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Dedicated to Maria S. Pop on her 60th anniversary

THE GENERALIZED RATIO TEST REVISITED

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Abstract. A new extension of the generalized ratio test [1]- [3], [5] is obtained by taking over an ideea from [8], where the ratio u_{n+1}/u_n is placed by the ratio u_{n+k}/u_n, with k fixed, $k \geq 1$.
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1. INTRODUCTION

A generalization of the well-known ratio or D'Alembert test for positive series was obtained in [1].

Theorem 1. *Let* $\sum\limits_{n=1}^{\infty} u_n$ *be a series of positive terms.*

(1) If there exist a convergent series of non-negative terms $\sum\limits_{n=1}^{\infty} v_n$, *and a constant number* N *such that*

$$\frac{u_{n+1}}{u_n + v_n} \leq q < 1 \quad \text{for,} \quad n \geq N, \tag{1}$$

then the series $\sum\limits_{n=1}^{\infty} u_n$ *is convergent;*

(2) If there exists a decreasing sequence of positive numbers such that for $n \geq N$ *(fixed) we have*

$$(i) \qquad u_n > v_n; \qquad (ii) \qquad \frac{u_{n+1} + v_n - v_{n+1}}{u_n} \geq 1,$$

then the series $\sum\limits_{n=1}^{\infty} u_n$ *is divergent.*

A typical exemple to show that the generalized ratio test applies wh[...] the ratio test fails is the pair $u_n = \frac{1}{n^2}, v_n = \frac{1}{n(n+1)}, \quad n \geq 1:$

$$\frac{u_{n+1}}{u_n + v_n} = \frac{n^2}{2n^2 + 3n + 1} < \frac{1}{2}, \ \forall \, n \geq 1.$$

As mentioned in [1]-[3], if we have a series $\sum\limits_{n=1}^{\infty} u_n$ for which the rati[...] test fails, then generally it is not easy to find a suitable "comparison" serie[...] $\sum\limits_{n=1}^{\infty} v_n$.

The ratio test is obtained from Theorem 1 for $v_n = 0, \ n \geq 1$.

Nurcombe [8] extended the same ratio test as follows:

Theorem 2. *Let* $\sum a_n$ *be a series of positive terms, and* k, *a fixed positive integer. If* $\lim \frac{a_{n+k}}{a_n} < 1 \ (> 1)$ *then* $\sum a_n$ *converges (diverges).*

The main aim of the present paper is to extend both Theorem 1 and Theorem 2.

2. A NEW EXTENSION OF THE GENERALIZED RATIO TEST

By restricting ourselves to the convergence part of tests, we can prove

Theorem 3. *Let* $\sum\limits_{n=1}^{\infty} u_n$ *be a series of positive terms and* $\sum\limits_{n=1}^{\infty} v_n$ *a convergent series of non-negative terms. If there exists a positive (constant) integer* k *such that*

$$\frac{u_{n+k}}{u_n + v_n} \leq q < 1, \ for \ n \geq N \ (fixed)$$

then the series $\sum\limits_{n=1}^{\infty} u_n$ *converges.*

Proof. Consider the k subseries,

$$\sum_{n=1}^{\infty} u_{1+nk}, \sum_{n=1}^{\infty} u_{2+nk}, \ldots, \sum_{n=1}^{\infty} u_{nk}, \qquad (2)$$

which collectively comprise all the terms of $\sum_{n=1}^{\infty} u_n$. Every subseries from converges by the generalized ratio test and, since the sum of series of ive terms is invariant under alterations in the order of the terms, $\sum u_n$ verges.

Remarks. 1) The proof given above is essentially adapted after [8];

For $k = 1$, Theorem 3 becomes Theorem 1 and for $v_n \equiv 0$, it becomes orem 2.

2) The divergence part of Theorem 3 may be now easily formulated, as as the corresponding version of another extension of the generalized ratio (Theorem 2, [3]).

Theorem 4. *Let $\sum_{n=1}^{\infty} u_n$ be a series of positive terms. If there exist a vergent series $\sum_{n=1}^{\infty} v_n$ and a sequence $(w_n)_{n\geq 1}$ satisfying*

i) $0 < w_n < 1$, $n \geq 1$;

ii) $w_{n+m} \leq w_n \cdot w_m$, $\forall n, m \geq 1$;

and such that

$$\frac{u_{n+1}}{u_n + v_n} \leq \frac{w_{n+1}}{w_n},\ n \geq 1 \tag{3}$$

then the series $\sum_{n=1}^{\infty} u_n$ is convergent.

3) For $w_n = q^n$, $0 \leq q < 1$, from (3) we obtain (1).

4) As the generalized ratio test has at least the power of the Raabe test e [5], Proposition 1), we suggest to the reader to compare all the tests the ratio test family given here, with other significant comparison tests Raabe's, Gauss's, and various test of the logarithmic scale).

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Dedicated to Mari

A GENERALIZED CON NEWTON'S METHO

Abstract. The aim of t
a result due to J.-S. Pang
Newton's method and some r
equations using generalized J
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method applied to the contac

MSC 2000: 26A27, 35J
Keywords: B-differentiab
gradient, Newton's method

1. INTRODUCTION

The notion of B-differen
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proximated by F-derivative.
The main objective of
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Newton's method and some
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2. PROPERTIES OF

Definition 1. A funct
point z if there exists a fu
of f at z, which is positiv
$\lambda Bf(z)(v)$, $\forall v \in R^n$ and a