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A CONVERGENCE THEOREM FOR SOME NEWTON TYPE METHODS UNDER WEAK SMOOTHNESS CONDITIONS

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Abstract. It is shown that some Newton type iterative methods converge to the unique solution of the scalar nonlinear equation $f(x) = 0$, under weak smoothness conditions, involving only the function f and its first derivative f'.

1. Preliminaries

When modelling a certain physical phenomenon, we are often led to solve a certain nonlinear equation for which we do not have exact methods. So, we need to apply an appropriate *iterative* method. Newton's method or Newton-Raphson method, as it is generally called in the case of scalar equations $f(x) = 0$, is one of the most used iterative procedures for solving such nonlinear equations. It is defined by the iterative sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n \geq 0, \tag{1}$$

under appropriate assumptions on f and its first derivatives. Notice that there is a close connection between Newton type iterative methods and fixed point theory, in the sense that (1) can be also viewed as the sequence of successive approximations of the Newton iteration function

$$G(x) = x - \frac{f(x)}{f'(x)},$$

and moreover, under appropriate conditions, α is a solution of the equation $f(x) = 0$ if and only if α is a fixed point of the iteration function G.

There exist several convergence theorems in literature for the Newton's method, see for example [13], [14], [16], which, in order to ensure a quadratic convergence for the iterative process (1), are requiring strong smoothness assumptions, that involve f, f' and f''. These theorems usually also provide appropriate error estimates.

Theorem 1. ([13]) *Let $f : [a, b] \to \mathbb{R}$, $a < b$, be a function such that the following conditions are satisfied*
1) $f(a)\, f(b) < 0$; 2) $f \in C^2[a, b]$ and $f'(x)\, f''(x) \neq 0$, $x \in [a, b]$;

Then the sequence $\{x_n\}$, *defined by (1) and* $x_0 \in [a,b]$, *converges to* α, *the unique solution of* $f(x)=0$ *in* $[a,b]$, *and the following estimation*

$$|x_n-\alpha| \le \frac{M_2}{2m_1}\left|x_n-x_{n-1}\right|,\ n\ge 1, \tag{2}$$

holds, where

$$m_1=\min_{x\in[a,b]}\left|f'(x)\right| \text{ and } M_2=\max_{x\in[a,b]}\left|f''(x)\right|.$$

For concrete applications, Theorem 1 is widely used but there exist more general results, based on weaker smoothness conditions. We state here such a result, due to Ostrowski ([15], Theorem 7.2, pp. 60), based on weaker conditions on f but still involving the second derivative f''.

Theorem 2. ([15]) *Let* $f(x)$ *be a real function of the real variable* x, $f(x_0)f'(x_0)\neq 0$, *and put* $h_0=-f(x_0)/f'(x_0)$, $x_1=x_0+h_0$.

Consider the interval $I_0=[x_0,x_0+2h_0]$ *and assume that* $f''(x)$ *exists in* I_0, *that* $\max_{x\in I_0}|f''(x)|=M_2$ *and*

$$2\left|h_0\right|M_2\le\left|f'(x_0)\right|.$$

Then the sequence $\{x_n\}$ *given by (1) lie in* I_0 *and* $x_n\to\alpha$ $(n\to\infty)$, *where* α *is the unique zero of* f *in* I_0.

The smoothness assumptions in Theorem 2 are still very sharp, as shown by the next Example.

Example 1. ([2]) Let $f:[-1,1]\to\mathbb{R}$ be given by $f(x)=-x^2+2x$, if $x\in[-1,0)$, and $f(x)=x^2+2x$, if $x\in[0,1]$. The Newton iteration (1) converges to the unique solution of $f(x)=0$ in $[-1,1]$ but Theorem 2 cannot be applied, because f'' does not exist in $0\in I_0=[-1,1]$.

In a series of papers [1] - [11], the last author obtained more general convergence theorems for what was called there the *extended Newton's method*, for both scalar equations ([1] - [8], [10] - [11]) and n-dimensional equations [9], theorems that can be applied to weakly smooth functions, including the function in the previous example. The term *extended Newton method* was adopted in view of the fact that the iterative process (1) has been extended from $[a,b]$ to the whole real axis $\mathbb{R}$, in order to cover possible overflowing of $[a,b]$ at a certain step. A sample scalar variant of these results is contained in the following theorem.

Theorem 3. ([4]-[5]) *Let* $f:[a,b]\to\mathbb{R}$, $a<b$, *be a function such that the following conditions are satisfied*

(f_1) $f(a)\,f(b)<0$;

(f_2) $f\in C^1[a,b]$ *and* $f'(x)\neq 0$, $x\in[a,b]$;

(f_3) $2m>M$, *where*

$$m=\min_{x\in[a,b]}\left|f'(x)\right| \text{ and } M=\max_{x\in[a,b]}\left|f'(x)\right|. \tag{3}$$

Then the Newton iteration $\{x_n\}$, defined by (1) and $x_0 \in [a,b]$, converges to α, the unique solution of $f(x) = 0$ in $[a,b]$, and the following estimation

$$|x_n - \alpha| \leq \frac{M}{m}\left|x_n - x_{n+1}\right|,\ n \geq 0, \tag{4}$$

holds.

A slightly more general variant of Theorem 3 has been obtained in ([4], Theorem 5), by replacing condition (f_3) by the next one

$$(f_3') \quad 2m \geq M.$$

All proofs in [2], [4] - [7] are based on a rather classical technique, which focuses on the behavior of the Newton sequence (1). In an other paper [3], without large circulation, the last author succeeded to prove Theorem 3 using an elegant fixed point argument.

Very recently, Sen at all [17] have extended Theorem 3 to the case of a Newton-like iteration of the form

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + M_1 f(x)},\ n \geq 0, \tag{5}$$

where $M_1 f(x) = \operatorname{sgn} f'(x) \cdot M$, with M defined by (3).

This result was then extended by Sen et all [18] to the n-dimensional case. In both cases an *extended* Newton-like algorithm was used.

In a very recent paper [12], we obtained a convergence theorem for the process (5), by means of a fixed point argument, under the same general assumptions like those in Theorem 3, that involve only f and its first derivative f'.

It is the main aim of this paper to obtain, by using a similar technique of proof, a unitary convergence theorem for several iterative Newton type methods like *regula falsi* method, the modified Newton's method or Steffensen's method.

2. The convergence theorem

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ $(a, b \in \mathbb{R}, a < b)$, be a function such that the following conditions are satisfied*
(f_1) $f(a)f(b) < 0$; (f_2) $f \in C^1[a,b]$ *and* $f'(x) \neq 0$, $x \in [a,b]$;
(f_3) $2m > M$, *where*

$$m = \min_{x\in[a,b]} |f'(x)| \text{ and } M = \max_{x\in[a,b]} |f'(x)|.$$

Assume $g : [a,b] \to \mathbb{R}$ satisfies the following conditions
(g_1) $g \in C[a,b]$, $g(x) > 0$, $x \in [a,b]$; (g_2) $\max_{x\in[a,b]} g(x) \leq \frac{2m}{M}$;
(g_3) *The sequence $\{x_n\}$ given by*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} g(x_n),\ n \geq 0, \tag{6}$$

remains in $[a,b]$, for all $x_0 \in [a,b]$.

Then the Newton type iteration $\{x_n\}$ *converges to* α*, the unique solution of* $f(x)=0$ *in* $[a,b]$*, and the following stopping inequality*

$$|x_n-\alpha| \leq \frac{M}{k\,m}\left|x_n-x_{n+1}\right|,\, n\geq 0, \tag{7}$$

holds, where $k=\min\limits_{x\in[a,b]} g(x)$.

Proof. By (f_1) and (f_2) it follows that the equation $f(x)=0$ has a unique solution α in (a,b). Since $f(\alpha)=0$, we get

$$x_{n+1}-\alpha = x_n-\alpha-\frac{f(x_n)-f(\alpha)}{f'(x_n)}g(x_n),\; n\geq 0,$$

which by the mean value theorem yields

$$x_{n+1}-\alpha=\left[1-\frac{f'(c_n)}{f'(x_n)}g(x_n)\right](x_n-\alpha),\; n\geq 0, \tag{8}$$

where $c_n=\alpha+\theta(x_n-\alpha),\ 0<\theta<1$.

In a similar way we obtain

$$x_{n+1}-x_n=\frac{f'(c_n)}{f'(x_n)}g(x_n)(x_n-\alpha),\; n\geq 0. \tag{9}$$

Using (f_2), it results that f' preserves sign on $[a,b]$. By (g_1) and (g_2) we then get

$$1-\frac{f'(c_n)}{f'(x_n)}g(x_n)<1,\ \ n\geq 0 \tag{10}$$

and

$$\frac{f'(c_n)}{f'(x_n)}g(x_n)\leq Mg(x_n),\ \ n\geq 0$$

which leads to the conclusion that

$$1-\frac{f'(c_n)}{f'(x_n)}g(x_n)>-1,\ \ n\geq 0. \tag{11}$$

If we denote

$$c_2=\min_{x\in[a,b]}\left[1-\frac{M}{m}g(x)\right],$$

then we have that $c_2>-1$. Therefore, if we denote

$$A=\max\left\{|c_2|\,,\,\left|1-\frac{mk}{M}\right|\right\}$$

then it follows that $0\leq A<1$. Now, by combining (8), (10) and (11), we obtain

$$|x_{n+1}-\alpha|\leq A\,|x_n-\alpha|\,,\, n\geq 0,$$

which by induction yields

$$|x_{n+1}-\alpha|\leq A^n\,|x_0-\alpha|\,,\, n\geq 0,$$

an inequality which shows that $\{x_n\}$ converges to α.

We now use (9) and immediately get the desired estimation (7). □

Note that, we did not use explicitly any fixed point argument in proving Theorem 4. However, from the above proof it follows that the Newton type iteration function is a quasi-contraction, see [12].

3. Particular cases

1) If $g \equiv 1$, then by Theorem 4 we obtain a convergence theorem for the classical Newton-Raphson method, under weak differentiability conditions. Note that, under such weak differentiability assumptions, it is no guarantee that the Newton type sequence $\{x_n\}$ given by (6) lies in $[a, b]$ at any step n. For this reason we had to consider the assumption g_3 in Theorem 4.

However, in the papers [1], [2], [4], [5], [7]-[10], all dealing with Newton's method, it was not necessarily to assume explicitly that the Newton iteration lies in the interval $[a, b]$ at each step, even if we worked under the same weak smoothness conditions.

The explanation comes from the fact that we actually used the so called *extended Newton method*, which is actually the usual Newton's method extended to the whole real axis, and which was able, each time a certain iteration went out from $[a, b]$, to send it back in $[a, b]$ at the very next step.

2) If

$$g(x) = \frac{f'(x)}{f(x) - f(b)} (x - b),$$

then by Theorem 4 we obtain a convergence theorem under weak smoothness conditions for the *regula falsi* method.

3) If

$$g(x) = \frac{f'(x_0)}{f(x)},$$

where x_0 is the first approximation, then by Theorem 4 we obtain a convergence theorem under weak smoothness conditions for the modified Newton method.

4) If

$$g(x) = \frac{f(x)f'(x)}{f(x + f(x)) - f(x)},$$

then by Theorem 4 we obtain a convergence theorem under weak smoothness conditions for the Steffensen's method.

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