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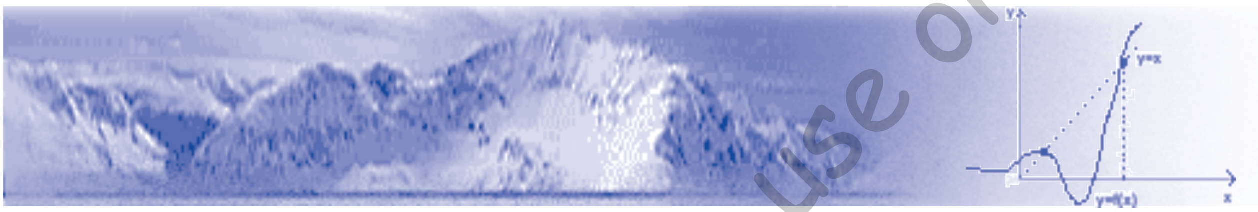

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Some remarks on a fixed point theorem for Ćirić-type almost contractions

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ABSTRACT. It is pointed out that the contraction condition used in a fixed point theorem for Ćirić-type almost contractions [Berinde, V., *General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces*, Carpathian J. Math., **24** (2008), No. 2, 10-19] must be slightly modified in order to always ensure the existence of fixed points.

1. INTRODUCTION

The main result in [4], included there as Theorem 3.2, has the following statement

Theorem 1.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a Ćirić almost contraction, that is, a mapping for which there exist two constants $\alpha \in [0, 1)$ and $L \geq 0$ such that*

$$d(Tx, Ty) \leq \alpha \cdot M(x, y) + L\, d(y, Tx)\,, \quad \text{for all } x, y \in X\,, \tag{1.1}$$

where

$$M(x, y) = \max\left\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ d(x, Ty),\ d(y, Tx)\right\}.$$

Then

1) $Fix\,(T) = \{x \in X : Tx = x\} \neq \emptyset$;

2) *For any $x_0 = x \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by*

$$x_{n+1} = Tx_n\,, \quad n = 0, 1, 2, \ldots \tag{1.2}$$

converges to some $x^* \in Fix\,(T)$;

3) *The following estimate holds*

$$d(x_n, x^*) \leq \frac{\alpha^n}{(1-\alpha)^2}\, d(x, Tx)\,, \quad n = 1, 2, \ldots \tag{1.3}$$

The Ćirić almost contraction condition (1.1) has been obtained by combining the definition of Ćirić quasi-contractions [12]: there exists $0 \leq h < 1$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq h \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \tag{1.4}$$

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on the one hand, and that of the almost contractions [7]: there exist the constants $\delta \in [0,1)$ and $L \geq 0$ such that

$$d(Tx,Ty) \leq \delta \cdot d(x,y) + Ld(y,Tx)\,, \quad \text{for all } x,y \in X, \tag{1.5}$$

introduced in [7], on the other hand. The latter has been studied in some other papers [2], [3], [5], [6], [8], [19] etc., for the case of both single-valued and multi-valued mappings.

Condition (1.4) is known as one of the most general contractive conditions for which the unique fixed point can be approximated by means of Picard iteration. It is numbered (24) in Rhoades' classification [23] that include 25 classical contractive conditions.

On the other hand, condition (1.5) is a very general contractive condition that allows the operator T to have more than one unique fixed point which still could be approximated by means of Picard iteration. It includes many contractive conditions from Rhoades' classification [23], among which we mention: Banach's contraction condition, numbered (1), Kannan's condition [17], numbered (4), Chatterjea's condition [9], numbered (7), Zamfirescu's condition [29], numbered (19) etc., but not fully includes Ćirić's quasi-contraction condition (1.4), see for example [7] and [5].

Another contractive condition due to Ćirić [10], and numbered (21) in Rhoades' classification is the following one: there exists $0 \leq h < 1$ such that for all $x, y \in X$,

$$d(Tx,Ty) \leq h \cdot \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Tx)]\right\}. \tag{1.6}$$

Condition (1.6) is weaker than the aforementioned ones from Rhoades' classification but it is sharper than (1.4), that is, (1.6) implies (1.4), but not vice-versa, see [15]. For other fixed and common fixed point theorems that use the contractive condition (1.6) see also [1], [13], [20], [28].

The main aim of this note is to show that a Ćirić type almost contraction satisfying (1.1) could fail to have a fixed point in the case

$$M(x,y) = d(x,Ty)$$

as shown by Example 3.1, and to show that if we consider instead of Ćirić's type almost contractions, the class of the so called *Ćirić's strong almost contractions*, obtained by replacing $M(x,y)$ in (1.1) by

$$M_1(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\left[d(x,Ty)+d(y,Tx)\right]\right\}, \tag{1.7}$$

then we do not need to require explicitly the existence of a fixed point.

2. Main results

Ćirić's strong almost contractions always have a fixed point, as shown by Theorem 2.2. We thus obtain the following correct version of Theorem 3.2 in [4].

Theorem 2.2. *Let (X,d) be a complete metric space and let $T : X \to X$ be a strong Ćirić almost contraction, that is, a mapping for which there exist two constants $\alpha \in [0,1)$*

and $L \geq 0$ *such that*

$$d(Tx, Ty) \leq \alpha \cdot M_1(x, y) + L\, d(y, Tx)\,, \quad \text{for all} \;\; x, y \in X\,, \tag{2.8}$$

where $M_1(x, y)$ *is given by (1.7). Then*

1) $Fix\,(T) = \{x \in X : Tx = x\} \neq \emptyset$;
2) *For any* $x_0 = x \in X$*, the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *given by (1.2) converges to some* $x^* \in Fix\,(T)$;
3) *The following estimate holds*

$$d(x_{n+i-1}, x^*) \leq \frac{\alpha^i}{1-\alpha}\, d(x_n, x_{n-1})\,, \quad n = 0, 1, 2, \ldots;\, i = 1, 2, \ldots \tag{2.9}$$

Proof. Let $x \in X$ be arbitrary and let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration defined by (1.2) with $x_0 = x$. By taking $x := x_{n-1}$, $y := x_n$ in (2.8), we obtain

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha \cdot M_1(x_{n-1}, x_n)\,,$$

that is,

$$d(x_n, x_{n+1}) \leq \alpha \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}[d(x_{n-1}, x_{n+1}) + 0]\right\},$$

since $d(x_n, Tx_{n-1}) = 0$. Now, by the triangle inequality

$$d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1})$$

and using the inequality $\dfrac{a+b}{2} \leq \max\{a, b\}$, we deduce that, either

$$\max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}d(x_{n-1}, x_{n+1})\right\} = d(x_{n-1}, x_n) \tag{2.10}$$

or

$$\max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}d(x_{n-1}, x_{n+1})\right\} = d(x_n, x_{n+1}). \tag{2.11}$$

The case (2.11) cannot hold because it would lead to the contradiction

$$d(x_n, x_{n+1}) \leq h\, d(x_n, x_{n+1}).$$

Hence, always (2.10) must hold, and this leads to

$$d(x_n, x_{n+1}) \leq h\, d(x_{n-1}, x_n).$$

The rest of the proof is similar to that of Theorem 3.2 in [4]. □

Note that the error estimate (2.9) obtained in Theorem 2.2 for Ćirić strong almost contractions is different from the estimate (1.3) that has been deduced in [4] for Ćirić strong almost contractions. While the former is the same as in Banach's contraction theorem, see for example [8], the latter is slightly different.

It is possible to force the uniqueness of the fixed point of a Ćirić strong almost contraction, like in the case of almost contractions [7], by imposing an additional contractive condition, quite similar to (1.5), as shown by the next theorem.

Theorem 2.3. *Let (X, d) be a complete metric space and let $T : X \to X$ be a Ćirić strong almost contraction for which there exist $\theta \in [0, 1)$ and some $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \theta \cdot d(x, y) + L_1 \cdot d(x, Tx), \quad \text{for all } x, y \in X. \tag{2.12}$$

Then

1) *T has a unique fixed point, i.e. $Fix(T) = \{x^*\}$;*
2) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.2) converges to x^*, for any $x_0 \in X$;*
3) *The error estimate (2.9) holds.*
4) *The rate of convergence of the Picard iteration is given by*

$$d(x_n, x^*) \leq \theta\, d(x_{n-1}, x^*), \quad n = 1, 2, \ldots \tag{2.13}$$

Proof. Assume T has two distinct fixed points $x^*, y^* \in X$. Then by (2.12), with $x := x^*$, $y := y^*$ we get

$$d(x^*, y^*) \leq \theta \cdot d(x^*, y^*) \Leftrightarrow (1 - \theta)\, d(x^*, y^*) \leq 0,$$

so contradicting $d(x^*, y^*) > 0$.

Now letting $y := x_n$, $x := x^*$ in (2.12), we obtain the estimate (2.13).
The rest of proof follows by Theorem 2.2. □

A stronger but simpler contractive condition that ensures the uniqueness of the fixed point has been obtained by Babu et al. [2] for almost contractions. We state the fixed point theorem corresponding to this uniqueness condition in the case of Ćirić strong almost contractions.

Theorem 2.4. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping for which there exist $\alpha \in [0, 1)$ and some $L \geq 0$ such that for all $x, y \in X$*

$$d(Tx, Ty) \leq \alpha \cdot M_1(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \tag{2.14}$$

Then

1) *T has a unique fixed point, i.e., $Fix(T) = \{x^*\}$;*
2) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.2) converges to x^*, for any $x_0 \in X$;*
3) *The error estimate (2.9) holds.*

To close this section we mention some important particular cases of the above fixed point theorems.

All fixed point theorems in [2] and [7] are particular cases of Theorems 2.2-2.4 in the present paper. These theorems also generalize, amongst many other fixed point theorems, the Banach's contraction principle, Kannan's fixed point theorem [17], Chatterjea's fixed point theorem [9], Reich's fixed point theorems [21], [22], Hardy and Rogers fixed point theorem [16], Zamfirescu's fixed point theorem [29], Ćirić's fixed point theorem [10].

3. Examples and conclusions

Example 3.1 illustrates a case when Theorem 1.1 does not ensure the existence of a fixed point, so requiring the replacement of Ćirić's almost contraction condition (1.1) by the Ćirić's strong almost contraction condition (2.8), while Example 3.2 shows that a Ćirić's strong almost contraction need not be neither a Ćirić quasi-contraction nor any other contraction type mappings in Rhoades' classification.

Example 3.1. Let $X = \mathbb{N} = \{0, 1, 2, ...\}$ with the usual norm and let T be defined by $T(n) = n + 1$. Then T does satisfy (1.6) with $\alpha = \frac{1}{2}$ and $L = 2$ but T is fixed point free. Indeed, if we take $x = n$, $y = m$, $m > n$, then $d(Tx, Ty) = m - n$, $M(x, y) = m - n + 1$, $d(y, Tx) = m - n - 1$. Thus condition (1.6) reduces to

$$m - n \leq \alpha(m - n + 1) + 2(m - n - 1) = \frac{5}{2}(m - n) - \frac{3}{2}$$

which is true, since $m - n \geq 1$.

Example 3.2. Let $[0, 1]$ be the unit interval with usual norm and let $T : [0, 1] \to [0, 1]$ be given by $Tx = \frac{x}{2}$ for $x \in [0, 1)$ and $T(1) = 1$.

As T has two fixed points, that is, $Fix\,(T) = \{0, 1\}$, it does not satisfy neither Ćirić's quasi-contractive condition (1.4), nor Banach, Kannan, Chatterjea or Zamfirescu contractive conditions etc., but T satisfies the contraction condition (1.5), and therefore the contractive condition (1.6), too.

Indeed, for $x, y \in [0, 1)$, condition (1.5) is satisfied with $a = \frac{1}{2}$ and $L \geq 0$ arbitrary, while, for $x \in [0, 1)$ and $y = 1$, condition (1.5) reduces to

$$\left|\frac{x}{2} - 1\right| \leq a|x - 1| + L\left|1 - \frac{x}{2}\right|,$$

which is obviously true if we take $a < 1$ arbitrary and $L \geq 1$. Therefore T satisfies (1.5), for all $x, y \in [0, 1]$ with $a = \frac{1}{2}$ and $L \geq 1$.

It is clear from their definition that any Ćirić's strong almost contraction is a Ćirić's almost contraction. As the main aim of introducing Ćirić's almost contractions has been to try to include the whole class of Ćirić's quasi contractions, the following problem still remains open: is any Ćirić quasi-contraction a Ćirić strong almost contraction ?

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