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From a Dieudonné theorem concerning the Cauchy problem to an open problem in the theory of weakly Picard operators

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ABSTRACT. Let (X, d) be a complete metric space and let $f : X \to X$ be a self operator. In this paper we study the following two problems:

Problem 1. Let f be such that its fixed points set is a singleton, i.e., $F_f = \{x^*\}$. Under which conditions the next implication does hold:
f is asymptotically regular $\Rightarrow$ f is a Picard operator?

Problem 2. Let f be such that, $F_f \neq \phi$. Under which conditions the following implication does hold:
f is asymptotically regular $\Rightarrow$ f is a weakly Picard operator?

The case of operators defined on a linear L^*-space is also studied.

1. INTRODUCTION

Let $f \in C([a,b] \times \mathbb{R}^m, \mathbb{R}^m)$. We consider the following Cauchy problem:

$$y' = f(x,y),\ y(a) = y_0 \tag{1.1}$$

and the corresponding sequence of successive approximations

$$y_{n+1}(x) := y_0 + \int_a^x f(s, y_n(s))ds, n = 0, 1, \ldots, \tag{1.2}$$

The following result was given by J. Dieudonné in 1945 ([17]).

Theorem D.
We suppose that the Cauchy problem (1.1) has a unique solution. Then there exists $h \in]0, b-a[$ such that the sequence of successive approximations (1.2) converges to the unique solution of the Cauchy problem (1.1) uniformly on $[a, a+h]$, if and only if the sequence $\{y_{n+1} - y_n\}_{n\in\mathbb{N}}$ converges to the null function uniformly on $[a, a+h]$.

This theorem was generalized in 1965 by T. Wazewski ([47]) for the case of a Cauchy problem in Banach spaces. An abstract result was given by A. Pelczar in 1969 ([31]).

The aim of this paper is to study the following problems (see [3] and [35]; see also [19], [13], [16], [30], [32], [42], [44], [20], [29], ...).

Problem 1. *If f has at most one fixed point, under which conditions we have that:*
f is asymptotically regular $\Rightarrow$ f is a Picard operator?

Problem 2. *If $F_f \neq \emptyset$, under which conditions we have that:*
f is asymptotically regular $\Rightarrow$ f is a weakly Picard operator?

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2. PRELIMINARIES

The considerations throughout the paper refer to operators defined on a metric space (X, d), respectively on an L-space $(X, \to)$, sometimes the change of the framework space being implicit. In the sequel we shall present the necessary notions for both framework spaces.

2.1. **Operators on metric spaces.** Let (X, d) be a metric space and $f : X \to X$ an operator. We denote by F_f the set of all fixed points of f. The operator f is a weakly Picard operator (WPO) ([34], [41],...) if the sequence $\{f^n(x)\}$ converges for all $x \in X$ and its limit (which generally may depend on x) is a fixed point of f. If an operator f is a WPO and $F_f = \{x^*\}$, then, by definition, f is called Picard operator (PO). The operator f is ψ-WPO if there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous at 0 with $\psi(0) = 0$ such that

$$d(x, f^\infty(x)) \leq \psi(d(x, f(x)), \ \forall\, x \in X.$$

An operator $f : X \to X$ is called asymptotically regular at x_0 iff $d(f^{n+1}(x_0), f^n(x_0)) \to 0$ as $n \to \infty$. If f is asymptotically regular at any $x \in X$, then it is said to be asymptotically regular on X, or simply asymptotically regular.

If $f : X \to X$ is a WPO then we define the operator $f^\infty : X \to X$, given by $f^\infty(x) := \lim\limits_{n\to\infty} f^n(x)$, for all $x \in X$.

2.2. **Operators on L-spaces.** Let X be a nonempty set and denote by

$$s(X) := \{(x_n)_{n\in\mathbb{N}} \,|\, x_n \in X, n \in \mathbb{N}\}$$

the set of all sequences in X.

Let $c(X) \subset s(X)$ be a subset of $s(X)$ and $Lim : c(X) \to X$ be an operator. Following Fréchet (see [21], [22], [38], [41], ...) the triple $(X, c(X), Lim)$ is called an L-space iff the following conditions are satisfied:

(i) If $x_n = x, \forall\, n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.

(ii) If $(x_n)_{n\in\mathbb{N}} \in c(X)$, $Lim(x_n)_{n\in\mathbb{N}} = x$, and $(x_{n_i})_{i\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$, then $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $Lim(x_{n_i})_{i\in\mathbb{N}} = x$.

An element of $c(X)$ is called a convergent sequence, $x = Lim(x_n)_{n\in\mathbb{N}}$ is called the limit of this sequence and we shall write $x_n \to x$ as $n \to \infty$.

In what follows we shall denote an L-space by $(X, \to)$.

An L-space is called an L^*-space if, in addition, it satisfies the following axiom:

(iii) If $(x_n)_{n\in\mathbb{N}}$ does not converge to x, then there exists a subsequence, $(x_{n_i})_{i\in\mathbb{N}}$, of this sequence, with the property that any subsequence of it does not converge to x.

In general the L^*-space convergence is not topological, in the sense that there is generally no topology which generates this convergence. In spite of this fact, one defines the continuity, compactness etc., in the usual way with respect to the convergence.

The notions of weakly Picard operator and, respectively, of Picard operator are defined on an L-space similarly to the case of a metric space.

2.3. **Daneš-Pasicki measure of noncompactness.** Let (X, d) be a metric space and let $\mathcal{P}_b(X), \mathcal{P}_{cp}(X)$ be the family of all nonempty bounded subsets, and nonempty compact subset of X, respectively. By definition (see [36], ...) a functional $\alpha_{DP} : P_b(X) \to \mathbb{R}_+$ is called a Daneš-Pasicki measure of noncompactness if

(i) $\alpha_{DP}(Y) = 0 \Rightarrow \overline{Y} \in P_{cp}(X), \forall\, Y \in P_b(X)$;

(ii) $Y_1, Y_2 \in P_b(X), Y_1 \subset Y_2 \Rightarrow \alpha_{DP}(Y_1) \leq \alpha_{DP}(Y_2)$;

(iii) $Y \in P_b(X), x \in X \Rightarrow \alpha_{DP}(Y \cup \{x\}) = \alpha_{DP}(Y)$.

We remark that Kuratowski's measure of noncompactness and Hausdorff's measure of noncompactness are examples of Daneš-Pasicki measure of noncompactness.

2.4. **Asymptotic regularity and fixed points.** Theorem D of Dieudonné mentioned in the Introduction is not the only fixed point result involving the property of asymptotic regularity of an operator. There are several other results, as for example:

Theorem 2.1 (Folklore Lemma). *Let $(X, +, K, \to)$ $(K := \mathbb{R} \vee \mathbb{C})$ be a linear L^*-space, $Y \subset X$ a nonempty subset of X and $f : Y \to Y$ be an operator. We suppose that:*

(i) *Y is compact;*
(ii) *f is continuous;*
(iii) *f is asymptotically regular.*

Then

(a) $\omega_f(x) \neq \emptyset, \forall\, x \in X$;
(b) $\omega_f(x) \subset F_f, \forall\, x \in X$;
(c) *if* $\operatorname{card} F_f \leq 1$, *then f is PO;*
(d) *if* $\operatorname{card} \omega_f(x) \leq 1, \forall\, x \in X$, *then f is WPO.*

Let (X, d) be a metric space and $f : X \to X$ a mapping. Then, for $x \in X$, $\mathcal{O}_f(x) = \{x, fx, f^2x, \dots\}$ is called the orbit of x under f. The metric space (X, d) is said to be orbitally complete if each Cauchy sequence in $\mathcal{O}_f(x)$ converges in X. If (X, d) is a complete metric space then it is orbitally complete but the reverse is not generally true. The mapping f is an α-graphic contraction if it has closed graph and there exist $\alpha \in]0, 1[$ such that

$$d(f(x), f^2(x)) \leq \alpha d(x, f(x)), \forall x \in X.$$

The mapping $f : X \to X$ is called orbitally continuous at a point $z \in X$ if, for any sequence $\{x_n\} \subset \mathcal{O}_f(x)$ with $x \in X$, $x_n \to z$ as $n \to \infty$ implies $f(x_n) \to f(z)$ as $n \to \infty$.

Theorem 2.2 (Ishikawa (1976); see [13]). *Let X be a Banach space, $Y \subset X$ a bounded, closed and convex subset of X and $f : Y \to Y$ a nonexpansive operator. For $\lambda \in]0, 1[$ we consider the operator $f_\lambda := (1 - \lambda)1_Y + \lambda f$. Then f_λ is asymptotically regular w.r.t. $\xrightarrow{\|\cdot\|}$.*

Theorem 2.3 (Belluce-Kirk (1969); [3]). *Let (X, d) be a compact metric space and $f : X \to X$ a nonexpansive operator. The following statements are equivalent:*

(i) *f has diminishing orbital diameters on X;*
(ii) *f is asymptotically regular on X;*
(iii) *f is not an isometry on $\mathcal{O}_f(x)$ if*

$$\delta_d(\mathcal{O}_f(x)) > 0, \textit{ for all } x \in X.$$

Theorem 2.4 (I. A. Rus [38]). *Let X be a nonempty set and $f : X \to X$ be an operator. Then the following statements are equivalent:*

(i) $F_{f^n} = F_f \neq \emptyset, \forall\, n \in \mathbb{N}^*$;
(ii) *$F_f \neq \emptyset$ and there exists a metric on X with respect to which f is asymptotically regular;*
(iii) *there exists an L-space structure on $(X, \to)$, such that f is WPO with respect to $\to$;*
(iv) *there exist $\alpha \in]0, 1[$ and a complete metric d on X such that:*
  (a) *f is orbitally continuous with respect to d;*
  (b) *f is an α-graphic contraction with respect to d.*

3. THE CASE card $F_f \leq 1$

First of all we shall improve some results given in [34] and [35]. We have

Theorem 3.5. *Let (X,d) be a complete metric space, α_{DP} be a Daneš-Pasicki measure of noncompactness on (X,d) and $Y \subset X$ be a bounded and closed subset of X. Let $f : Y \to Y$ be an operator. We suppose that:*
(i) *f is continuous;*
(ii) *$\alpha_{DP}(f(A)) < \alpha_{DP}(A)$, for $A \subset Y$ such that $\alpha_{DP}(A) \neq 0$ and $f(A) \subset A$;*
(iii) *f is asymptotically regular;*
(iv) $\operatorname{card} F_f \leq 1$.
Then f is a PO.

Proof. Let $x \in Y$. We remark that

$$\alpha_{DP}(\mathcal{O}_f(x)) = \alpha_{DP}(f \mid \mathcal{O}_f(x)).$$

From the condition (ii) it follows that, $\alpha_{DP}(\mathcal{O}_f(x)) = 0$. From the definition of α_{DP} we have $\overline{\mathcal{O}_f(x)}$ is a compact subset of Y. From Theorem 2.1 (c) $f|_{\overline{\mathcal{O}_f(x)}}$ is a PO. So, from (iv) $f : Y \to Y$ is a PO. □

4. APPLICATIONS OF CARBONE'S LEMMA

In a paper published in 1988, A. Carbone gives an interesting general fixed point result which involves the asymptotic regularity of the operator at some point of the space. In the same paper [14] he also applies this result to studying the fixed points for Banach contractions and, respectively, for φ-contractions in the sense of Matkowski [23] and Rus [33].

Lemma 4.1 (Carbone, [14])**.** *Let $f : X \to X$ be an operator, where (X,d) is an orbitally complete metric space. Assume that:*
(ι) *for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $d(x, f(x)) < \delta(\epsilon)$, then $f[B(x,\epsilon)] \subset B(x,\epsilon)$;*
$(\iota\iota)$ *$d(f^n(x_0), f^{n+1}(x_0)) \to 0$ as $n \to \infty$, for some $x_0 \in X$.*
Then the sequence $f^n(x_0)$ converges to a fixed point of f.

The result of Carbone inspires us to consider the following lemma, which brings us closer to the Problems 1 and 2 formulated in the Introduction of the paper. Its proof is similar to that of Carbone's Lemma.

Lemma 4.2 (Lemma of the invariant ball)**.** *Let $f : X \to X$ be an operator, where (X,d) is an orbitally complete metric space. Assume that:*
(i) *there exists $t : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each $r > 0$, $d(x, f(x)) \leq t(r)$ implies $f(\overline{B}(x;r)) \subset \overline{B}(x;r)$;*
(ii) *f is asymptotically regular on X.*
Then f is a WPO.

Remark 4.1. If $cardF_f \leq 1$ and f satisfies the conditions in Lemma 4.2, then f is a PO.

Remark 4.2. It is obvious that any graphic contraction is asymptotically regular, so we can formulate a result similar to Lemma 4.2 in which condition (ii) is replaced by:
(ii') f is a graphic contraction.
This also implies that f is a WPO and, under the assumption that $cardF_f \leq 1$, it is a PO.

Lemma 4.2 and the above considerations give an answer to *Problems* 1 and 2 posed in Introduction. They also enable new proofs of several fixed point theorems for known classes of generalized contractions and not only. Condition (ii) has been verified for several classes of generalized contractions, see for example [4], as well as condition (ii'), see for example [24]. The remaining task is to study which classes of generalized contractions satisfy condition (i) of Lemma 4.2.

A typical example of generalized contractions, which do not imply continuity, are the Kannan operators, that is, operators $f : X \to X$ such that

$$d(f(x), f(y)) \leq a\left[d(x, f(x)) + d(y, f(y))\right], \forall x, y \in X,$$

where $a \in [0, \frac{1}{2})$.

Example 4.1. Any Kannan operator satisfies condition (i) of Lemma 4.2.

Proof. Let $t : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $t(r) = \dfrac{1-2a}{1+a}r$, $\forall r > 0$. Let $r > 0$ and let $x \in X$ such that $d(x, f(x)) \leq t(r)$, that is, $d(x, f(x)) \leq \dfrac{1-2a}{1+a}r$.

Take $z \in \overline{B}(x; r)$. We aim to show that $f(z) \in \overline{B}(x; r)$. We have:

$$\begin{aligned} d(x, f(z)) &\leq d(x, f(x)) + d(f(x), f(z)) \\ &\leq t(r) + ad(x, f(x)) + ad(z, f(z)) \\ &\leq (1+a)\frac{1-2a}{1+a}r + ad(z, f(z)) \\ &\leq (1-2a)r + a\left[d(z, , x) + d(x, f(z))\right] \\ &= (1-2a)r + ar + ad(x, f(z)) \\ &= (1-a)r + ad(x, f(z)). \end{aligned}$$

It follows immediatelly that $d(x, f(z)) \leq r$, so $f(z) \in \overline{B}(x; r)$.

In conclusion, $f(\overline{B}(x; r)) \subset \overline{B}(x; r)$.

□

Another interesting class of generalized contractions are the strict almost contractions, introduced in [7]. In [28] there is shown that the conditions in the definition of strict almost contractions are equivalent to the following: there exist two constants $\delta \in [0, 1)$ and $L \geq 0$ such that

$$d(f(x), f(y)) \leq \delta d(x, y) + L\ min\{d(x, f(x)), d(x, f(z)), d(z, f(x)), d(z, f(z))\},$$

for all $x, y \in X$.

Example 4.2. Any strict almost contraction satisfies condition (i) of Lemma 4.2.

Proof. Let $t : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $t(r) = \dfrac{1-\delta}{1+L}r$, $\forall r > 0$. Let $r > 0$ and let $x \in X$ such that $d(x, f(x)) \leq t(r)$, that is, $d(x, f(x)) \leq \dfrac{1-\delta}{1+L}r$.

Take $z \in \overline{B}(x;r)$. We aim to show that $f(z) \in \overline{B}(x;r)$. We have:

$$\begin{aligned} d(x,f(z)) &\leq d(x,f(x)) + d(f(x),f(z)) \\ &\leq t(r) + \delta d(x,z) + Lmin\{d(x,f(x)), d(x,f(z)), d(z,f(x)), d(z,f(z))\} \\ &\leq t(r) + \delta r + Ld(x,f(x)) \\ &\leq t(r) + \delta \cdot r + L \cdot t(r) \\ &\leq \frac{1-\delta}{1+L} r(1+L) + r\delta. \end{aligned}$$

It follows immediatelly that $d(x,f(z)) \leq r$, so $f(z) \in \overline{B}(x;r)$.

In conclusion, $f(\overline{B}(x;r)) \subset \overline{B}(x;r)$.

□

Remark 4.3. The next examples present a WPO (Example 4.3) and a mapping which is neither WPO nor PO (Example 4.4).

Example 4.3. Let $X = [0,1]$ be endowed with the usual norm and let $f : X \to X$ be given by $f(x) = \dfrac{2}{3}x$, for $x \in [0,1/2)$ and $f(x) = \dfrac{2}{3}x + \dfrac{1}{3}$, for $x \in [1/2,1]$. Then, see [27], f is an almost contraction with $\delta = 2/3$ and $L = 6$, f has two fixed points, $F_f = \{0,1\}$ and f is a WPO.

Example 4.4. Let $X = [-1,1]$ be endowed with the usual norm and let $f : X \to X$ be given by $f(x) = -x$, for all $x \in X$. Then

1) $F_f = \{0\}$;
2) $\{f^n(x)\}$ is asymptotically regular if and only if $x = 0$;
3) $\{f^n(x)\}$ converges if and only if $x = 0$;
4) Condition (i) in Lemma 4.2 holds if and only if $x = 0$;

We prove 4) as 1)-3) are immediate. Indeed, since

$$\overline{B}(x;r) = [-r+x, x+r] \text{ and } f(\overline{B}(x;r)) = [-r-x, -x+r],$$

the inclusion $f(\overline{B}(x;r)) \subset \overline{B}(x;r)$ implies

$$-r + x \leq -x - r \text{ and } r - x \leq x + r,$$

which lead to $x \leq 0$ and $x \geq 0$, i.e., $x = 0$.

Remark 4.4. Lemma 4.2 cannot be applied to the function in Example 4.4, since condition (ii) does not hold for all $x \in X$. This suggests us to consider the following weaker condition:

(ii') f is asymptotically regular at some point $x_0 \in X$.

For some other similar results which give answers to Problem 1, see [1], [2], [25] and [46].

5. (ψ, χ)-OPERATORS

Let (X,d) be a complete metric space and $f : X \to X$ be an operator with $F_f \neq \emptyset$. Let $\chi : X \to F_f$ be a set retraction and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(0) = 0$ and ψ is continuous at 0. By definition, the operator f is a (ψ,χ)-operator if ψ and χ are as above and

$$d(x,\chi(x)) \leq \psi(d(x,f(x))),\ \forall\, x \in X.$$

Example 5.5. If $f : X \to X$ is a ψ-WPO, then f is a (ψ, f^∞)-operator.

Example 5.6. Let (X, d) be a complete metric space, $f : X \to X$ with $F_f \neq \emptyset$ and $\chi : X \to F_f$ a set retraction such that $\chi(\mathcal{O}_f(x)) = \{\chi(x)\}$, $\forall\, x \in X$.
We suppose that there exists $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, $\theta(0) = 0$ and continuous at 0, and $0 < \beta < 1$ such that

$$(*) \qquad d(f(x), \chi(x)) \leq \theta(d(x, f(x)) + \beta d(x, \chi(x)), \ \forall\, x \in X.$$

Then f is a (ψ, χ)-operator with $\psi = (1-\beta)^{-1}(1_{\mathbb{R}_+} + \theta)$.
Indeed,

$$\begin{aligned} d(x, \chi(x)) &\leq d(x, f(x)) + d(f(x), \chi(x)) \\ &\leq d(x, f(x)) + \theta(d(x, f(x)) + \beta d(x, \chi(x)), \end{aligned}$$

and hence

$$d(x, \chi(x)) \leq \frac{1+\theta}{1-\beta} d(x, f(x)),$$

as required

In the following example, we present a large class of (ψ, f^∞)-operators, see [7], which do not satisfy always condition $(*)$.

Example 5.7. Let (X, d) be a complete metric space and $f : X \to X$ an almost contraction (see [7], [5], [6],), that is, a mapping for which there exist the constants $\delta \in (0, 1)$ and $L \geq 0$ such that

$$(5.3) \qquad d(Tx, Ty) \leq \delta\, d(x, y) + L\, d(y, Tx)\,, \quad \text{for all } \ x, y \in X\,.$$

Then f is a (ψ, f^∞)-operator, with $\psi(t) = \dfrac{t}{1-\delta}$, $t \in \mathbb{R}_+$.

We prove directly that f is a (ψ, f^∞)-operator, without using (5.3). Indeed, by the continuity of the distance we have

$$(5.4) \qquad d(x, f^\infty(x)) = d(x, \lim_{n\to\infty} f^n(x)) = \lim_{n\to\infty} d(x, f^n(x)).$$

On the other hand, by (5.3) we get in particular

$$d(f^{k-1}(x), f^k(x)) \leq \delta^{k-1} d(x, f(x)), \forall k \geq 1$$

and hence

$$d(x, f^n(x)) \leq d(x, f(x)) + d(f(x), f^2(x)) + \cdots + d(f^{n-1}(x), f^n(x))$$

$$(5.5) \qquad \leq (1 + \delta + \cdots + \delta^{n-1}) d(x, fx) = \frac{1-\delta^n}{1-\delta} d(x, f(x)).$$

So, by (5.4) and (5.5), we obtain

$$d(x, f^\infty(x)) = \lim_{n\to\infty} d(x, f^n(x)) \leq \lim_{n\to\infty} \frac{1-\delta^n}{1-\delta} d(x, f(x)) = \frac{1}{1-\delta} d(x, f(x)),$$

which shows that f is a (ψ, f^∞)-operator, with $\psi(t) = \dfrac{t}{1-\delta}$, $t \in \mathbb{R}_+$.

Note also that f in Example 4.3 satisfies condition (*) with $\theta(t) = t$ and $\beta = 1/3$. However, in order to prove that condition (*) holds, we must have $\delta + L < 1$ which is not generally true for any almost contraction.

We also have a kind of reverse of the previous theoretical results:

Theorem 5.6. *Let (X, d) be a complete metric space and $f : X \to X$ an operator. We suppose that:*

(i) *f is a (ψ, χ)-operator;*

(ii) *f is asymptotically regular.*

Then, f is ψ-WPO and $\chi = f^\infty$.

Proof. From $(i) + (ii)$ it follows that

$$d(f^n(x), \chi(x)) \leq \psi(d(f^n(x), f^{n+1}(x))) \to 0 \text{ as } n \to \infty.$$

So,

$$f^n(x) \to \chi(x) \text{ as } n \to \infty.$$

□

6. Open problems

1. In Section 4, the application of Carbone Lemma has been illustrated for the case of almost contractions possessing a unique fixed point, i.e., essentially for POs. But an almost contraction is in general a WPO.

So, the question is if Lemma 4.1 or Lemma 4.2 could be applied in the case of almost contractions possessing two or more fixed points.

2. In Section 5, we presented an example of (ψ, χ)-operator for the case of almost contractions. In particular, the identity map on a compact interval of the real axis, see [7], which is in fact a non expansive map, is an almost contraction whose set of fixed points is the whole interval.

The question is wether or not nonexpansive mappings in gereral are (ψ, χ)-operators.

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