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Dedicated to Costică MUSTĂŢA on his 60th anniversary

ON THE STABILITY OF SOME FIXED POINT PROCEDURES

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Abstract. Following the concept of T-stability defined in [1] for a fixed point iteration procedure, we establish in a more simpler manner than in [1], the same stability results for the Picard iteration and Mann iteration considered in metric spaces and normed linear spaces, respectively.

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Introduction.

Let (X, d) be a metric space, $T : X \to X$ a map and let $F(T) = \{p \in X \mid Tp = p\}$ denote the set of fixed points of T. The literature abounds with metrical fixed point theorems for mapping satisfying a variety of contractive conditions. In most cases the contractive conditions are strong enough to ensure not only the existence of the fixed point but also its uniqueness and, moreover, the convergence to that fixed point of various fixed point iteration procedures.

The most used iteration procedure to approximate fixed points is the method of successive approximations (or Picard iteration), given by

$$x_{n+1} = T\,x_n\,, \quad n = 0, 1, 2, \dots \tag{1}$$

and $x_0 \in X$.

Since the Picard iteration does not converge to a fixed point for all kind of contractive mappings (as the nonexpansive mappings, for example), to over come these difficulties, other fixed point iteration procedures were considered: Mann iteration, Ishikawa iteration, Kirk iteration etc. (see [3],[5], [6], [7],[8],[10] and references therein).

An important practical feature of a given fixed point iteration procedure consists in its numerical stability. Harder and Hicks [5] introduced a concept of stability of fixed points iteration procedures and established some stability results for the Picard, Mann and Kirk interations under various contractive conditions. Rhoades [9, 11] extended the results of Harder and Hicks to other classes of contractive mappings.

The main aim of this paper is to prove in a more simpler manner than in [5] the stability results established there.

1. Preliminaries.

Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by an iteration procedure involving the operator T,

$$x_{n+1} = f(T, x_n)\,, \qquad n = 0, 1, 2, \ldots . \tag{2}$$

where $x_0 \in X$ is the initial approximation and f is some function.

For example, the Picard iteration is obtained from (2) for $f(T, x_n) = T\,x_n$, while the Mann iteration is obtained for $f(T, x_n) = (1 - a_n)x_n + a_n\,T\,x_n$, with $\{a_n\}$ a sequence in $[0, 1]$ and X a normed linear space. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. When calculating $\{x_n\}_{n=0}^{\infty}$, then we cover the following steps:

1. We chose the initial approximation $x_0 \in X$;

2. Then we compute $x_1 = f(T, x_0)$ but, due to various errors (rounding errors, numerical approximations of functions, derivatives or integrals), we do not get the exact value of x_1, but a different and one y_1 which is very closed to x_1, $y_1 \approx x_1$.

3. Consequently, when computing $x_2 = f(T, x_1)$ we shall have actually $x_2 = f(T, y_1)$ and instead of the theoretical value x_2 we shall obtain a closed value $y_2 \approx x_2$, and so on.

In this way, instead of the theoretical sequence $\{x_n\}_{n=0}^{\infty}$ generated by the iterative method, we get an approximant sequence $\{y_n\}_{n=0}^{\infty}$. We shall consider the iteration method is stable if and only if for y_n, closed enough to x_n, $\{y_n\}_{n=0}^{\infty}$ still converges to the fixed point p of T. Following this idea, Harder and Hicks introduced the following concept of stability[5].

Definition 1. Let (X, d) be a metric space, $T : X \to X$ a self map, $x_0 \in X$ and the iteration procedure defined by (2), such that the generated sequence $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X, and set

$$\varepsilon_n = d(y_{n+1}, f(T, y_n)), \text{ for } n = 0, 1, 2, \ldots \;. \tag{3}$$

We say that the iteration (2) is *T-stable* or *stable with respect to* T if and only if

$$\lim_{n\to\infty} \varepsilon_n = 0 \Rightarrow \lim_{n\to\infty} y_n = p.$$

Using this concept, Harder and Hicks [5] emphasized four stability results, one of them being due to Ostrowski. Our main aim in this paper is to obtain in a more simpler way the same stability results for the same iterations and contractive conditions considered by Harder and Hicks in [5]. The following Lemma will be used in the proofs of Theorem 1-4 which follow.

Lemma 1. *If δ is a real number such that $0 \le \delta < 1$, and $\{\varepsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim\limits_{n\to\infty} \varepsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \le \delta\, u_n + \varepsilon_n\,, \quad n = 0, 1, 2, \dots \tag{4}$$

we have

$$\lim_{n\to\infty} u_n = 0.$$

Proof. For $\delta = 0$, the conclusion is obvious. So we assume that $0 < \delta < 1$ and rewrite (4) as

$$u_{k+1}\,\delta^{-k-1} \le u_k \delta^{-k} + \delta^{-k-1}\varepsilon_k\,, \quad k = 0, 1, 2, \dots$$

and sum these inequalities for $k = 0, 1, 2, \dots, n+1$. After doing all cancelations, we obtain

$$0 \le u_{n+1} \le \delta^{n+1} \cdot u_0 + \sum_{k=0}^{n} \delta^{n-k} \varepsilon_k. \tag{4'}$$

Now, using Lemma 1 [5], it results that

$$\lim_{n\to\infty} \left(\sum_{k=0}^{n} \delta^{n-k} \varepsilon_k \right) = 0$$

and hence from (4'), we obtain that $\lim\limits_{n\to\infty} u_n = 0$.

2. Stability of Picard iteration for strict contractions.

The Banach's fixed point theorem (or the contraction mapping principle) is the most important metrical fixed point theorem. We give here its full statement and prove only the stability part, which is rather simpler than that contained in Theorem 1 [5]. For the proof of (i)-(iv) in Theorem 1, we refer to [1].

Theorem 1. *Let (X, d) be a complete metric space and $T : X \to X$ be an a contraction, that is,*

$$d(Tx,Ty) \le \alpha \cdot d(x,y)\,, \ \text{ for each } x,y \in X \tag{5}$$

where α is a constant such thar $0 \le \alpha < 1$. Then:

(i) $F(T) = \{p\}$;

(ii) The Picard iteration $\{x_n\}_{n=0}^{\infty}$, $x_{n+1} = f(T, x_{n)} := T\,x_n$, $x = 0, 1, \ldots$, converges to p for each $x_0 \in X$;

(iii) $d(x_n, p) \le \frac{\alpha^n}{1-\alpha} \cdot d(x_0, x_1)\,, \quad n \ge 1$;

(iv) $d(x_n, p) \le \frac{\alpha}{1-\alpha} \cdot d(x_n, x_{n-1})\,, \quad n \ge 1$;

(v) If $\{y_n\}_{n=0}^{\infty}$ is a sequence in X and $\{\varepsilon_n\}_{n=0}^{\infty}$ the sequence defined by (3), then

$$\lim_{n\to\infty} \varepsilon_n = 0 \ \Leftrightarrow \lim_{n\to\infty} y_n = p. \tag{6}$$

Proof. (v) We prove firstly the "$\Rightarrow$ " implication in (6). Using the triangle inequality, we get

$$d(y_{n+1}, p) \le d(y_{n+1}, T\,y_n) + d(T\,y_n, p).$$

But $p \in F(T)$, hence by the contraction condition (5) we get

$$d(T\,y_n, p) = d(T\,y_n, T\,p) \le \alpha \cdot d(y_n, p).$$

This yields

$$d(y_{n+1}, p) \le \alpha \cdot d(y_n, p) + \varepsilon_n, \quad n = 0, 1, 2, \ldots \tag{7}$$

and by Lemma 1 we deduce $\lim\limits_{n\to\infty} d(y_{n+1}, p) = 0$, that is

$$\lim_{n\to\infty} y_{n+1} = p.$$

The reverse implication is an immediate consequence of the inequalities

$$\varepsilon_n = d(y_{n+1}, T\,y_n) \le d(y_{n+1}, p) + d(p, T\,y_n) \le d(y_{n+1}, p) + \alpha \cdot d(y_n, p).$$

The proof is complete.

Remarks. 1) In Theorem 1, (iii) gives the *a priori* estimation of the convergence rate for Picard iteration, while (iv) gives the *a posteriori* error estimation. The last one is very usefull in applications, because it provides a direct stopping criterion for the iterative procedure;

2) Our condition (7) is essentially simpler than conclusions (1) and (2) in Theorem 1 [5], needed to ensure the $T-$stability of the Picard iteration. In fact (v) from Theorem 1, shows that the Picard iteration corresponding to a strict contraction T, is $T-$stable.

3. Stability of Picard and Mann iterations for Zamfirescu contractions

Definition 2. A selfmap $T : X \to X$ is said to be *a Zamfirescu contraction* if there exist real numbers α, β and γ satisfying $0 \leq \alpha < 1\,,\ 0 \leq \beta < 0.5$, and $0 \leq \gamma \leq 0.5$ such that for each $x, y \in X$, at least one of the following is true:

(z_1) $d(T\,x, T\,y) \leq \alpha \cdot d(x, y)$;

(z_2) $d(Tx, Ty) \leq \beta[d(x, T\,y) + d(y, T\,y)]$;

(z_3) $d(Tx, Ty) \leq \gamma[d(x, T\,y) + d(y, Tx)]$.

It was showed [13] that any Zamfirescu contraction in a complete metric space X has a unique fixed point p and that the Picard iteration converges to p for any initial approximation $x_0 \in X$.

The next theorem offers information on its stability.

Theorem 2. *The Picard iteration corresponding to a Zamfirescu contraction $T : X \to X$ on a complete metric space (X, d) is $T-$stable.*

Proof. Let $F(T) = \{p\}$ and $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration associated to T. Let $\{y_n\}_{n=0}^{\infty}$ be a sequence in X and $\{\varepsilon_n\}_{n=0}^{\infty}$ the sequence given by (3). To prove that the Picard iteration is T-stable, we have to prove that (5) is true. To this end, we firstly prove (in a similar manner to [5], Theorem 2) that if T is a Zamfirescu contraction, then the following condition

$$d(T\,x, T\,y) \leq 2\delta\, d(x, T\,x) + \delta\, d(x, y), \tag{8}$$

holds for each $x, y \in X,$ where

$$\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}. \tag{9}$$

Now, by triangle inequality and (8) we get

$$\begin{aligned} d(y_{n+1}, p) &\leq d(y_{n+1}, T\,y_n) + d(Ty_n, p) = \varepsilon_n + d(T\,y_n, T\,p) \leq \\ &\leq \varepsilon_n + 2\delta\, d(p, T\,p) + \delta\, d(y_n, p)\,, \end{aligned}$$

and since $p = Tp$, it results that

$$d(y_{n+1}, p) \leq \delta\, d(y_n, p) + \varepsilon_n\,, \quad n = 0, 1, 2, \dots\ . \tag{10}$$

Recall that $0 \leq \alpha < 1\,,\ 0 \leq \beta < 0.5$ and $0 \leq \gamma < 0.5$.

So $\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$ implies that $0 \leq \delta < 1$.

Suppose $\lim\limits_{n\to\infty} \varepsilon_n = 0$, then by (10) and Lemma 1 we obtain

$$\lim_{n\to\infty} y_n = 0.$$

The reverse implication is immediate:

$$\begin{aligned}\varepsilon_n &= d(y_{n+1}, T\,y_n) \leq d(y_{n+1}, p) + d(T\,y_n, p) = d(y_{n+1}, p) + d(T\,y_n, T\,p) \leq \\ &\leq d(y_{n+1}, p) + 2\,\delta\, d(p, T\,p) + \delta \cdot d(y_n, p)\end{aligned}$$

and hence

$$0 \leq \varepsilon_n \leq d(y_{n+1}, p) + \delta \cdot d(y_n, p)$$

which shows that $\lim\limits_{n\to\infty} y_n = p$ implies $\lim\limits_{n\to\infty} \varepsilon_n = 0$.

Remark. 1) Our stability result established here is significantly simpler than the similar one from [5], Theorem 2, based on condition (1) there; based on condition (1) in Theorem 2 from [5];

2) Having in view that many contractive conditions (see [12] and [5]) imply the Zamfirescu conditions, it result that Theorem 2 establishes $T-$stability of Picard iteration for all such mappings.

Let us now restrict ourselves to a normed linerar space $(X, \|\cdot\|\,)$ and let $T : X \to X$ be a self map. The Mann iteration is defined by (2) with

$$f(T, x_n) = (1-\alpha_n)\,x_n + \alpha_n\,T\,x_n\,,\ \ n = 0, 1, \ldots \tag{11}$$

where $x_0 \in X$ and $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of real numbers, $0 \leq \alpha_n \leq 1,\ n = 0, 1, 2, \ldots$.

There exist various convergence theorems for the Mann iteration [ ] based on certain assumptions on the sequence $\{\alpha_n\}_{n=0}^{\infty}$, which must ensure, among other facts, that

$$\sum_{n=0}^{\infty} \alpha_n = \infty. \tag{12}$$

For example, the sequence $\{\alpha_n\}_{n=0}^{\infty}$ for which there exists the constants a and b such that

$$0 < a \leq \alpha < b < 1\,,\ \text{ for } n = 0, 1, \ldots \tag{13}$$

does satisfy (12).

Theorem 3. *Let $(X, \|\cdot\|\,)$ be a normed linear space and $T : X \to X$ be a Zamfirescu contraction. Suppose there exists $p \in F(T)$ such that the Mann iteration $\{x_n\}_{n=0}^{\infty}$ with $x_0 \in X$ and $\{\alpha_n\}_{n=0}^{\infty}$ satisfying (12), converges to p.*

Then the Mann iteration procedure is $T-$stable.

Proof. Let $\{y_n\}_{n=0}^{\infty}$ be a sequence in X and

$$\varepsilon_n = \|y_{n+1} - [(1-\alpha_n)\, y_n + \alpha_n\, T\, y_n]\| \ , \ n = 0, 1, 2, \dots \ .$$

We have to prove that

$$\lim_{n\to\infty} y_n = p \Longleftrightarrow \lim_{n\to\infty} \varepsilon_n = 0.$$

From the proof of Theorem 2 we know that for each $x, y \in X$

$$\|T\,x - T\,y\| \leq 2\,\delta \cdot \|x - T\,x\| + \delta \cdot \|x - y\| \qquad (14)$$

holds, with δ given by (9). Now suppose $\lim_{n\to\infty} \varepsilon_n = 0$. Then

$$\begin{aligned} \|\, y_{n+1} - p\,\| &\leq \|\, y_{n+1} - f(T, y_n)\,\| + \|\, f(T, y_n) - p\,\| = \\ &= \varepsilon_n + \|\, (1-\alpha_n)\, y_n + \alpha_n\, T\, y_n - [\,(1-\alpha_n) + \alpha_n] \cdot p\,\| < \\ &< \varepsilon_n + (1-\alpha_n)\ \|\, y_n - p\,\| + \alpha_n\ \|\ T\, y_n - p\,\| \ , \end{aligned}$$

and using (13), we get

$$\begin{aligned} \|\ T\, y_n - p\,\| &= \|\, T\, y_n - T\, p\,\| \leq 2\,\delta \cdot \|\, p - T\, p\,\| + \delta \cdot \|\, y_n - p\,\| = \\ &= \delta \cdot \|\ y_n - p\,\|\,, \end{aligned}$$

and then

$$\|\ y_{n+1} - p\ \| \leq (1 - \alpha_n + \alpha_n\, \delta)\ \|\ y_n - p\,\| + \varepsilon_n\,, \quad n = 0, 1, \dots\ . \qquad (15)$$

Since $0 \leq 1 - \alpha_n + \alpha_n\, \delta < 1$, by using Lemma 1, it follows that

$$\lim_{n\to\infty} y_n = p.$$

Remarks. 1) Theorem 3 in this paper corresponds to Theorem 3 in [5]. The inequality (1) in [5]from which it is deduced the $T-$stability of the Mann iteration is much more complicated then our inequality obtained from (4') and (14) as in the proof of Lemma 1.

2) In a similar way one can prove the $T-$stability of the Kirk's iteration [13] for the case of strict contractions in a Banach space.

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