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Convergence theorems for fixed point iterative methods defined as admissible perturbations of a nonlinear operator

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ABSTRACT. The aim of this paper is to prove some convergence theorems for a general fixed point iterative method defined by means of the new concept of *admissible perturbation* of a nonlinear operator, introduced in [Rus, I. A., *An abstract point of view on iterative approximation of fixed points*, Fixed Point Theory **13** (2012), No. 1, 179–192]. The obtained convergence theorems extend and unify some fundamental results in the iterative approximation of fixed points due to Petryshyn [Petryshyn, W. V., *Construction of fixed points of demicompact mappings in Hilbert space*, J. Math. Anal. Appl. **14** (1966), 276–284] and Browder and Petryshyn [Browder, F. E. and Petryshyn, W. V., *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl. **20** (1967), No. 2, 197–228].

1. INTRODUCTION AND PRELIMINARIES

There exists a vast literature on the iterative approximation of fixed points, see for example the recent monographs [2], [7] and [16] and references therein. The fundamental problem of this field of research could be briefly stated as follows.

We have to solve a certain nonlinear fixed point equation

$$x = Tx, \tag{1.1}$$

where T is a given self operator of a space X. Suppose X and T are such that the equation (1.1) has at least one solution (usually called a *fixed point* of T).

A typical situation of this kind is illustrated by the well known Browder-Gohde-Kirk fixed point theorem (see Theorem 3.1 in [2]), stated here in a Hilbert space, because in this paper we are particularly interested to work in this setting.

Theorem 1.1. *Let C be a closed bounded and convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive operator. Then T has at least one fixed point.*

In case X and T are general enough, like in Theorem 1.1 (or even more general, e.g., X is a uniformly Banach space and T is nonexpansive), when T has at least one fixed point, however, the Picard iteration associated to (1.1), that is, the sequence defined by $x_0 \in X$ and

$$x_{n+1} = Tx_n,\ n = 0, 1, 2, \ldots \tag{1.2}$$

does not converge in general or, even if it converges, its limit is not a fixed point of T.

In such circumstances, it is necessary to consider more reliable fixed point iterative methods, like Krasnoselskij iteration, Mann iteration, Ishikawa iteration etc. For the sake of completeness, we present below the definitions of Krasnoselskij, Mann and Ishikawa iteration procedures (for more details and convergence results, see [2], [7]).

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Let E be a real vector space and $T : E \to E$ a given operator. Let $x_0 \in E$ be arbitrary and $\{\alpha_n\} \subset [0, 1]$ a sequence of real numbers.
The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by $x_0 \in E$ and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n\,, \quad n = 0, 1, 2, \dots \tag{1.3}$$

is called the *Mann iteration* or *Mann iterative procedure*. The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by

$$\begin{cases} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n\,, \quad n = 0, 1, 2, \dots \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n\,, \quad n = 0, 1, 2, \dots\,, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0, 1]$, and $x_0 \in E$ arbitrary, is called the *Ishikawa iteration* or *Ishikawa iterative procedure*.

Remark 1.1. For $\beta_n \equiv 0$ the Ishikawa iteration (1.4) reduces to Mann iteration (1.3), while, for $\alpha_n = \lambda$ (constant), the Mann iteration (1.3) reduces to the so called *Krasnoselskij iteration*. Picard iteration is obtained from the latter for $\lambda = 1$.

In order to state a very important convergence theorem for the Krasnoselskij iteration in the class of nonexpansive mappings, due to Petryshyn [18], we need the following concept.

Definition 1.1. (Petryshyn [18]) Let H be a Hilbert space and C a subset of H. A mapping $T : C \to H$ is called *demicompact* if it has the property that whenever $\{u_n\}$ is a bounded sequence in H and $\{Tu_n - u_n\}$ is strongly convergent, then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which is strongly convergent.

Theorem 1.2. (Petryshyn [18]) *Let C be a bounded closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive and demicompact operator. Then the set $Fix\,(T)$ of fixed points of T is a nonempty convex set and for any given x_0 in C and any fixed number λ with $0 < \lambda < 1$ the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ given by*

$$x_{n+1} = (1 - \lambda)\,x_n + \lambda\,T x_n\,, \quad n = 0, 1, 2, \dots \tag{1.5}$$

converges (strongly) to a fixed point of T.

In a very recent paper, I. A. Rus [22] considered a new approach to fixed point iterative methods, based on the concept of admissible perturbation of a self operator. The theory of admissible perturbations of an operator opened a new direction of research that unifies the most important aspects of the iterative approximation of fixed points for single valued self and nonself operators. In the case of nonself operators the theory of admissible perturbations has been studied in [4].

The problems tackled in [22] and [4] are the following: a) The admissible perturbation of an operator; b) Iterative algorithms in terms of admissible perturbations; c) Gronwall lemmas; d) Comparison lemmas; e) Data dependence; f) Stability of an iterative algorithm.

In the present paper our aim is more limited: just to use the theory of admissible perturbations in order to establish convergence theorems for Krasnoselskij type fixed point iterations, thus obtaining very general and interesting results that extend Theorem 1.2 and unify many related results in literature.

2. ADMISSIBLE PERTURBATIONS OF AN OPERATOR

Definition 2.2. ([22]) Let X be a nonempty set. A mapping $G : X \times X \to X$ is called *admissible* if it satisfies the following two conditions:
(A_1) $G(x, x) = x$, *for all* $x \in X$;
(A_2) $G(x, y) = x$ *implies* $y = x$.

Definition 2.3. ([22]) Let X be a nonempty set. If $f : X \to X$ is a given operator and $G : X \times X \to X$ is an admissible mapping, then the operator $f_G : X \to X$, defined by

$$f_G(x) = G(x, f(x)), \ \forall x \in X, \tag{2.6}$$

is called the *admissible perturbation* of f.

Remark 2.2. The following property of admissible perturbations is fundamental in the iterative approximation of fixed points: if $f : X \to X$ is a given operator and $f_G : X \to X$ denotes its admissible perturbation, then

$$Fix\,(f_G) = Fix\,(f) := \{x \in X | x = f(x)\}, \tag{2.7}$$

that is, the admissible perturbation f_G of f has the same set of fixed points as the operator f itself.

Note that, in general,

$$Fix\,(f_G^n) \neq Fix\,(f^n), \ n \geq 2. \tag{2.8}$$

Example 2.1. ([22]) Let $(V, +, \mathbb{R})$ be a real vector space, $X \subset V$ a convex subset, $\lambda \in (0, 1)$, $f : X \to X$ and $G : X \times X \to X$ be defined by

$$G(x, y) := (1 - \lambda)x + \lambda y, \ x, y \in X.$$

Then f_G is an admissible perturbation of f. We shall denote f_G by f_λ and call it the Krasnoselskij perturbation of f.

Example 2.2. ([22]) Let $(V,+,\mathbb{R})$ be a real vector space, $X \subset V$ a convex subset, $\chi : X \times X \to (0, 1)$, $f : X \to X$ and $G(x, y) := (1 - \chi(x, y))x + \chi(x, y)y$.

Then f_G is an admissible perturbation of f which reduces to the Krasnoselskij perturbation in the case $\chi(x, y)$ is a constant function.

For other important examples of admissible mappings and admissible perturbations of nonlinear operators, see [22] (for the case of self mappings) and [4] (for the case of nonself mappings).

Definition 2.4. ([22])

Let $f : X \to X$ be a nonlinear operator and $G : X \times X \to X$ an admissible mapping. Then the iterative algorithm $\{x_n\}$ given by $x_0 \in X$ and

$$x_{n+1} = G(x_n, f(x_n)), n \geq 0, \tag{2.9}$$

is called the *Krasnoselskij algorithm* corresponding to G or the *GK-algorithm*.

Definition 2.5. Let H be a Hilbert space and $T : H \to H$ be an operator with $Fix\,(T) \neq \emptyset$. We say that the admissible mapping $G : H \times H \to H$ *has the property (C) with respect to* T if there exists $\lambda \in (0, 1)$ such that

$$\|G(x, Tx) - p\| \leq \lambda^2 \cdot \|x - p\|^2 + (1 - \lambda)^2 \cdot \|Tx - p\|^2 + 2\lambda(1 - \lambda) \langle Tx - p, x - p \rangle, \tag{2.10}$$

for all $x \in H$ and all $p \in Fix\,(T)$.

Remark 2.3. In the particular case

$$G(x,y) := (1-\lambda)x + \lambda y,\ x,y \in X, \tag{2.11}$$

the GK-algorithm (2.9) reduces to the classical Krasnoselskij algorithm (1.5).

Note that in a Hilbert space H, the admissible mapping $G : H \times H \to H$ given by (2.11) has the property (C) with respect to any selfmap $T : H \to H$ with $Fix\,(T) \neq \emptyset$, as a direct consequence of the next lemma.

Lemma 2.1. ([2], Lemma 1.8) *Let x, y, z be points in a Hilbert space and $\lambda \in [0,1]$. Then*

$$\|\,\lambda x + (1-\lambda)\,y - z\,\|^2 = \lambda\ \|\,x - z\,\|^2 + (1-\lambda)\ \|\,y - z\,\|^2 - \lambda\,(1-\lambda)\ \|\,x - y\,\|^2 .$$

Starting from these concepts and results, the main aim of the next section is to extend Theorem 1.2 from the case of classical Krasnoselskij algorithm (1.5) to the general case of the GK-algorithm (2.9).

3. Convergence theorems

The main result of this paper is the following strong convergence theorem for the GK-algorithm associated to nonexpansive operators in Hilbert spaces.

Theorem 3.3. *Let C be a bounded closed convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive and demicompact operator. Then the set $Fix\,(T)$ of fixed points of T is a nonempty convex set.*

Moreover, if $G : H \times H \to H$ is an admissible mapping which has the property (C) with respect to T, then the GK-algorithm $\{x_n\}_{n=0}^{\infty}$ given by x_0 in C and

$$x_{n+1} = G(x_n, f(x_n)), n \geq 0, \tag{3.12}$$

converges (strongly) to a fixed point of T.

Proof. By Theorem 1.1, $Fix\,(T)$ is nonempty. In order to prove that $Fix\,(T)$ in convex, let us consider $p, q \in Fix\,(T)$ and $\lambda \in [0,1]$. Then, by Lemma 2.1 we have

$$\|T\,[\lambda p + (1-\lambda)q] - [\lambda p + (1-\lambda)q]\|^2 = \|\lambda p + (1-\lambda)q - T\,[\lambda p + (1-\lambda)q]\|^2 \tag{3.13}$$
$$= \lambda\,\|p - T\,[\lambda p + (1-\lambda)q]\|^2 + (1-\lambda)\,\|q - T\,[\lambda p + (1-\lambda)q]\|^2 - \lambda(1-\lambda)\,\|p-q\|^2 .$$

Using the fact that $p = Tp$, $q = Tq$ and T is nonexpansive, one obtains

$$\|p - T\,[\lambda p + (1-\lambda)q]\| \leq (1-\lambda)\,\|p-q\|\,;\ \|q - T\,[\lambda p + (1-\lambda)q]\| \leq \lambda\,\|p-q\|\,,$$

and therefore, by (3.13)

$$\|T\,[\lambda p + (1-\lambda)q] - [\lambda p + (1-\lambda)q]\|^2 \leq \lambda(1-\lambda)^2\,\|p-q\|^2 + (1-\lambda)\lambda^2\,\|p-q\|^2$$
$$-\lambda(1-\lambda)\,\|p-q\|^2 = \left[\lambda(1-\lambda)^2 + (1-\lambda)\lambda^2 - \lambda(1-\lambda)\right]\|p-q\|^2 = 0,$$

which shows that

$$T\,[\lambda p + (1-\lambda)q] = \lambda p + (1-\lambda)q,$$

that is, $\lambda p + (1-\lambda)q \in Fix\,(T)$, which proves that $Fix\,(T)$ is convex.

In order to prove the second part of the theorem, let p be a fixed point of T.

We first show that the sequence $\{x_n - Tx_n\}_{n \in \mathbb{N}}$ converges strongly to zero.

Indeed, since G has the property (C) with respect to T, we have

$$\|x_{n+1} - p\|^2 = \|T_G(x_n) - p\|^2 = \|G(x_n, Tx_n) - p\|^2 \tag{3.14}$$
$$\leq \lambda^2 \cdot \|x_n - p\|^2 + (1-\lambda)^2 \cdot \|Tx_n - p\|^2 + 2\lambda(1-\lambda)\,\langle Tx_n - p, x_n - p\rangle .$$

On the other hand,

$$\|x_n - Tx_n\|^2 = \|x_n - p\|^2 + \|Tx_n - p\|^2 - \langle Tx_n - p, x_n - p\rangle \,. \tag{3.15}$$

By (3.14) and (3.15) and by using the nonexpansiveness of T and the fact that $Tp = p$, for any real number a we have

$$\| x_{n+1} - p\|^2 + a^2 \| x_n - Tx_n\|^2 \leq [2a^2 + \lambda^2 + (1-\lambda)^2] \cdot \| x_n - p\|^2 +$$

$$+2[\lambda(1-\lambda) - a^2] \cdot \langle Tx_n - p, x_n - p\rangle \,.$$

If we choose now a nonzero a such that $a^2 \leq \lambda(1-\lambda)$, then from the last inequality we obtain

$$\| x_{n+1} - p\|^2 + a^2 \| x_n - Tx_n\|^2 \leq \tag{3.16}$$

$$\leq \left(2a^2 + \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) - 2a^2\right) \|x_n - p\|^2 = \|x_n - p\|^2 \,.$$

(we used the Cauchy-Schwarz inequality,

$$\left. \langle Tx_n - p, x_n - p\rangle \;\leq\; \| Tx_n - p\| \cdot \| x_n - p\| \leq \| x_n - p\|^2 \right) .$$

So, by (3) we get

$$a^2 \|x_n - Tx_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \,,\; n \geq 0. \tag{3.17}$$

Now, take $n = 0, 1, \dots, N$ in (3.17) and sum up all the obtained inequalities to get

$$a^2 \cdot \sum_{n=0}^{N} \|x_n - Tx_n\|^2 \leq \|x_0 - p\|^2 - \|x_{N+1} - p\|^2 \leq \|x_0 - p\|^2 \,, \tag{3.18}$$

which shows that the series

$$\sum_{n=0}^{N} \|x_n - Tx_n\|^2$$

is convergent, and hence

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.19}$$

This shows that $x_n - Tx_n \to 0$ (strongly), and since T is demicompact, it follows that there exists a subsequence $\{x_{n_k}\} \subset C$ and a point $q \in C$ such that

$$\lim_{k\to\infty} x_{n_k} = q.$$

But T is nonexpansive, hence continuous. This implies

$$\lim_{k\to\infty} Tx_{n_k} = Tq.$$

But, by virtue of (3.19), $0 = \lim_{k\to\infty} (x_{n_k} - Tx_{n_k}) = q - Tq$, which shows that $q \in Fix\,(T)$.

Using the inequality (3.17), with $p := q$, we deduce that the sequence of nonnegative real numbers $\{\|x_n - q\|\}_{n\geq 0}$ is nonincreasing, hence convergent.

Since its subsequence $\{\|x_{n_k} - q\|\}_{k\geq 0}$ converges to 0, it follows that the entire sequence $\{\|x_n - q\|\}_{n\geq 0}$ converges to 0, that is, the sequence $\{x_n\}$ converges strongly to q, as $n\to\infty$.

□

Remark 3.4. 1) If the admissible mapping $G(x,y)$ is given by (2.11), then by Theorem 3.3 we obtain Theorem 1.2, which is actually Theorem 6 of Petryshyn [18] (reformulated in Browder and Petryshyn [6]);

2) As the class of demicompact operators contains the compact operators, by Theorem 3.3 we obtain, in particular, two theorems in Hilbert spaces that extend the original results of Krasnoselskij [13] and Schaefer [23] (established there in the more general context of uniformly Banach spaces) from the case of classical Krasnoselskij algorithm (1.5) to the general case of the GK-algorithm (2.9);

Remark 3.5. 1) The first part of the proof of Theorem 3.3 is inspired by the proof of Proposition 4.1 in [14], while the second part of the proof follows the arguments in the proof from Browder and Petryshyn [6] (see also [2], Theorem 3.2);

2) It is possible to obtain the conclusion that the sequence $\{x_n - Tx_n\}_{n\in\mathbb{N}}$ converges strongly to zero in the proof of Theorem 3.3 in a simpler manner, suggested by the proof of Proposition 3.2 in [14]. Indeed, using the inequality (3.17), we deduce that the sequence of nonnegative real numbers $\{\|x_n - p\|\}_{n\geq 0}$ is nonincreasing, hence convergent.

By the same inequality (3.17) we have

$$0 \leq \|x_n - Tx_n\|^2 \leq \frac{1}{a^2}\left[\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right],\ n \geq 0, \tag{3.20}$$

from which, by letting $n \to \infty$, we get exactly the desired conclusion.

Definition 3.6. ([2], pp. 67) A map T of $C \subset H$ into H is said to be *demicompact at* u if, for any bounded sequence $\{x_n\}$ in C such that $x_n - Tx_n \to u$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\}$ and an x in C such that $x_{n_j} \to x$ as $j \to \infty$ and $x - Tx = u$.

Remark 3.6. Clearly, if T is demicompact on C, then it is demicompact at 0 but the converse is not true.

The demicompactness on the whole C of T in Theorem 3.3 may be weakened to the demicompactness at 0 by simultaneously adding another compensating assumption.

Theorem 3.4. *Let H be a Hilbert space, C a closed bounded convex subset of H, and $T : C \to C$ a nonexpansive mapping such that T satisfies any one of the following two conditions:*
(i) $I - T$ maps closed sets in C into closed sets in H;
(ii) T is demicompact at 0.

If $G : H \times H \to H$ is an admissible mapping which has the property (C) with respect to T, then the GK-algorithm $\{x_n\}_{n=0}^{\infty}$ given by x_0 in C and

$$x_{n+1} = G(x_n, f(x_n)), n \geq 0, \tag{3.21}$$

converges (strongly) to a fixed point of T.

Proof. Note that in the proof of Theorem 3.3 we actually used the demicompactness of T at 0, so the arguments used there could be applied unchanged here. □

Remark 3.7. Let us observe that the if $\{x_n\}$ is the GK-algorithm $\{x_n\}_{n=0}^{\infty}$ given by x_0 in C and

$$x_{n+1} = G(x_n, f(x_n)), n \geq 0,$$

then $x_n = T_G^n(x_0)$, for all $n \geq 0$, where T_G is the admissible perturbation of T.

From the proof of Theorem 3.3 it results that T_G is *asymptotically regular*, i.e.,

$$\| T_G^n(x) - T_G^{n+1}(x)\| \to 0,$$

as $n \to \infty$, for any $x \in C$, that is,

$$x_n - x_{n+1} \to 0, \quad \text{as} \quad n \to \infty, \tag{3.22}$$

for any $x_0 \in C$.

The existence of the previous limit alone does not imply generally the convergence of the sequence $\{x_n\}_{n=0}^{\infty}$ to a fixed point of T. For example, in Theorems 3.3 and 3.4 certain additional assumptions were necessary, i.e., the demicompactness of T and the demicompactness of T at 0, respectively).

But there are other possible additional assumptions to ensure the convergence of $\{x_n\}_{n=0}^{\infty}$ under the hypothesis of asymptotic regularity. For example, in the case of the real line, $C = [a, b]$ the closed bounded interval and $T : C \to C$ a continuous function, Hillam [10] showed that the Picard iteration associated to T converges if and only if it is asymptotically regular.

Definition 3.7. Let $G : X \times X \to X$ be an admissible mapping on a normed space X. We say that G is *affine Lipschitzian* if there exists a constant $\mu \in [0, 1]$ such that

$$\|G(x_1, y_1) - G(x_2, y_2)\| \le \mu \|x_1 - x_2\| + (1 - \mu) \|y_1 - y_2\|, \tag{3.23}$$

for all $x_1,\ x_2,\ y_1,\ y_2 \in X$.

Example 3.3. The admissible mapping $G(x, y)$ given by

$$G(x, y) := (1 - \lambda)x + \lambda y,\ x, y \in X,$$

which corresponds to the classical Krasnoselskij iteration, is an affine Lipschitzian mapping.

In fact, the admissible mappings corresponding to other important fixed point iterative algorithms, like Picard, Mann, Ishikawa etc., are all affine Lipschitzian mappings.

If in Theorems 3.3 and 3.4 we remove the demicompactness assumption, then, as in the case of classical Krasnoselskij iteration (see [2]), the GK-algorithm does not longer converge strongly, in general, but it could converge (at least) weakly to a fixed point, as shown by the next theorem, which extends Theorem 3.3 in [2].

Theorem 3.5. *Let H be a Hilbert space, C a closed bounded convex subset of H, and $T : C \to C$ a nonexpansive mapping such that $F_T = \{p\}$. If $G : H \times H \to H$ is a affine Lipschitzian admissible mapping which has the property (C) with respect to T, then the GK-algorithm $\{x_n\}_{n=0}^{\infty}$ given by x_0 in C and*

$$x_{n+1} = G(x_n, f(x_n)), n \ge 0, \tag{3.24}$$

converges weakly to p.

Proof. It suffices to show that if $\{x_{n_j}\}_{j=0}^{\infty},\ x_{n_j} = T_G^{n_j} x$, converges weakly to a certain p_0, then p_0 is a fixed point of T_G (and hence of T) and therefore $p_0 = p$. Suppose that $\{x_{n_j}\}_{j=0}^{\infty}$ does not converge weakly to p. Then, since G is affine Lipschitzian and T is nonexpansive, we have

$$\|G(x, Tx) - G(y, Ty)\| \le \mu \|x - y\| + (1 - \mu) \|Tx - Ty\| \le \mu \|x - y\| + (1 - \mu) \|x - y\| = \|x - y\|,$$

which shows that the admissible perturbation T_G of T is nonexpansive and hence

$$\| x_{n_j} - T_G p_0\| \le \| T_G x_{n_j} - T_G p_0\| + \| x_{n_j} - T_G x_{n_j}\| \le$$
$$\le \| x_{n_j} - p_0\| + \| x_{n_j} - T_G x_{n_j}\|$$

and, using the arguments in the proof of Theorem 3.3, it results

$$\| x_{n_j} - T_G x_{n_j} \| \to 0, \quad \text{as} \quad n \to \infty,$$

and so the last inequality implies that

$$\lim\ \sup \left(\| x_{n_j} - T_G p_0 \| - \| x_{n_j} - p_0 \| \right) \leq 0. \tag{3.25}$$

But, like in the proof of Theorem 3.3, we have

$$\| x_{n_j} - T_G p_0 \|^2 = \| (x_{n_j} - p_0) + (p_0 - T_G p_0) \|^2 =$$

$$= \| x_{n_j} - p_0 \|^2 + \| p_0 - T_G p_0 \|^2 + 2 \left\langle x_{n_j} - p_0, p_0 - T_G p_0 \right\rangle,$$

which shows, together with $x_{n_j} \rightharpoonup p_0$ (as $j \to \infty$), that

$$\lim_{n\to\infty} \left[\| x_{n_j} - T_G p_0 \|^2 - \| x_{n_j} - p_0 \|^2 \right] = \| p_0 - T_G p_0 \|^2 . \tag{3.26}$$

On the other hand, we have

$$\| x_{n_j} - T_G p_0 \|^2 - \| x_{n_j} - p_0 \|^2 = \left(\| x_{n_j} - T_G p_0 \| - \| x_{n_j} - p_0 \| \right) \cdot \tag{3.27}$$

$$\cdot \left(\| x_{n_j} - T_G p_0 \| + \| x_{n_j} - p_0 \| \right) .$$

Since C is bounded, the sequence

$$\{ \| x_{n_j} - T_G p_0 \| + \| x_{n_j} - p_0 \| \}$$

is bounded, too, and so by the relations (3.25) – (3.27) we get

$$\| p_0 - T_G p_0 \| \leq 0, \quad \text{i.e.,} \quad T_G p_0 = p_0 \Leftrightarrow p_0 \in F_T = \{p\}.$$

□

Remark 3.8. The assumption $F_T = \{p\}$ in Theorem 3.5 may be removed in order to obtain a more general result, similar to Theorem 3.4 in [2].

Theorem 3.6. *Let C be a bounded closed convex subset of a Hilbert space and $T : C \to C$ be a nonexpansive operator. If $G : H \times H \to H$ is a affine Lipschitzian admissible mapping which has the property (C) with respect to T, then the GK-algorithm $\{x_n\}_{n=0}^{\infty}$ given by x_0 in C and*

$$x_{n+1} = G(x_n, f(x_n)), n \geq 0,$$

converges weakly to a fixed point of T in C.

Proof. We follow essentially the steps and arguments in the proof of Theorem 3.4 in [2] but transposed from the classical Krasnoselskij iteration to the GK-algorithm. For each $p \in Fix\,(T)$ and each n we have, by the proof of Theorem 3.3,

$$\| x_{n+1} - p \| \leq \| x_n - p \| ,$$

which shows that the function $g(p) = \lim\limits_{n\to\infty} \| x_n - p \|$ is well defined and is a lower semi-continuous convex function on the nonempty convex set $Fix\,(T)$. Let

$$d_0 = \inf\{ g(p) : p \in Fix\,(T) \}.$$

For each $\varepsilon > 0$, the set

$$F_\varepsilon = \{ y : g(y) \leq d_0 + \varepsilon \}$$

is closed, convex, and, hence, weakly compact. Therefore $\bigcap\limits_{\varepsilon>0} F_\varepsilon \neq \emptyset$, and in fact

$$\bigcap_{\varepsilon>0} F_\varepsilon = \{ y : g(y) = d_0 \} \equiv F_0.$$

Moreover, F_0 contains exactly one point. Indeed, since F_0 is convex and closed, for $p_0, p_1 \in F_0$, and $p_\lambda = (1-\lambda)p_0 + \lambda p_1$,

$$g^2(p_\lambda) = \lim_{n\to\infty} \| p_\lambda - x_n\|^2 = \lim_{n\to\infty} (\|\lambda(p_1 - x_n) + (1-\lambda)(p_0 - x_n)\|^2) =$$

$$= \lim_{n\to\infty} (\lambda^2 \|p_1 - x_n\|^2 + (1-\lambda)^2 \| p_0 - x_n\|^2 + 2\lambda(1-\lambda) \langle p_1 - x_n, p_0 - x_n\rangle)$$

$$= \lim_{n\to\infty} (\lambda^2 \|p_1 - x_n\|^2 + (1-\lambda)^2 \| p_0 - x_n\|^2 + 2\lambda(1-\lambda) \| p_1 - x_n\| \cdot \| p_0 - x_n\|) +$$

$$+ \lim_{n\to\infty} \{2\lambda(1-\lambda) \, [\langle p_1 - x_n, p_0 - x_n\rangle - \| p_1 - x_n\| \cdot \| p_0 - x_n\|]\} =$$

$$= g^2(p) + \lim_{n\to\infty} \{2\lambda(1-\lambda) \ \langle p_1 - x_n, p_0 - x_n\rangle - \| p_1 - x_n\| \cdot \| p_0 - x_n\|\} .$$

Hence

$$\lim_{n\to\infty} \{2\lambda(1-\lambda) \, [\langle p_1 - x_n, p_0 - x_n\rangle - \| p_1 - x_n\| \cdot \| p_0 - x_n\|]\} = 0.$$

Since

$$\|p_1 - x_n\| \to d_0 \text{ and } \| p_0 - x_n\| \to d_0,$$

the latter relation implies that

$$\| p_1 - p_0\|^2 = \| (p_1 - x_n) + (x_n - p_0\|^2 = \| p_1 - x_n\|^2 +$$

$$+ \| x_n - p_0\|^2 - 2 < p_1 - x_n, \, p_0 - x_n > \to d_0^2 + d_0^2 - 2d_0^2 = 0,$$

giving a contradiction.

Now, in order to show that $x_n = T_G^n x_0 \rightharpoonup p_0$, it suffices to assume that $x_{n_j} \rightharpoonup p$ for an infinite subsequence and then prove that $p = p_0$. By the arguments in Theorem 3.3, $p \in Fix\,(T)$. Considering the definition of g and the fact that $x_{n_j} \to p$, we have

$$\| x_{n_j} - p_0\|^2 = \| x_{n_j} - p + p - p_0\|^2 = \| x_{n_j} - p\|^2 + \| p - p_0\|^2 -$$

$$-2 \left\langle x_{n_j} - p, \, p - p_0\right\rangle \to g^2(p) + \| p - p_0\|^2 = g^2(p_0) = d_0^2.$$

Since $g^2(p) \geq d_0^2$, the last inequality implies that

$$\| p - p_0\| \leq 0,$$

which means that $p = p_0$.

□

4. Conclusions and further study

The results established in the present paper are generalizations of several important results in literature, amongst which we mention the following ones: Theorem 3.3 generalizes Theorem 6 of Petryshyn [18] (which may also be found reformulated in Browder and Petryshyn [6]); Theorem 3.4 extends Corollary 3.1 from [19], while Theorem 3.5 and Theorem 3.6 are generalizations of Theorem 7 and Theorem 8, respectively, from [6].

Similar results to those established in the present paper could be obtained for most of the convergence theorems in [1]-[5], [8]-[12], [15]-[21], [24], [25] and many other related papers.

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