CARPATHIAN J. MATH. 27 (2011), No. 1, 13 - 23
Online version available at http://carpathian.ubm.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Stability of Picard iteration for contractive mappings satisfying an implicit relation

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ABSTRACT. We obtain new and very general stability results for Picard iteration associated to self operators satisfying an implicit relation. Our stability results unify, extend, generalize, enrich and complement a multitude of related stability results from recent literature.

1. INTRODUCTION

Let (X, d) be a metric space, $T : X \to X$ a self operator with $Fix\,(T) := \{x \in X : Tx = x\} \neq \emptyset$ and let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration, that is, the sequence defined by $x_{n+1} = Tx_n$, $n = 0, 1, \dots$, with $x_0 \in X$, arbitrary. If (X, d) is complete and T is a contraction, i.e., there exists a constant $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq a\, d(x, y), \text{ for all } x, y \in X, \tag{1.1}$$

then, by Banach contraction mapping principle, we know that T has a unique fixed point p and, for any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ converges to p. Moreover, the following error estimate, which is very useful in concrete applications for stopping the iterative process,

$$d(x_{n+1}, p) \leq \frac{\alpha^n}{1-\alpha}\, d(x_0, x_1)\,, \quad n = 0, 1, 2, \dots,$$

holds.

However, when solving concrete problems, because of rounding errors, numerical approximations of functions, derivatives or integrals, discretization etc., instead of the theoretical sequence $\{x_n\}_{n=0}^{\infty}$, defined by the given iterative method, we will practically obtain an *approximate sequence* $\{y_n\}_{n=0}^{\infty}$, satisfying the following approximation bounds:

$$y_0 := x_0,\ d(y_1, Ty_0) \leq \epsilon_1,\ \dots,\ d(y_n, Ty_{n-1}) \leq \epsilon_n,\ \dots,$$

where the positive quantity ϵ_n can be interpreted as the "round-off error" of x_n, see [18].

The problem of the numerical stability of Picard iteration is now whether this approximate sequence $\{y_n\}_{n=0}^{\infty}$ is still convergent to the fixed point p of T, provided $\epsilon_n \to 0$ or $\sum\limits_{n=1}^{\infty} \epsilon_n < \infty$.

Received: 01.10.2010; In revised form: 02.02.2011; Accepted: 23.02.2011
2010 *Mathematics Subject Classification.* 47H10, 47H09, 54H25.
Key words and phrases. *Metric space, fixed point, Picard iteration, implicit relation, error estimate, rate of convergence, stability, almost stability, summable almost stability.*

This question has been answered in the positive in 1967 by Ostrowski [24] who thus established the first stability theorem for a fixed point iteration procedure, by using the following estimate

$$d(y_n, p) \leq \frac{\alpha^n}{1-\alpha}\, d(x_0, x_1) + \sum_{k=1}^{n} \alpha^{n-k} \epsilon_k, \quad n = 0, 1, 2, \ldots,$$

from which easily follows that $y_n \to p$, provided $\epsilon_n \to 0$.

In 1988, Harder and Hicks [15], [16], introduced the notion of stability for a general fixed point iteration procedure and started the systematic study of this concept, thus obtaining various stability results for Picard iteration that extended Ostrowski's theorem to mappings satisfying more general contractive conditions and also established some stability results for other fixed point iteration procedures in the class of Banach contractions, Zamfirescu operators etc.

Further, Rhoades [31], [32], [33] extended the results of Harder and Hicks by considering more general contractive mappings. More specifically, Rhoades [32] considered the following explicit contractive condition which extends both (1.1) and Zamfirescu contractive condition [37]: there exists a constant c, $0 \leq c < 1$, such that for all $x, y \in X$:

$$d(Tx, Ty) \leq c \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, d(x,Ty), d(y,Tx)\right\}. \tag{1.2}$$

In [32] Rhoades has shown that any mapping T satisfying (1.2) also satisfies the inequality

$$d(Tx, Ty) \leq a\, d(x,y) + L\, d(x,Tx), \tag{1.3}$$

where $a = c$ and $L = \dfrac{c}{1-c}$.

Osilike [21] extended Rhoades' results by considering mappings T for which $Fix\,(T) \neq \emptyset$ and satisfy condition (1.3), with $a \in [0,1)$ and $L \geq 0$, arbitrary, thus also extending all results of Harder and Hicks [15], [16].

All the stability results previously mentioned are basically established in connection with a corresponding fixed point theorem: Banach, Kannan, Chatterjea, Zamfirescu etc., see for example [10] for more details. Note that a mapping T satisfying (1.3) does not have a fixed point, in general, but, if T has a fixed point, this fixed point is certainly unique.

Alternatively, Jachymski [18] extended Ostrowski's theorem to the class of φ-contractions, which is independent from the classes of operators discussed above, by using Browder's fixed point theorem [13].

On the other hand, several classical fixed point theorems and common fixed point theorems have been recently unified by considering general contractive conditions expressed by an implicit condition. This approach has been initiated by Popa [25], [26]. Following Popa's papers, a consistent literature (that cannot be completely cited here) on fixed point, common fixed point and coincidence point theorems, for both single valued and multi-valued mappings, in various ambient spaces, has been developed, see [2]-[3], [25]-[27] and references therein, for a very selective list of references on this topic. For a similar but different approach to implicit contractions see also the papers by M. Turinici [35] and [36].

As for these new fixed point theorems do not exist corresponding stability results, yet, the main aim of this paper is to fill this gap and establish stability theorems for fixed point iteration procedures associated to contractive mappings defined by an implicit relation.

The fixed point theorems and the stability results we shall obtain in this way are extremely general. They unify, extend, generalize, enrich and complement a multitude of related results from recent literature: [6]-[10], [14]–[25], [29]-[34], [37], [39] and most of the references therein.

The paper is organized as follows: in Section 2 we present the basic concepts and results concerning the stability of fixed point iteration procedures associated to self mappings that satisfy explicit contractive conditions. In Section 3 we introduce the implicit relations that will be used in the paper and establish a basic fixed point theorem based on such relations, while in Section 4, the main stability results of this paper are presented. In Section 5, we end this paper by some concluding remarks and also discuss some directions for further study.

2. Stability of fixed point iteration procedures

Let (X, d) be a metric space, $T : X \to X$ a self operator with $Fix\,(T) \neq \emptyset$ and let $\{x_n\}_{n=0}^{\infty}$ be a fixed point iteration procedure of the general form

(2.4) $$x_{n+1} = f(T, x_n), \quad n = 0, 1, 2, ...,$$

where $f(T, x_n)$ is given (For example, in the case of Picard iteration we have $f(T, x_n) := Tx_n$).

Definition 2.1. (Harder and Hicks, [15]) Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and set

(2.5) $$\varepsilon_n = d(y_{n+1},\, f(T, y_n)), \quad \text{for} \quad n = 0, 1, 2, ...$$

We shall say that the fixed point iteration procedure (2.4) is *T-stable* or *stable with respect to T* if

(2.6) $$\lim_{n\to\infty} \varepsilon_n = 0 \Rightarrow \lim_{n\to\infty} y_n = p.$$

In view of their generalization in this paper, we state here in an abbreviated form, two illustrative samples of stability results: Theorem 1 from [33] and the main result of [21]. For other related results we refer to [6], [7], and [10].

Theorem 2.1. (Rhoades, [33]) *Let (X, d) be a complete metric space, $T : X \to X$ a self mapping satisfying* (1.2) *and let p be the (unique) fixed point of T. Then the Picard iteration associated to T is T-stable.*

Theorem 2.2. (Osilike, [21]) *Let (X, d) be a complete metric space, $T : X \to X$ a self mapping satisfying* (1.3) *with $Fix\,(T) \neq \emptyset$ and let p be the fixed point of T. Then Picard iteration $\{x_n\}_{n=0}^{\infty}$ associated to T is T-stable.*

As Picard iteration and other fixed point iteration procedures are not stable with respect to some classes of contractive operators, various weak stability concepts have been also introduced, see [6], [10], [22], [38]. For example Osilike [22] introduced the concept of *almost stability*, while Berinde [6] introduced the concept of *summable almost stability*, two notions which are presented in the following.

Definition 2.2. (Osilike, [22]) Let (X,d) be a metric space, $T:X\to X$ a self operator with $Fix\,(T)\neq\emptyset$ and let $\{x_n\}_{n=0}^{\infty}$ be a fixed point iteration procedure given by (2.4), supposed to converge to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and let $\{\varepsilon_n\}$ be defined by (2.5). We shall say that the fixed point iteration procedure (2.4) is *almost T-stable* or *almost stable with respect to T* if

$$\sum_{n=1}^{\infty}\varepsilon_n<\infty\Rightarrow\lim_{n\to\infty}y_n=p. \tag{2.7}$$

Definition 2.3. (Berinde, [7]) Let (X,d) be a metric space, $T:X\to X$ a self operator with $Fix\,(T)\neq\emptyset$ and let $\{x_n\}_{n=0}^{\infty}$ be a fixed point iteration procedure given by (2.4), supposed to converge to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and let $\{\varepsilon_n\}$ be defined by (2.5). We shall say that the fixed point iteration procedure (2.4) is *summable almost T-stable* or *summable almost stable with respect to T* if

$$\sum_{n=1}^{\infty}\varepsilon_n<\infty\Rightarrow\sum_{n=1}^{\infty}d(y_n,p)<\infty. \tag{2.8}$$

It is clear from Definitions 2.1-2.3 that:

1) any stable iteration procedure is almost stable;

2) any summable almost stable procedure is almost stable,

but the reverses of these assertions are not generally true, see Example 1 in [7]. Moreover, in general, the class of stable iteration procedures is independent of the class of summable almost stable procedures.

In order to prove our main stability results in this paper we shall need Lemma 1.6 from [10]:

Lemma 2.1. *Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers and a constant q, $0\leq q<1$, so that*

$$a_{n+1}\leq qa_n+b_n,\ \textit{for all}\ n\geq 0.$$

(i) If $\lim\limits_{n\to\infty}b_n=0$, then $\lim\limits_{n\to\infty}a_n=0$.

(ii) If $\sum\limits_{n=0}^{\infty}b_n<\infty$, then

$$\sum_{n=0}^{\infty}a_n<\infty.$$

3. Some fixed point theorems for mappings satisfying an implicit relation

A simple and natural way to unify and prove in a simple manner several metrical fixed point theorems is to consider an implicit contraction type condition instead of the usual explicit contractive conditions. V. Popa in 1997, [25] and [26], initiated this direction of research which, produced so far a consistent literature (that cannot be completely cited here) on fixed point, common fixed point and coincidence point theorems, for both single valued and multi-valued mappings, in various ambient spaces.

Let $\mathcal{F}$ be the set of all continuous real functions $F : {\mathbb{R}_+}^6 \to \mathbb{R}_+$, for which we consider the following conditions:

(F_{1a}) F is non-increasing in the fifth variable and
$F(u, v, v, u, u + v, 0) \leq 0$ for $u, v \geq 0 \Rightarrow \exists h \in [0, 1)$ such that $u \leq hv$;

(F_{1b}) F is non-increasing in the fourth variable and
$F(u, v, 0, u + v, u, v) \leq 0$ for $u, v \geq 0 \Rightarrow \exists h \in [0, 1)$ such that $u \leq hv$;

(F_{1c}) F is non-increasing in the third variable and
$F(u, v, u + v, 0, v, u) \leq 0$ for $u, v \geq 0 \Rightarrow \exists h \in [0, 1)$ such that $u \leq hv$;

(F_2) $F(u, u, 0, 0, u, u) > 0$, for all $u > 0$.

The following functions correspond to well known fixed point theorems and satisfy most of the conditions (F_{1a})-(F_2) above.

Example 3.1. The function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2,$$

where $a \in [0, 1)$, satisfies (F_2) and (F_{1a})-(F_{1c}), with $h = a$.

Example 3.2. Let $b \in \left[0, \frac{1}{2}\right)$. Then the function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - b(t_3 + t_4),$$

satisfies (F_2) and (F_{1a})-(F_{1c}), with $h = \dfrac{b}{1 - b} < 1$.

Example 3.3. Let $c \in \left[0, \frac{1}{2}\right)$. Then the function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - c(t_5 + t_6),$$

satisfies (F_2) and (F_{1a})-(F_{1c}), with $h = \dfrac{c}{1 - c} < 1$.

Example 3.4. The function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \max\left\{t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2}\right\},$$

where $a \in [0, 1)$, satisfies (F_2) and (F_{1a})-(F_{1c}), with $h = a$.

Example 3.5. The function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6),$$

where $a, b, c \in [0, 1)$ and $a + 2b + 2c < 1$, satisfies (F_2) and (F_{1a})-(F_{1c}), with $h = \dfrac{a + b + c}{1 - b - c} < 1$.

Example 3.6. The function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \max\left\{t_2, \frac{t_3 + t_4}{2}, t_5, t_6\right\},$$

where $a \in [0, 1)$, satisfies (F_2) and (F_{1b}), (F_{1c}), with $h = a$ and (F_{1a}), with $h = \dfrac{a}{1 - a} < 1$, if $a < 1/2$.

Example 3.7. The function $F \in \mathcal{F}$, given by

$$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - Lt_3,$$

where $a \in [0,1)$ and $L \geq 0$, satisfies (F_2) and (F_{1b}), with $h = a$, but, in general, does not satisfy (F_{1a}) and (F_{1c}).

Example 3.8. The function $F \in \mathcal{F}$, given by

$$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - Lt_6,$$

where $a \in [0,1)$ and $L \geq 0$, satisfies (F_{1a}), with $h = a$, but, in general, does not satisfy (F_{1b}), (F_{1c}) and (F_2).

The following theorem, which is an enriched version of Theorem 3 of Popa [25], unifies the most important metrical fixed point theorems for contractive mappings in Rhoades' classification [30].

Theorem 3.3. *Let (X,d) be a complete metric space, $T : X \to X$ a self mapping for which there exists $F \in \mathcal{F}$ such that for all $x, y \in X$*

$$F\left(d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right) \leq 0. \tag{3.9}$$

If F satisfies (F_{1a}) and (F_2) then:

$(p1)$ T has a unique fixed point x^* in X;

$(p2)$ The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n\,, \quad n = 0,1,2,\dots \tag{3.10}$$

converges to $\overline{x}$, for any $x_0 \in X$.

$(p3)$ The following estimate holds:

$$d(x_{n+i-1},\overline{x}) \leq \frac{h^i}{1-h}\, d(x_n,x_{n-1})\,, \quad n = 0,1,2,\dots;\ i = 1,2,\dots, \tag{3.11}$$

where h is the constant appearing in (F_{1a}).

$(p4)$ If, additionally, F satisfies (F_{1c}), then the rate of convergence of Picard iteration is given by:

$$d(x_{n+1},\overline{x}) \leq h d(x_n,\overline{x})\,, \quad n = 0,1,2,\dots. \tag{3.12}$$

Proof.

$(p1)$ Let x_0 be an arbitrary point in X and $x_{n+1} = Tx_n$, $n = 0,1,\dots$, be the Picard iteration. If we take $x := x_{n-1}$ and $y := x_n$ in (3.9), by denoting $u := d(x_n,x_{n+1})$, $v := d(x_{n-1},x_n)$ we get

$$F\left(u,v,v,u,d(x_{n-1},x_{n+1}),0\right) \leq 0.$$

By triangle inequality, $d(x_{n-1},x_{n+1}) \leq d(x_{n-1},x_n) + d(x_n,x_{n+1}) = u + v$ and, since F is non-increasing in the fifth variable, we have

$$F\left(u,v,v,u,u+v,0\right) \leq F\left(u,v,v,u,d(x_{n-1},x_{n+1}),0\right) \leq 0$$

and hence, in view of assumption (F_{1a}), there exists $h \in [0,1)$ such that $u \leq hv$, that is,

$$d(x_n,x_{n+1}) \leq h d(x_{n-1},x_n). \tag{3.13}$$

In a straightforward way, (3.13) leads to the conclusion that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

Since (X,d) is complete, there exists a $\overline{x}$ in X such that

$$\lim_{n\to\infty} x_n = \overline{x}. \tag{3.14}$$

By taking $x := x_n$ and $y := \overline{x}$ in (3.9) we get

$$F\left(d(Tx_n, T\overline{x}), d(x_n, \overline{x}), d(x_n, Tx_n), d(\overline{x}, T\overline{x}), d(x_n, T\overline{x}), d(\overline{x}, Tx_n)\right) \leq 0. \tag{3.15}$$

As F is continuous, by letting $n \to \infty$ in (3.15) we obtain

$$F\left(d(\overline{x}, T\overline{x}), 0, d(\overline{x}, T\overline{x}), d(\overline{x}, T\overline{x}), d(\overline{x}, T\overline{x}), 0)\right) \leq 0$$

which, by assumption (F_{1a}), yields $d(\overline{x}, T\overline{x}) \leq 0$, that is, $\overline{x} = T\overline{x}$. Now, we shall prove that $\overline{x}$ is the unique fixed point of T. Assume the contrary, i.e., there exists $\overline{y} \in Fix\,(T)$, $\overline{x} \neq \overline{y}$. Then by taking $x := \overline{x}$ and $y := \overline{y}$ in (3.9) and by denoting $\delta := d(\overline{x}, \overline{y}) > 0$ we get

$$F(\delta, \delta, 0, 0, \delta, \delta) \leq 0,$$

which contradicts (F_2). This proves that T has a unique fixed point.

$(p2)$ It follows by the proof of $(p1)$.

$(p3)$ By using (3.13) we inductively get

$$d(x_{n+i-1}, x_{n+i}) \leq h^i d(x_{n-1}, x_n),\ i = 1, 2, \ldots \tag{3.16}$$

and hence by triangle inequality

$$d(x_{n+i-1}, x_{n+i+p}) \leq d(x_{n+i-1}, x_{n+i}) + \cdots + d(x_{n+i+P-1}, x_{n+i+p}) \leq$$

$$\leq \sum_{k=0}^{p-1} h^{i+k} d(x_{n-1}, x_n) = \frac{h^i}{1-h}(1-h^p) d(x_{n-1}, x_n),\ i, n, p = 1, 2, \ldots,$$

which, by letting $p \to \infty$, yields exactly the desired estimate (3.11).

$(p4)$ By taking $x := x_n$ and $y := \overline{x}$ in (3.9) we get

$$F\left(d(Tx_n, \overline{x}), d(x_n, \overline{x}), d(x_n, Tx_n), d(\overline{x}, \overline{x}), d(x_n, \overline{x}), d(\overline{x}, Tx_n)\right) \leq 0.$$

that is,

$$F\left(d(x_{n+1}, \overline{x}), d(x_n, \overline{x}), d(x_n, x_{n+1}), 0, d(x_n, \overline{x}), d(\overline{x}, x_{n+1})\right) \leq 0. \tag{3.17}$$

Denote $u := d(x_{n+1}, \overline{x})$, $v := d(x_n, \overline{x})$. Then, by triangle inequality we have $d(x_n, x_{n+1}) \leq d(x_n, \overline{x}) + d(x_{n+1}, \overline{x}) = u + v$ and hence, in view of assumption (F_{1c}), by (3.17) we obtain

$$F\left(u, v, u+v, 0, v, u)\right) \leq F\left(u, v, d(x_n, x_{n+1}), 0, v, u\right) \leq 0,$$

which again by (F_{1c}) implies the existence of a $h \in [0, 1)$ such that $u \leq hv$, which is exactly the desired estimate (3.12). □

Remark 3.1. Theorem 3.3 completes Theorem 3 in Popa [25] with the additional information regarding the iterative method available for approximating the fixed point $\overline{x}$, with the estimate (3.12) of the rate of convergence of Picard iteration, and by providing the unifying error estimate (3.11) inspired from [39], from which one can deduce both the *a priori* estimate

$$d(x_n, \overline{x}) \leq \frac{h^n}{1-h}\, d(x_0, x_1)\,, \quad n = 0, 1, 2, \ldots$$

and the *a posteriori* estimate

$$d(x_n,\overline{x}) \leq \frac{h}{1-h}\, d(x_n,x_{n-1})\,, \quad n=1,2,\dots$$

Remark 3.2.

(a) If F is the function in Example 3.1, then by Theorem 3.3 we obtain the well known Banach contraction mapping principle, in its complete form, see Theorem B in [9].

(b) If F is the function in Example 3.2, then by Theorem 3.3 we obtain Theorem 1 in [9], that completes the well known Kannan fixed point theorem [19].

(c) If F is the function in Example 3.3, then by Theorem 3.3 we obtain a fixed point theorem that completes Chatterjea fixed point theorem [14].

(d) If F is the function in Example 3.4, then by Theorem 3.3 we obtain Theorem 2 in [9], that completes the well known Zamfirescu fixed point theorem [37].

(e) If F is the function in Example 3.5, then by Theorem 3.3 we obtain a fixed point theorem that extends the Reich fixed point theorem [29].

4. Stability of Picard iteration for mappings satisfying an implicit relation

The first main result of this paper is the following general stability theorem for Picard iteration.

Theorem 4.4. *Let (X,d) be a complete metric space, $T:X\to X$ a self mapping for which there exists $F\in\mathcal{F}$ such that for all $x,y\in X$*

$$F\left(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\right)\leq 0. \tag{4.18}$$

If F satisfies (F_1), (F_{2a}), and (F_3), then T has a unique fixed point. If, additionally, F satisfies (F_{2b}), then Picard iteration is: a) T-stable; b) summable almost T-stable.

Proof. Let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration associated to T and defined by (2.4), converging to the fixed point $\overline{x}$ of T, which exists and is unique by virtue of Theorem 3.3. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and let $\{\varepsilon_n\}$ be defined by

$$\varepsilon_n = d(y_{n+1},\, f(T,y_n)), \quad \text{for} \quad n=0,1,2,...$$

In order to show that Picard iteration is T-stable we shall prove that the implication

$$\lim_{n\to\infty}\varepsilon_n = 0 \Rightarrow \lim_{n\to\infty} y_n = p,$$

holds. Indeed, assume $\lim\limits_{n\to\infty}\varepsilon_n=0$. Then

$$d(y_{n+1},\overline{x}) \leq d(y_{n+1},Ty_n)+d(Ty_n,\overline{x}) = \epsilon_n + d(Ty_n,\overline{x}). \tag{4.19}$$

Take $x:=\overline{x}$ and $y:=y_n$ in (4.18) to obtain

$$F(u,v,0,w,v,u)\leq 0, \tag{4.20}$$

where $u:=d(Ty_n,\overline{x})$, $v:=d(y_n,\overline{x})$, $w:=d(y_n,Ty_n)$. By triangle inequality, $d(y_n,Ty_n)\leq d(Ty_n,\overline{x})+d(y_n,\overline{x})$, that is, $w\leq u+v$. Now, since F is non-increasing in the fourth variable, we have

$$F\left(u,v,0,u+v,v,u\right)\leq F\left(u,v,0,w,v,u\right)\leq 0$$

which implies, by assumption (F_2) that there exists $h \in [0,1)$ such that $u \leq qv$, that is,

$$d(Ty_n, \overline{x}) \leq hd(y_n, \overline{x}),$$

which, by (4.19) yields

$$d(y_{n+1}, \overline{x}) \leq hd(y_n, \overline{x}) + \varepsilon_n.$$

Now both conclusions follow by applying Lemma 1. □

Remark 4.3. Note that for the class of mappings satisfying the hypotheses of Theorem 4.4, all the three concepts of stability given by Definitions 2.1-2.3 coincide.

In a similar way to the proof of Theorem 4.4 one can prove a slightly more general version of Theorem 4.4, in order to include other stability results as particular cases.

Theorem 4.5. *Let (X,d) be a complete metric space, $T : X \to X$ a self mapping with $Fix\,(T) \neq \emptyset$, for which there exists $F \in \mathcal{F}$ such that for all $x, y \in X$* (4.18) *holds.*

If F satisfies (F_{2b}), then Picard iteration is: a) T-stable; b) summable almost T-stable.

Corollary 4.1. *Let (X,d) be a complete metric space and $T : X \to X$ a self mapping with $Fix\,(T) \neq \emptyset$, for which* (4.18) *holds, for all $x, y \in X$, with $F \in \mathcal{F}$ from Example 3.7.*

Then Picard iteration is: a) T-stable; b) summable almost T-stable.

Proof. As the function $F \in \mathcal{F}$ from Example 3.7 satisfies (F_1), (F_3) and (F_{2b}), with $h = a$, we can apply Theorem 4.5 (but not Theorem 4.4) to get the desired conclusion. □

Remark 4.4. Corollary 4.1 is the main result in [21] and [23], see also [6].

Corollary 4.2. *Let (X,d) be a complete metric space, $T : X \to X$ a self mapping with $Fix\,(T) \neq \emptyset$, which satisfies* (4.18) *for all $x, y \in X$, with $F \in \mathcal{F}$ from Example 3.8.*

Then Picard iteration is: a) T-stable; b) summable almost T-stable.

Proof. As the function $F \in \mathcal{F}$ from Example 3.8 satisfies (F_1), (F_3) and (F_{2b}), with $h = a$, we can apply Theorem 4.5 (but not Theorem 4.4) to get the desired conclusion. □

Remark 4.5. Corollary 4.2 is the main result in [17], see also [20]. If $\varphi(t) = Lt$, with the constant $L \geq 0$, then Corollary 4.2 reduces to Corollary 4.1.

5. Concluding remarks and some directions for further study

The results obtained in this paper are significant generalizations of a multitude of both fixed point theorems and stability theorems for Picard iteration existing in literature: [6]-[10], [14]–[25], [29]-[34], [37], [39] and most of the references therein.

Note that all contractive conditions obtained from (3.9) with F in Examples 3.1-3.7 imply the contraction condition (1.2) used by Rhoades in [31], [32] and [33].

Thus, Theorem 4.4 extends Theorem 1 [33] and Theorem 1 [31] of Rhoades, Theorem 2 of Harder and Hicks [16] and Theorem 2 of Ostrowski [24]. For example, Theorem 1 in [33] is obtained from Theorem 4.4 in this paper, if F is the function in Example 3.6, Theorem 2 in [16] is obtained from Theorem 4.4 in this paper, if F is the function in Example 3.4, while Theorem 2 in [24] is obtained from Theorem 4.4 in this paper, if F is the function in Example 3.1.

Moreover, our Theorem 4.5 extends several stability results obtained by M. O. Osilike and M. O. Olatinwo and co-workers, in a series of papers from which we quote [21] and [17], respectively. Osilike's stability results for Picard iteration are obtained from Theorem 4.5 in this paper, if F is the function in Example 3.7, while Olatinwo's stability results for Picard iteration are obtained from Theorem 4.5 in this paper, if F is the function in Example 3.8.

On the other hand, our Theorem 3.3 extends several classical fixed point theorems, among which we mention that of Banach, Kannan, Chatterjea, Zamfirescu, and many others, which are obtained from Theorem 3.3 if F is as in Example 3.1, Example 3.2, Example 3.3, Example 3.4, respectively, see also [1], [4], [5], [11], [12].

We end this paper by noting that some weaker concepts of stability of fixed point iteration procedures were also considered in literature: a notion of *weak stability*, introduced and studied in [10], Chapter 7, and a different concept of *weak stability* together with that of *pseudo stability*, introduced and studied in [38], for the Ishikawa fixed point iteration procedure in the case of Φ-hemicontractive and accretive operators.

As noted in [38], if an iteration procedure is T-stable, then it is weakly T-stable and, if the iteration procedure is weakly T-stable, then it is both almost and pseudo T-stable. But if an iteration procedure is either almost or pseudo T-stable, it may fail to be weakly T-stable.

Consequently, it is of important theoretical interest to study the weak stability of fixed point iteration procedures for those classes of contractive mappings for which Picard and other fixed point iteration procedures are not stable (almost stable).

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