; Ioan Despi, *Relational databases*, 1998; Eugen Creţu, *Design and Analysis of Parallel Algorithms in Fuzzy Sets Computations*, 2000; Robert Reisz, *Mathematical Modelling Using Stochastic Automata on Infinite Words*, 2000; Teodor Florin Fortiş, *Contextual Grammars with Concatenation*, 2001; Lucian Luca, *Processes Administration*, 2002; Adriana Popovici, *N-dimensional Cell Automata. Applications and Generalizations*, 2003; Mircea Drăgan, *Concurrent Solution of Sonlinear Systems*, 2004; Călin Şandru, *Multi-Agent Architecture Using a Task-Oriented Approach to Solving Problems*, 2004; Daniel Pop, *Intelligent Systems for Modelling and Extracting Knowledge*, 2006; Cosmin Bonchiş, *Arithmetic and Information Iheory in Membrane Computation*, 2009; Emil Horia Popa, *Multiagent Models in Information Retrieval Systems*, 2009; Cristina Popîrlan (Chiţu), *Numerical Methods for Approximating Fixed Points of Demicontractive Operators*, 2009; Cornel Izbaşa, *Information Theory and Multistage Computing*, 2010.

2 The place of demicontractive mappings in the research activity of Professor Măruşter

It is certainly not easy to summarize the whole research activity and professional achievements of Professor Măruşter, because this would require a whole book. This is why, from the list of his main scientific interests (software, programming languages, compilers; theory of computing, formal languages; mathematics of computation: numerical methods for nonlinear equations; convergence and stability; mathematical software; convex feasibility problem etc.), we restrict ourselves in this article to present just a few data about *demicontractive mappings*.

The story of how Ştefan Măruşter has discovered the class of demicontractive mappings is presented in his autobiographical paper [8]. We translate below a few important phrases.

"Coming back to my research interest regarding the problem itself, i.e., the computation of the best approximation polynomials, after several explorations and numerical tests (having at my disposal a computer, I had a great advantage to perform numerical tests), I eventually realized that such a polynomial is the solution of an operator equation sophisticated enough, and I even succeeded to find explicitly this equation in some sense. The more natural setting to consider this problem has been chosen to be a Hilbert space... The operator that was involved in this equation turned out to have some monotonicity and closeness properties, but it was not continuous, more precisely, it was discontinuous on an infinite set of points. Using the Remez algorithm as a model, I considered, from the very beginning, an iterative algorithm of convex combinations type (between the current iteration and the value of the operator at the current iteration). With a considerable endeavour, I have succeeded to prove the weak convergence of the sequence generated in this way to the fixed point of the operator, that is, to the solution of the equation, but I did not succeed to obtain a strong convergence result, which would have been really interesting from the practical point of view.

Anyway, after many efforts and endeavours, I have found a relatively strange and completely not natural condition, but which had the merit to ensure the strong convergence. Later on, I found out that the respective iterative method had been previously considered by R. W. Mann and that a series of papers had been dedicated exactly to the same topic, i.e., the weak convergence of the iterative method and additional conditions that ensure the strong convergence. It is interesting that approximately in the same period I succeeded to identify another problem, later called convex feasibility problem, which possessed exactly the same features, that is, its solution was nothing else but the solution of an operator equation of the same type. Most of these results have been included in my PhD thesis, that I defended at the University of Cluj-Napoca, in 1974, under the supervision of Professor D.D. Stancu. About the same period, I finished a paper devoted to the convergence of iterative methods of convex combinations type (Mann) and I submitted it to a journal from the United States, Proceedings of the American Mathematical Society. The postal correspondence with that journal lasted a lot of time (since a letter posted from Timişoara to New York usually reached its destination after a few months), and so the paper was eventually published in 1977."

In the paper mentioned above [7], the PAMS editors are mentioning the complete submission information: "Received by the editors March 26, 1975 and, in revised form, July 24, 1975 and September 24, 1976". In this paper the "relatively strange condition" would mark in fact the birth of what we are calling now demicontractive mappings.

Definition 2.1 (Măruşter, [7], 1977)**.** *Let H be a real Hilbert space, C a closed convex subset of H and $T : C \to C$ a mapping with the property that it has at least one fixed point, i.e., $Fix\,(T) \neq \emptyset$. T is said to satisfy condition (A) if there exists $\lambda > 0$ such that*

$$\langle x - Tx, x - x^* \rangle \geq \lambda \|Tx - x\|^2, \forall x \in C, x^* \in Fix\,(T). \tag{2.1}$$

In the same year 1977 but in a different journal, *J. Math. Anal. Appl.*, Hicks and Kubicek [5] introduced and studied, in the same context as in [7], a class of mappings called by them *demicontractive.*

Definition 2.2. *[Hicks and Kubicek,* [5]*, 1977] Let H be a real Hilbert space, C a closed convex subset of H, and $T : C \to C$ a mapping satisfying $Fix\,(T) \neq \emptyset$. T is said to be demicontractive if there exists $k < 1$ such that*

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|Tx - x\|^2, \forall x \in C, x^* \in Fix\,(T). \tag{2.2}$$

Condition (2.1) had been considered previously in a paper published by Ştefan Măruşter in 1973 in *Canadian J. Math.* [6] (and received by the editors on December 21st, 1971). Note that in [6] the condition (2.1) was not written in a fixed point form like in [7] but for an operator equation $U(x) = 0$, that is, with $U(x)$ instead of $x - Tx$.

The first papers that effectively payed attention to the class of demicontractive mappings are due to Charles E. Chidume [3], who started his work from the paper by Hicks and Kubicek [5] but afterwards [4] also cited the paper of Ştefan Măruşter [7].

Charles E. Chidume has also the indirect merit of identifying the relevance of demicontractive maps as defined by Ştefan Măruşter in [7], since one of his students, Chika Moore [10], was the first one to show in 1998 that the two classes of mappings introduced by Definitions 2.1 and 2.2 are actually equivalent. His proof is based on the fact that, in a Hilbert space one has

$$\|x - x^*\|^2 + k\|x - Tx\|^2 - \|Tx - x^*\|^2 =$$

$$2\langle x - x^*, x - Tx\rangle - (1-k)\|x - Tx\|^2,$$

and hence, with $\lambda = \dfrac{1-k}{2}$, (2.1) and (2.2) are:

$$\|x - x^*\|^2 + k\|Tx - x\|^2 - \|Tx - x^*\|^2 \geq 0, \forall x \in C, x^* \in Fix\,(T)$$

and

$$\langle x - Tx, x - x^*\rangle \geq \lambda\|Tx - x\|^2, \forall x \in C, x^* \in Fix\,(T),$$

respectively, from which we easily see their equivalence.

So, in view of the arguments above, we have all reasons to credit Ştefan Măruşter as the true father of demicontractive mappings.

If we search the electronic databases we find out that the paper [Măruşter, Şt., *The solution by iteration of nonlinear equations in Hilbert spaces.* Proc. Amer. Math. Soc. 63 (1977), no. 1, 69–73.] collected so far 80 Google Scholar citations; 39 Web of Science citations and 56 SCOPUS citations.

But if we search the same databases for the syntagm ”demicontractive mapping” we find the following figures: 123 papers with ”demicontractive mapping” in their title (Google Scholar); 80 papers with ”demicontractive mapping” in their topic (Web of Science) and 358 papers with ”demicontractive mapping” in any field (SCOPUS).

It appears that Ştefan Măruşter himself has not been aware of the importance of his contribution to iterative approximation of fixed points. Rather late, he revisited this area of research and published the following papers, most of them directly related to that topic.

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- Berinde, V., Măruşter, Ş., Rus, Ioan A., *Saturated contraction principles for non self operators. Generalizations and applications*, Filomat 31 (2017), No. 11, 3391–3406.

3 The relevance of demicontractive mappings

In order to point out the importance of demicontractive mappings in the framework of iterative approximation of fixed points, we summarize some of the most important results in the area of metric fixed point theory.
Consider the fixed point problem $x = Tx$, where $T : X \to X$ is a self mapping and (X, d) is a metric space. The first metric fixed point theorem, i.e., Banach's contraction mapping principle, is based on the property of T to be a *(strict) contraction*, i.e., T is such that there exists $0 \le k < 1$ with $d(Tx, Ty) \le k\, d(x, y)$, for all $x, y \in X$. Denote by $\mathcal{C} = \{T : X \to X\colon T$ is a strict contraction$\}$.
Banach's contraction mapping principle asserts that, if (X, d) is complete and $T \in \mathcal{C}$, then:
(i) the equation $x = Tx$ has a unique solution x^* in X;
(ii) the Picard iteration, that is, the sequence $\{x_n\}$ defined iteratively by $x_{n+1} = Tx_n$, $n \ge 0$, converges to x^* for any initial value $x_0 \in X$.

If (X, d) is actually a normed space, C is a subset of X and $T : C \to C$, then if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$, T is called *nonexpansive*. Denote by $\mathcal{N} = \{T : C \to C\colon T$ is nonexpansive$\}$. Obviously, $\mathcal{C} \subset \mathcal{N}$ but if $T \in \mathcal{N}$ then, in general, T does not possess a fixed point and, moreover, even if $Fix\,(T) \ne \emptyset$, the Picard iteration does not converge (to a fixed point of T). The nonexpansive mapping principle (Browder-Göhde-Kirk, 1965) asserts that if $(X, \|\cdot\|)$ is a uniformly convex Banach space, $C \subset X$ is bounded closed convex and $T \in \mathcal{N}$, then $Fix\,(T) \ne \emptyset$.

Under these assumptions, the Picard iteration $x_0 \in X$, $x_{n+1} = Tx_n$, $n \geq 0$ still does not converge or, if it converges, $\lim\limits_{n\to\infty} x_n \notin Fix\,(T)$. To ensure convergence, some additional assumptions are needed, like in the nonexpansive mapping principle (Petryshyn, 1966; Browder and Petryshyn, 1967), which asserts that if H is a Hilbert space, $C \subset H$ is bounded closed and convex, $T \in \mathcal{N}$ and T is demicompact (i.e., if $\{u_n\} \subset H$ bounded and $\{Tu_n - u_n\}$ is strongly convergent, there exists $\{u_{n_k}\}$ of $\{u_n\}$ which is strongly convergent), then $Fix\,(T) \neq \emptyset$ and the Krasnoselskij iteration $x_{n+1} = (1-\lambda)x_n + \lambda Tx_n, n \geq 0$ converges to $x^* \in Fix\,(T)$, for any $x_0 \in C$ and any $\lambda \in (0, 1)$. Other related contractive conditions:

- $T : C \to C$ is *pseudocontractive* if there exists $k < 1$ such that $\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)(x) - (I - T)(y)\|$, $x, y \in C$. Denote by $\mathcal{PC} = \{T : C \to C$: T is pseudocontractive$\}$.
- $T : C \to C$ is *strictly pseudocontractive* if $\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)(x) - (I - T)(y)\|$, $x, y \in C$. Denote by $\mathcal{SPC} = \{T : C \to C$: T is pseudocontractive$\}$.
- $T : C \to C$ is *asymptotically nonexpansive* if there exists $\{k_i\}_{i=1}^{\infty}$, $\lim k_i = 1$: $\|Tx - Ty\| \leq k_i\|x - y\|$, $x, y \in C$, $i = 1, 2, \ldots$. Denote by $\mathcal{AN} = \{T : C \to C$: T is asymptotically nonexpansive$\}$.
- $T : C \to C$ is called *quasi-nonexpansive* if for $p \in Fix\,(T) \neq \emptyset$, $\|Tx - p\| \leq \|x - p\|$, for all $x \in C$. Denote by $\mathcal{QN} = \{T : C \to C$: T is quasi-nonexpansive$\}$.

The class of demicontractive mappings (see Definition 2.2), which shall be denoted by $\mathcal{DC}$ in the sequel, is indeed very large, as shown by the next inclusions, valid for specific settings:

- $\mathcal{QN} \subset \mathcal{DC}$; $\mathcal{PC} \subset \mathcal{DC}$; $\mathcal{SPC} \subset \mathcal{DC}$;
- for $T \in \mathcal{N}$ with $Fix\,(T) \neq \emptyset$ holds $T \in \mathcal{QN} \Rightarrow T \in \mathcal{DC}$;
- $\mathcal{C} \subset \mathcal{DC}$; $\mathcal{AN} \subset \mathcal{DC}$ etc.

We end this paper by recalling Măruşter's demicontractive mapping principle ([7], 1977).

Theorem 3.1. *Let H be a Hilbert space, $C \subset H$ closed and convex, $T : C \to C$, $T \in \mathcal{DC}$ with constant λ. If $I - T$ is demiclosed at* 0 *($S : C \to C$ is said to be demiclosed at* 0 *in C if $\{u_k\}$ is a sequence in C which converges weakly to $u \in C$, and if $\{Su_k\}$ converges strongly to zero, then $Su = 0$).*

Then the Mann iteration $x_{n+1} = (1 - t_k)x_n + t_kTx_n, n \geq 0$, where $0 < a \leq t_k \leq b < 2\lambda$, converges to $x^* \in Fix\,(T)$, for any $x_0 \in C$.

Hicks and Kubicek's demicontractive mapping principle ([5], 1977) is not so general as Măruşter's demicontractive mapping principle ([7], 1977), due to the strong assumptions on the parameter sequence $\{t_k\}$ involved in the Mann recurrence relation:

Theorem 3.2. *Let H be a Hilbert space, $C \subset H$ closed convex $T : C \to C$, $T \in \mathcal{DC}$ with constant k. If $I - T$ is demiclosed at 0, then the Mann iteration $x_{n+1} = (1 - t_k)x_n + t_k T x_n, n \geq 0$, where $t_k \to t$ and $0 < t < 1 - k$, converges to $x^* \in Fix\,(T)$, for any $x_0 \in C$.*

4 Conclusions

The concept of demicontractive mappings, which is essentially due to Ştefan Măruşter ([6], [7]), is of crucial importance in the iterative approximation of fixed points. There is an impressive literature devoted to this area to which have contributed renown mathematicians like C.E. Chidume, P.E. Maingé, S.S. Chang, S.M. Kang, L.W. Liu, Y.J. Cho, S. Suantai, M.O. Osilike, N. Hussain, W.S. Du, P. Kumam, Y. Shehu, A. Rafiq, M. Eslamian, J.G. Peng, L.J. Qin, Y.C. Tang, L. Yang, etc.

To see how actual the interest for the study of demicontractive mappings is, we searched in Scopus and found 60 papers, one of which is published in 2018; 10 papers published in 2017; 9 papers published in 2016; 10 papers published in 2015 and 5 papers published in 2014.

In Web of Science we found 60 papers as well, one of which published in 2018; 12 papers published in 2017; 7 papers published in 2016; 10 papers published in 2015 and 4 papers published in 2014. When searching in Zentralblatt MATH, we found 83 documents indexed with the most recent published as follows: 2017 (5 articles); 2016 (7); 2015 (11); 2014 (6); 2013 (9); 2012 (8); 2011 (4); 2010 (4); 2009 (1); 2008 (10).

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