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Coincidence point theorem and common fixed point theorem for nonself single-valued almost contractions

by

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Abstract

Let X be a Banach space, K a nonempty closed subset of X and let $T, S : K \to X$ be two nonself almost contractions. In this paper, we prove the existence of coincidence points and common fixed points of almost contractions T and S. The main result shows that if $S(K)$ is closed, the pair (T, S) has property (M') and T and S satisfy some suitable conditions, then T and S have a unique common fixed point in K. This theorem generalizes several fixed point theorems for nonself mappings and also extend many great results in the fixed point theory of self mappings to the case on nonself mappings. Also, we give an example to support the validity of our results.

Key Words: Banach space, nonself almost contraction, weakly compatible mapping, coincidence point, common fixed point, property (M').
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1 Introduction

A great number of the most important nonlinear problems of applied mathematics reduce to finding a solution of a given equation which in turn may be reduced to searching for the fixed points of a certain mapping or the common fixed points of two mappings. This explains why the study of fixed points and common fixed points of mappings satisfying some certain contractive conditions attracted many mathematicians and encouraged an impressive output in the last four decades.

Most of the research in metric fixed point theory deals with single-valued self mappings $T : X \to X$ and multi-valued self mappings $T : X \to P(X)$ satisfying a certain contraction type condition, where X is a set endowed with a certain metric structure (metric space, convex metric space, Banach space etc.). These results are mainly generalizations of the Banach contraction principle [9], which can be shortly stated as follows.

Theorem 1. *Let (X, d) be a complete metric space and $T : X \to X$ a contraction, i.e., a map satisfying*

$$d(Tx, Ty) \leq \alpha \cdot d(x, y), \quad \text{for all } x, y \in X, \tag{1.1}$$

where $0 < \alpha < 1$ is a constant. Then T has a unique fixed point in X, say x^*, and the Picard iteration $\{T^n x_0\}$ converges to x^* for all $x_0 \in X$ (that is, T is a Picard operator).

The Banach fixed point theorem is one of the most useful research works in nonlinear analysis, and has many applications in solving nonlinear functional equations, optimization problems, variational inequalities etc., by transforming them in the form of fixed point problem. However, under the present form it has at least two drawbacks: first, the contraction condition (1.1) forces T to be continuous and, secondly, the condition $T(X) \subset X$ makes it not applicable to most of the nonlinear problems where the associated operator T is actually a nonself operator.

This is the explanation why, in continuation and fulfillment to the plentiful fixed point theory for self mappings, produced in the last 50 years, it was also a great and challenging research topic to obtain fixed point theorems for nonself mappings.

In 1972, Assad and Kirk [8] extended Banach contraction mapping principle to nonself multi-valued contraction mappings $T : K \to P(X)$ in the case (X, d) is a convex metric space in the sense of Menger and K is a nonempty closed subset of X such that T maps ∂K (the boundary of K) into K. Next, in 1976, by using an alternative and weaker condition, by if T is metrically inward, Caristi [20] has shown that any nonself single-valued contraction has a fixed point. Later, in 1978, Rhoades [35] proved a fixed point result in Banach spaces for single-valued nonself mapping satisfying the following contraction condition:

$$d(Tx, Ty) \leq \lambda\ max \left\{ \frac{d(x, y)}{2}, d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{1 + 2\lambda} \right\}, \tag{1.2}$$

for all $x, y \in K$, where $0 < \lambda < 1$.

Notice that although the class of mappings satisfying (1.2) is large enough to include some discontinuous mappings, it however does not include contraction mappings satisfying (1.1) for $\frac{1}{2} \leq \lambda < 1$.

A more general result, which also solved a very hard problem that was open for more than 20 years, has been obtained by Ćirić [24], who considered instead of (1.2) the quasi-contraction condition previously introduced and studied by himself in [23]:

$$d(Tx, Ty) \leq \lambda\ max \left\{ d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx) \right\}, \tag{1.3}$$

for all $x, y \in K$, where $0 < \lambda < 1$. More recently, Ćirić et al. [25] have considered the following contraction condition which is more general than (1.2) and (1.3):

$$d(Tx, Ty) \leq max \left\{ \varphi(d(x, y)), \varphi(d(x, Tx)), \varphi(d(y, Ty)), \varphi(d(x, Ty)), \varphi(d(y, Tx)) \right\}, \tag{1.4}$$

for all $x, y \in K$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a certain comparison function.

For some other fixed point results for nonself mappings, see [4–7], [15], [32] and Problem 5 in [36].

Furthermore, the first author [10], introduced a new class of self mappings (usually called weak contractions, almost contractions or Berinde operators) that satisfy a simple but more general contraction condition that includes most of the conditions in Rhoades classification [34]. The corresponding fixed point theorems, established mainly in [10], have two important features that differentiate them from similar results in literature: 1) the fixed points set of almost contractions is not a singleton, in general; and 2) the fixed points of almost contractions can be obtained by means of Picard iteration, like in the case of Banach contractions and, moreover, the error estimate is of the same form as in the case of contraction mapping principle (this motivated the term of "almost contractions").

In 2010, V. Berinde [12] proved the existence of coincidence points and common fixed points of noncommuting self almost contractions in metric spaces. Moreover, a method for approximating the coincidence points or the common fixed points was also constructed. These results generalized, extended and unified several clasical and very recent related results in literature.

Next, in 2013, V. Berinde and M. Păcurar [17] obtained fixed point theorems for nonself almost contractions. This result generalized several fixed point theorems for nonself mappings and also extended several important results in the fixed point theory of self mappings to the case on nonself mappings.

Motivated by these works, by using the idea given by [12] and [17], the purpose of this paper is to prove the existence of coincidence points and common fixed points of nonself almost contractions in a nonempty closed subset of a Banach space. Our results extend the results in [17] and several associated research works. In addition, we also illustrate an example to support our theorems.

2 Preliminaries

We can see that any contraction mapping satisfying (1.1) is continuous. In 1968, Kannan [30] proved a fixed point theorem which extends Theorem 1 to mappings that need not be continuous on X (but are continuous at their fixed point).

From Kannan's theorem, a lot of research works were devoted to getting fixed point or common fixed point theorems for various classes of contractive type conditions that do not require the continuity of T.

For Chatterjea [21] is based on a condition similar to Kannan fixed point theorem.

On the other hand, Zamfirescu [41], in 1972, obtained a very interesting fixed point theorem which gather together all three contractive conditions mentioned above, that is, condition (1.1) of Banach, condition of Kannan and condition of Chatterjea, in a rather unexpected way: if T is such that, for any pair $x, y \in X$, at least one of the conditions (1.1), Kannan's condition and Chatterjea's condition holds, then T is a Picard operator. Notice that considering conditions (1.1), Kannan's condition and Chatterjea's condition all together is not trivial since, as shown later by Rhoades [34], the contractive conditions (1.1), Kannan's condition and Chatterjea's condition are independent of each other.

These fixed point results were then complemented by coresponding results about the existence of common fixed points. Then in 1976, Jungck [28] proved a common fixed point theorem for commuting maps, so extending Theorem 1. In the same spirit, recently M. Abbas and G. Jungck [1] obtained coincidence and common fixed point theorems for the class of Banach contractions, Kannan contractions and Chatterjea contractions, respectively, in cone metric spaces, without making use of the commutative property, but based on the so called concept of weakly compatible mappings, introduced by Jungck [29].

In 2009, V. Berinde [11] proved a common fixed point version of Zamfirescu's fixed point theorem in metric spaces, including also the error and rate of convergence estimates.

However, Zamfirescu's fixed point theorem [41] is a particular case of the next fixed point theorem [10].

Theorem 2. ([10, Theorem 2.1]) *Let (X, d) be a complete metric space and $T : X \to X$ an almost contraction, that is, a mapping for which there exist a constant $\delta \in [0, 1)$ and some*

$L \geq 0$ *such that*

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx), \quad for\ all\ x, y \in X. \tag{2.1}$$

Then

1) $Fix(T) = \{x \in X : Tx = x\} \neq \emptyset$;

2) *For any* $x_0 \in X$, *Picard iteration* $\{x_n\}_{n=0}^{\infty}$, $x_n = T^n x_0$, *converges to* $x^* \in Fix(T)$;

3) *The following estimate holds*

$$d(x_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta}\, d(x_n, x_{n-1}), \quad n = 0, 1, 2, ...; i = 1, 2, ... \tag{2.2}$$

Let us recall that a mapping T having properties 1) and 2) above is said to be a *weakly Picard operator*.

In 2010, the main results of paper [12] generalized Theorem 2 to coincidence points and common fixed points of two noncommuting self almost contractions in metric spaces.

In the same year, V. Berinde [13] extended paper [11] by studying a common fixed point version of Zamfirescu's fixed point theorem in cone metric spaces.

Later, the same author [14] investigated the existence of coincidence points and common fixed points of two noncommuting self almost contractions in cone metric spaces.

For some other fixed point, coincidence point and common fixed point theorems, see also [2], [3], [16], [18], [19], [26], [27], [31], [33], [37], [38], [39] and [40].

Moreover, recently V. Berinde and M. Păcurar [17] extended Theorem 2 to the case of nonself almost contractions. This result extended several important fixed point theorems of Banach [9], Kannan [30], Chatterjea [21], Zamfirescu [41] and Ćirić [22].

Therefore the aim of this paper is to extend and unify all the results in [17, Theorem 3.3 and Theorem 3.6] and several other related results in literature, by proving general results dealing with the existence and the uniqueness of coincidence points and common fixed points of two nonself almost contractions.

By using a concept of [1] and [29] we give two definitions and a result that will be used in our main results.

Definition 1. Let X be a metric space, K a nonempty closed subset of X and let $T, S : K \to X$ be two nonself mappings. If there exists $x \in K$ such that $Tx = Sx$, then x is called a *coincidence point* of T and S, and $y = Tx = Sx$ is called a *point of coincidence* of T and S. If $Tx = Sx = x$, then x is called a *common fixed point* of T and S.

Definition 2. Let X be a metric space, K a nonempty closed subset of X and let $T, S : K \to X$ be two nonself mappings. The pair of mappings T and S is said to be *weakly compatible* if they commute at their coincidence points.

Proposition 1. Let X be a metric space, K a nonempty closed subset of X and let T and $S : K \to X$ be weakly compatible nonself mappings. If T and S have a unique point of coincidence $y \in K$, then y is the unique common fixed point of T and S.

Proof. Similarly to Proposition 1.4 in [1]. □

3 Main Results

In this section, we prove the existence of coincidence points and common fixed point of nonself almost contractions T and S.

Let X be a Banach space, K a nonempty closed subset of X and $T, S : K \to X$ two nonself mappings. Let $S(K)$ be a closed subset of X. Let $X_{ST} = \{x \in K \mid Tx \notin S(K)\}$. For $x \in X_{ST}$, we can always choose $y \in \partial(S(K))$ such that

$$y = (1-\lambda)Sx + \lambda Tx, (0 < \lambda < 1) \tag{3.1}$$

and denote by Y_x the set of all points $y \in \partial(S(K))$ satisfying (3.1). We see that

$$d(Sx, Tx) = d(Sx, y) + d(y, Tx).$$

where we denoted $d(x, y) = \| x - y \|$.

In general, the set Y_x of points satisfying condition (3.1) may contain more than one element. In this circumstance we will need the following concept.

Definition 3. Let X be a Banach space, K a nonempty closed subset of X and $T, S : K \to X$ two nonself mappings. Let $S(K)$ be a closed subset of X. Let $X_{ST} = \{x \in K \mid Tx \notin S(K)\}$. For $x \in X_{ST}$, let $y \in \partial(S(K))$ be the corresponding elements given by (3.1). If, for any $x \in X_{ST}$, the inequality

$$d(Sy', Ty') \leq d(Sx, Tx) \tag{3.2}$$

is satisfied for at least one point $y \in Y_x$ where $y = Sy'$, with $y' \in K$, then we say that the pair (T, S) has property (M').

Theorem 3. *Let X be a Banach space, K a nonempty closed subset of X and let $T, S : K \to X$ be two nonself mappings for which there exist two constants $\delta \in (0, 1)$ and $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta \cdot d(Sx, Sy) + Ld(Sy, Tx), \quad \textit{for all } x, y \in K. \tag{3.3}$$

If $S(K)$ is closed, the pair (T, S) has property (M') and satisfies the condition: for any $x \in K$,

$$\textit{if } Sx \in \partial(S(K)), \textit{ then } Tx \in S(K), \tag{3.4}$$

then T and S have a point of coincidence in X.

Proof. Let $y \in \partial(S(K))$. Then there is $x_0 \in K$ such that $y = Sx_0$. By (3.4) we have $Tx_0 \in S(K)$. Then there is $x_1 \in K$ such that $Sx_1 = Tx_0$. Next, if $Tx_1 \in S(K)$, then there exists $x_2 \in K$ such that $Sx_2 = Tx_1$. If $Tx_1 \notin S(K)$, by property (M') we can choose $y_1 \in Y_{x_1}$ such that $y_1 \in \partial(S(K))$, $y_1 = Sx_2$ for some $x_2 \in K$ which

$$d(Sx_2, Tx_2) \leq d(Sx_1, Tx_1)$$

and

$$y_1 = Sx_2 = (1-\lambda_1)Sx_1 + \lambda_1 Tx_1, \quad \text{for some } \lambda_1 \in (0, 1).$$

Note that $Sx_2 \neq Tx_1$. Continuing in this manner, we get a sequence $\{Sx_n\}$ such that

(i) $Sx_n = Tx_{n-1}$, if $Tx_{n-1} \in S(K)$;

(ii) $Sx_n = (1-\lambda_{n-1})Sx_{n-1} + \lambda_{n-1}Tx_{n-1} \in \partial(S(K))$ $(0 < \lambda_{n-1} < 1)$, if $Tx_{n-1} \notin S(K)$.

Let us denote

$$P = \{Sx_k \in \{Sx_n\} \mid Sx_k = Tx_{k-1}\}$$

and

$$Q = \{Sx_k \in \{Sx_n\} \mid Sx_k \neq Tx_{k-1}\}.$$

We see that $\{Sx_n\} \subset S(K)$ and that, if $Sx_k \in Q$, then both Sx_{k-1} and Sx_{k+1} belong to the set P. Furthermore, by virtue of (3.4), we cannot have two consecutive terms of $\{Sx_n\}$ in the set Q (but we can have two consecutive terms of $\{Sx_n\}$ in the set P).

We claim that $\{Sx_n\}$ is a Cauchy sequence. To prove this, we have to discuss the following three distinct cases: Case I. $Sx_n,\ Sx_{n+1} \in P$; Case II. $Sx_n \in P,\ Sx_{n+1} \in Q$; Case III. $Sx_n \in Q,\ Sx_{n+1} \in P$;

Case I. $Sx_n,\ Sx_{n+1} \in P$.

In this case we get $Sx_n = Tx_{n-1}$, $Sx_{n+1} = Tx_n$ and by (3.3) we have

$$d(Sx_n, Sx_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \delta \cdot d(Sx_{n-1}, Sx_n) + Ld(Sx_n, Tx_{n-1})$$

that is,

$$d(Sx_{n+1}, Sx_n) \leq \delta \cdot d(Sx_n, Sx_{n-1}). \tag{3.5}$$

Case II. $Sx_n \in P,\ Sx_{n+1} \in Q$.

In this case we get $Sx_n = Tx_{n-1}$ but $Sx_{n+1} \neq Tx_n$ and

$$d(Sx_n, Sx_{n+1}) + d(Sx_{n+1}, Tx_n) = d(Sx_n, Tx_n).$$

Therefore

$$d(Sx_n, Sx_{n+1}) \leq d(Sx_n, Tx_n) = d(Tx_{n-1}, Tx_n)$$

and then by using (3.3) we obtain

$$\begin{aligned} d(Sx_n, Sx_{n+1}) \leq d(Tx_{n-1}, Tx_n) &\leq \delta \cdot d(Sx_{n-1}, Sx_n) + Ld(Sx_n, Tx_{n-1}) \\ &= \delta \cdot d(Sx_n, Sx_{n-1}), \end{aligned}$$

which satisfies again inequality (3.5).

Case III. $Sx_n \in Q,\ Sx_{n+1} \in P$.

In this situation, we get $Sx_{n-1} \in P$. Since the pair (T, S) has property (M'), it follows that

$$d(Sx_n, Sx_{n+1}) = d(Sx_n, Tx_n) \leq d(Sx_{n-1}, Tx_{n-1}).$$

Since $Sx_{n-1} \in P$, we get $Sx_{n-1} = Tx_{n-2}$ and by (3.3) we have

$$\begin{aligned} d(Tx_{n-2}, Tx_{n-1}) &\leq \delta \cdot d(Sx_{n-2}, Sx_{n-1}) + Ld(Sx_{n-1}, Tx_{n-2}) \\ &= \delta \cdot d(Sx_{n-2}, Sx_{n-1}), \end{aligned}$$

which yields that

$$d(Sx_n, Sx_{n+1}) \leq \delta \cdot d(Sx_{n-2}, Sx_{n-1}). \tag{3.6}$$

From above three cases, and (3.5) and (3.6), we obtain a sequence $\{Sx_n\}$ satisfying the inequality

$$d(Sx_n, Sx_{n+1}) \leq \delta \; max\{d(Sx_{n-2}, Sx_{n-1}), d(Sx_{n-1}, Sx_n)\}, \tag{3.7}$$

for all $n \geq 2$. From (3.7), by induction, we can show that

$$d(Sx_n, Sx_{n+1}) \leq \delta^{[n/2]} \; max\{d(Sx_0, Sx_1), d(Sx_1, Sx_2)\},$$

for all $n \geq 2$, where $[n/2]$ denotes the greatest integer not exceeding $n/2$.

Moreover, for $m > n > N$,

$$d(Sx_n, Sx_m) \leq \sum_{i=N}^{\infty} d(Sx_i, Sx_{i-1}) \leq 2\frac{\delta^{[N/2]}}{1-\delta} \; max\{d(Sx_0, Sx_1), d(Sx_1, Sx_2)\},$$

which yields that $\{Sx_n\}$ is a Cauchy sequence.

Since $\{Sx_n\} \subset S(K)$ and $S(K)$ is closed, $\{Sx_n\}$ converges to some point x^* in $S(K)$.

Let $\{Sx_{n_k}\} \subset P$ be an infinite subsequence of $\{Sx_n\}$ (such a subsequence always exists). Since $x^* \in S(K)$, there is $p \in K$ such that $Sp = x^*$. We note that

$$d(Sp, Tp) \leq d(Sp, Sx_{n_k+1}) + d(Sx_{n_k+1}, Tp) = d(Sp, Sx_{n_k+1}) + d(Tx_{n_k}, Tp)$$

and

$$d(Tx_{n_k}, Tp) \leq \delta \cdot d(Sx_{n_k}, Sp) + Ld(Sp, Tx_{n_k}),$$

hence, we have

$$d(Sp, Tp) \leq (1 + L)d(Sp, Sx_{n_k+1}) + \delta \cdot d(Sx_{n_k}, Sp), \tag{3.8}$$

for all $k \geq 0$. Taking $k \to \infty$ in (3.8), we obtain

$$d(Sp, Tp) = 0,$$

which shows that $Sp = Tp$, that is p is a coincidence point of T and S and x^* is a point of coincidence of T and S. □

Theorem 4. *Let X be a Banach space, K a nonempty closed subset of X and let $T, S : K \to X$ be two nonself mappings satisfying* (3.3) *for which there exist a constant $\theta \in (0, 1)$ and some $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \theta d(Sx, Sy) + L_1 d(Sx, Tx), \quad \textit{for all } x, y \in K. \tag{3.9}$$

If $S(K)$ is closed, the pair (T, S) has property (M') and satisfies the condition (3.4)*, then T and S have a unique point of coincidence in X. Moreover, if T and S are weakly compatible and a unique point of coincidence of T and S is in K, then T and S have a unique common fixed point in K.*

Proof. By Theorem 3, T and S have a point of coincidence, say $x^* = Tp = Sp$ for some $p \in K$. Now, let us show that T and S really have a unique point of coincidence. Suppose there exists $q \in K$ such that $Tq = Sq$. Then by (3.9) we have

$$d(Sq, Sp) = d(Tq, Tp) \leq \theta d(Sq, Sp) + L_1 d(Sq, Tq) = \theta d(Sq, Sp),$$

So $(1-\theta)d(Sq, Sp) \leq 0$, which implies $d(Sq, Sp) = 0$, that is $Sq = Sp = x^*$. Hence T and S have a unique point of coincidence, x^*.

Next, suppose that T and S are weakly compatible and $x^* \in K$. By Proposition 1, it follows that x^* is a unique common fixed point of T and S. □

Example 1. Let X be the set of real numbers with the usual norm, $K = [0,1]$ be the unit interval and let $T, S : [0,1] \to \mathbb{R}$ be given by

$$T(x) = \begin{cases} \frac{1}{2}x & \text{if} \quad x \in [0, \frac{1}{2}), \\ -\frac{1}{2} & \text{if} \quad x = \frac{1}{2}, \\ \frac{1}{2}x + \frac{1}{6} & \text{if} \quad x \in (\frac{1}{2}, 1] \end{cases}$$

and $S(x) = \frac{2}{3}x$ for all $x \in [0,1]$.

We see that $S(K) = S([0,1]) = [0, \frac{2}{3}] \subset [0,1] = K$ and $S(K)$ is closed. Moreover, $T(K) = [0, \frac{1}{4}) \cup (\frac{5}{12}, \frac{2}{3}] \cup \{-\frac{1}{2}\}$ and observe that $x = \frac{1}{2}$ is only one point in X_{ST} and $T(\frac{1}{2}) = -\frac{1}{2} \notin S(K)$ and $S(\frac{1}{2}) = \frac{1}{3}$. We note that $y = 0 \in \partial(S(K))$ such that $0 = (1-\lambda)S(\frac{1}{2}) + \lambda T(\frac{1}{2})$, for some $\lambda \in (0,1)$. Furthermore, $y = S(0)$ and $|S(0) - T(0)| = 0 \leq \left|S(\frac{1}{2}) - T(\frac{1}{2})\right|$. Therefore the pair (T, S) has property (M').

We also note that $S(0) = 0$ and $S(1) = \frac{2}{3}$ are only two points in $\partial(S(K))$ and we see that $T(0) = 0$, $T(1) = \frac{2}{3} \in S(K)$. Thus the pair (T, S) satisfies the condition (3.4).

Next, we show that T and S satisfy the condition (3.3). We will discuss the following possible 8 cases.

Case 1) $x, y \in [0, \frac{1}{2})$. Then $Tx = \frac{1}{2}x$, $Ty = \frac{1}{2}y$, $Sx = \frac{2}{3}x$ and $Sy = \frac{2}{3}y$. Then

$$|Tx - Ty| = \left|\frac{1}{2}x - \frac{1}{2}y\right| \leq \delta \left|\frac{2}{3}x - \frac{2}{3}y\right| + L\left|\frac{2}{3}y - \frac{1}{2}x\right| = \delta\,|Sx - Sy| + L\,|Sy - Tx|\,,$$

where $\delta \geq \frac{3}{4}$ and any $L \geq 0$.

Case 2) $x, y \in (\frac{1}{2}, 1]$. Then $Tx = \frac{1}{2}x + \frac{1}{6}$, $Ty = \frac{1}{2}y + \frac{1}{6}$, $Sx = \frac{2}{3}x$ and $Sy = \frac{2}{3}y$. Then

$$\begin{aligned} |Tx - Ty| = \left|\frac{1}{2}x + \frac{1}{6} - \frac{1}{2}y - \frac{1}{6}\right| = \left|\frac{1}{2}x - \frac{1}{2}y\right| &\leq \delta\left|\frac{2}{3}x - \frac{2}{3}y\right| + L\left|\frac{2}{3}y - \frac{1}{2}x - \frac{1}{6}\right| \\ &= \delta\,|Sx - Sy| + L\,|Sy - Tx|\,, \end{aligned}$$

where $\delta \geq \frac{3}{4}$ and any $L \geq 0$.

Case 3) $x \in [0, \frac{1}{2})$, $y \in (\frac{1}{2}, 1]$. Then $Tx = \frac{1}{2}x$, $Ty = \frac{1}{2}y + \frac{1}{6}$, $Sx = \frac{2}{3}x$ and $Sy = \frac{2}{3}y$. We note that

$$-\frac{2}{3} \leq \frac{1}{2}x - \frac{1}{2}y - \frac{1}{6} < -\frac{1}{6}$$

and

$$\frac{1}{12} < \frac{2}{3}y - \frac{1}{2}x \leq \frac{2}{3}.$$

It follows that

$$|Tx - Ty| = \left|\frac{1}{2}x - \frac{1}{2}y - \frac{1}{6}\right| \leq \frac{2}{3}$$

and

$$|Sy - Tx| = \left|\frac{2}{3}y - \frac{1}{2}x\right| > \frac{1}{12}.$$

Then $|Tx - Ty| \leq \frac{2}{3} \leq \delta\,|Sx - Sy| + L\,|Sy - Tx|$, where $L \geq 8$ and $\delta \in (0,1)$.

Case 4) $x \in (\frac{1}{2}, 1],\ y \in [0, \frac{1}{2})$. Then $Tx = \frac{1}{2}x + \frac{1}{6}$, $Ty = \frac{1}{2}y$, $Sx = \frac{2}{3}x$ and $Sy = \frac{2}{3}y$. We note that

$$\frac{1}{6} < \frac{1}{2}x + \frac{1}{6} - \frac{1}{2}y \leq \frac{2}{3}$$

and

$$-\frac{2}{3} \leq \frac{2}{3}y - \frac{1}{2}x - \frac{1}{6} < -\frac{1}{12}.$$

It follows that

$$|Tx - Ty| = \left|\frac{1}{2}x + \frac{1}{6} - \frac{1}{2}y\right| \leq \frac{2}{3}$$

and

$$|Sy - Tx| = \left|\frac{2}{3}y - \frac{1}{2}x - \frac{1}{6}\right| > \frac{1}{12}.$$

Then $|Tx - Ty| \leq \frac{2}{3} \leq \delta\,|Sx - Sy| + L\,|Sy - Tx|$, where $L \geq 8$ and $\delta \in (0,1)$.

Case 5) $x = \frac{1}{2},\ y \in [0, \frac{1}{2})$. Then $Tx = -\frac{1}{2}$, $Ty = \frac{1}{2}y$, $Sx = \frac{1}{3}$ and $Sy = \frac{2}{3}y$. Then

$$|Tx - Ty| = \left|-\frac{1}{2} - \frac{1}{2}y\right| < \frac{3}{4}$$

and

$$|Sy - Tx| = \left|\frac{2}{3}y + \frac{1}{2}\right| \geq \frac{1}{2}.$$

Hence $|Tx - Ty| < \frac{3}{4} \leq \delta\,|Sx - Sy| + L\,|Sy - Tx|$, where $L \geq \frac{3}{2}$ and $\delta \in (0,1)$.

Case 6) $x \in [0, \frac{1}{2}),\ y = \frac{1}{2}$. Thus $Tx = \frac{1}{2}x$, $Ty = -\frac{1}{2}$, $Sx = \frac{2}{3}x$ and $Sy = \frac{1}{3}$. Then

$$|Tx - Ty| = \left|\frac{1}{2}x + \frac{1}{2}\right| < \frac{3}{4}$$

and

$$|Sy - Tx| = \left|\frac{1}{3} - \frac{1}{2}x\right| > \frac{1}{12}.$$

Hence $|Tx - Ty| < \frac{3}{4} \leq \delta\,|Sx - Sy| + L\,|Sy - Tx|$, where $L \geq 9$ and $\delta \in (0, 1)$.

Case 7) $x = \frac{1}{2}$, $y \in (\frac{1}{2}, 1]$. Thus $Tx = -\frac{1}{2}$, $Ty = \frac{1}{2}y + \frac{1}{6}$, $Sx = \frac{1}{3}$ and $Sy = \frac{2}{3}y$. Therefore

$$|Tx - Ty| = \left|-\frac{1}{2} - \frac{1}{2}y - \frac{1}{6}\right| \leq \frac{7}{6}$$

and

$$|Sy - Tx| = \left|\frac{2}{3}y + \frac{1}{2}\right| > \frac{5}{6}.$$

Hence $|Tx - Ty| \leq \frac{7}{6} \leq \delta\,|Sx - Sy| + L\,|Sy - Tx|$, where $L \geq \frac{7}{5}$ and $\delta \in (0, 1)$.

Case 8) $x \in (\frac{1}{2}, 1]$, $y = \frac{1}{2}$. Thus $Tx = \frac{1}{2}x + \frac{1}{6}$, $Ty = -\frac{1}{2}$, $Sx = \frac{2}{3}x$ and $Sy = \frac{1}{3}$. Therefore

$$|Tx - Ty| = \left|\frac{1}{2}x + \frac{1}{6} + \frac{1}{2}\right| \leq \frac{7}{6}$$

and

$$|Sy - Tx| = \left|\frac{1}{3} - \frac{1}{2}x - \frac{1}{6}\right| > \frac{1}{12}.$$

Hence $|Tx - Ty| \leq \frac{7}{6} \leq \delta\,|Sx - Sy| + L\,|Sy - Tx|$, where $L \geq 14$ and $\delta \in (0, 1)$.

By concluding all possible cases, we summarize that T and S satisfy (3.3) with $\delta = \frac{3}{4}$ and $L = 14$.

Hence by Theorem 3 we can conclude that T and S have a point of coincidence in X. For this example, the points of coincidence of T and S are $0 = T(0) = S(0)$ and $\frac{2}{3} = T(1) = S(1)$. We also see that 0 is a common fixed point of T and S.

Moreover, we can show that T and S do not satisfy the condition (3.9) because

$$\left|T(0) - T\left(\frac{1}{2}\right)\right| = \frac{1}{2} \geq \frac{1}{3}\theta = \theta\left|S(0) - S\left(\frac{1}{2}\right)\right| + L\,|S(0) - T(0)|\,,$$

for any $\theta \in (0, 1)$ and $L \geq 0$.

4 Particular Cases and Conclusions

Our main theorems extend the results of V. Berinde and M. Păcurar [17] as follows.

Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a nonself mapping. If $x \in K$ is such that $Tx \notin K$, then we can always choose $y \in \partial K$ such that

$$y = (1 - \lambda)x + \lambda Tx, (0 < \lambda < 1) \tag{4.1}$$

and denote by Y the set of all points $y \in \partial K$ satisfying (4.1). We see that

$$d(x, Tx) = d(x, y) + d(y, Tx).$$

In general, the set Y of points satisfying condition (4.1) may contain more than one element. In this circumstance we will need the following concept.

Definition 4. Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a nonself mapping. Let $x \in K$ with $Tx \notin K$ and let $y \in \partial K$ be the corresponding elements given by (4.1). If, for any such element x, the inequality

$$d(y, Ty) \leq d(x, Tx) \tag{4.2}$$

is satisfied for at least one point $y \in Y$, then we say that T has property (M).

Corollary 1. ([17]) *Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a nonself almost contraction, that is, a mapping for which there exist two constants $\delta \in [0, 1)$ and $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx), \quad \text{for all } x, y \in K. \tag{4.3}$$

If T has property (M) and satisfies Rothe's boundary condition

$$T(\partial K) \subset K, \tag{4.4}$$

then T has a fixed point in K.

Proof. We take $S = I$, where $I : K \to K$ is the identity mapping. Then the inequality (4.3) becomes (3.3), property (M) becomes property (M') and condition (4.4) becomes (3.4). Therefore, by Theorem 3, T and S have a point of coincidence y in X. Thus, there exists $p \in K$ such that $y = Tp = Sp = p$. Hence T has a fixed point p in K. □

Corollary 2. ([17]) *Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a nonself almost contraction for which there exist $\theta \in (0, 1)$ and some $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \theta d(x, y) + L_1 d(x, Tx), \quad \text{for all } x, y \in K. \tag{4.5}$$

If T has property (M) and satisfies Rothe's boundary condition $T(\partial K) \subset K$, then T has a unique fixed point in K.

Proof. By taking $S = I$, where $I : K \to K$ is the identity mapping, all conditions of Theorem 4 are satisfied. By Theorem 4, T and S have a unique common fixed point p in K. Since S is an identity mapping, T has the unique fixed point p in K. □

Moreover, the following Corollaries can be obtained directly from our main results.

Corollary 3. *Let X be a Banach space and let $T, S : X \to X$ be two mappings for which there exist two constants $\delta \in (0,1)$ and $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta \cdot d(Sx, Sy) + Ld(Sy, Tx), \quad \text{for all } x, y \in X. \tag{4.6}$$

If $T(X) \subset S(X)$ and $S(X)$ is a complete subspace of X, then T and S have a point of coincidence in X.

Proof. The mappings $T, S : X \to X$ that satisfy (4.6) are the mappings that satisfy (3.3) in Theorem 3 in the case $K = X$. Since $S(X)$ is a complete subspace of X, $S(X)$ is closed. If $T(X) \subset S(X)$, then it is easy to see that the pair (T, S) has property (M') and satisfies the condition (3.4). Hence, by Theorem 3, T and S have a point of coincidence in X. □

Corollary 4. *Let X be a Banach space and let $T, S : X \to X$ be two mappings satisfying* (4.6) *for which there exist a constant $\theta \in (0,1)$ and some $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \theta d(Sx, Sy) + L_1 d(Sx, Tx), \quad \text{for all } x, y \in X. \tag{4.7}$$

If $T(X) \subset S(X)$ and $S(X)$ is a complete subspace of X, then T and S have a unique point of coincidence in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

Proof. The mappings $T, S : X \to X$ that satisfy (4.6) and (4.7) are the mappings in Theorem 4 that satisfy (3.3) and (3.9) in the case $K = X$. Since $S(X)$ is a complete subspace of X, $S(X)$ is closed. If $T(X) \subset S(X)$, then the pair (T, S) has property (M') and satisfies the condition (3.4). Hence, by Theorem 4, T and S have a unique point of coincidence in X.

Moreover, if T and S are weakly compatible, by Theorem 4, T and S have a unique common fixed point in X. □

Conclusions: Our main results are more general than that of [17] and other related results existing in literature. The proof of Theorem 3 is principally based on the hypothesis that the pair (T, S) has property (M') and satisfies the condition (3.4). These facts lead to the following open problems.

Open Problems:

1. We note that the property (M') is a sufficient condition for Theorem 3.
   Question: Is it a necessary condition?

2. It is also noted that all results in this paper are considered in a Banach space.
   Question: Can we extend our study in other spaces, such as a convex metric space?

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