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Dedicated to Professor Emeritus Ioan A. Rus on the occasion of his 80^{th} *anniversary*

Fixed point approximation of Prešić nonexpansive mappings in product of CAT (0) spaces

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ABSTRACT. We obtain a fixed point theorem for Prešić nonexpansive mappings on the product of $CAT\,(0)$ spaces and approximate this fixed points through Ishikawa type iterative algorithms under relaxed conditions on the control parameters. Our results are new in the literature and are valid in uniformly convex Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

An interesting generalization of Banach contraction principle has been obtained in 1965 by Prešić [26]:

Theorem 1.1. ([26]) *Let* (X, d) *be a complete metric space,* k *a positive integer,* $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_k \in \mathbb{R}_+, \sum_{i=1}^{k} \alpha_i = \alpha < 1$ *and* $f : X^k \to X$ *a mapping satisfying*

$$d\left(f\left(x_0, x_1, ..., x_{k-1}\right), f\left(x_1, x_2, ..., x_k\right)\right) \leq \sum_{i=0}^{k-1} \alpha_{i+1} d\left(x_i, x_{i+1}\right)$$

for all $x_0, x_1, ..., x_k \in X$. *Then* f *has a unique fixed point* x^*, *that is, there exists a unique* $x^* \in X$ *such that* $f\left(x^*, x^*, ..., x^*\right) = x^*$ *and the sequence defined by*

$$x_{n+1} = f\left(x_{n-k+1}, ..., x_n\right), \quad n = k-1, k, k+1, ...$$

converges to x^* *for any* $x_0, x_1, ..., x_{k-1} \in X$.

We note that Theorem 1.1 collapses into the classical Banach contraction principle for $k = 1$.

A mapping $T : X \to X$ is called an α-contraction if there exists $\alpha \in [0, 1)$ such that

$$d\left(Tx, Ty\right) \leq \alpha d\left(x, y\right), \text{for all } x, y \in X.$$

When $\alpha = 1$, T is called nonexpansive.

Some generalizations of Theorem 1.1 have been obtained in [7, 27] (see also [22, 23, 24, 25]). Nonexpansive mappings do not inherit much from contraction mappings. More precisely, if C is a nonempty closed subset of a Banach space E and $T : C \to C$ is nonexpansive, then T may not have a fixed point (unlike the case T is an $\alpha-$contraction). Even in the case when T has a fixed point, the Picard iteration associated with T may fail to converge to the fixed point of T.

For the above known facts and also many other reasons, a richer geometrical structure of the ambient space is needed in order to ensure the existence of a fixed point and the

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convergence of an iterative algorithm (more general than Picard iterative algorithm) to a fixed point of a nonexpansive mapping T.

Here are the details of a class of metric spaces with rich geometrical structure.

A geodesic path from x to y in a metric space (X, d) is a mapping $c : [0, d(x, y)] \to X$ such that $c(0) = x$, $c(d(x, y)) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, d(x, y)]$. The image of c is called a geodesic segment. The space X is known as uniquely geodesic space if every two points of X are joined by a unique geodesic segment.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space X consists of three points $x_1, x_2, x_3 \in X$ and a geodesic segment between each pair of these points. A comparison triangle for a geodesic triangle $\Delta(x_1, x_2, x_3)$ in X is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for all $i, j = 1, 2, 3$.

A geodesic space X is a $CAT(0)$ space if the following inequality

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$$

holds for all $x, y \in \Delta$ in X and $\bar{x}, \bar{y} \in \overline{\Delta}$ in $\mathbb{R}^2$.

Since a $CAT(0)$ space (X, d) is uniquely geodesic, we use $[x, y]$ to denote the geodesic segment between x and y and $(1 - \alpha) x \oplus \alpha y$ to denote the unique point $z \in [x, y]$ such that

$$d(z, x) = \alpha d(x, y) \text{ and } d(z, y) = (1 - \alpha) d(x, y).$$

A subset C of a $CAT(0)$ space is convex if $[x, y] \subset C$ for all $x, y \in C$.

A few examples of $CAT(0)$ spaces are: (i) Hilbert spaces–the only Banach spaces which are $CAT(0)$; (ii) $\mathbb{R}-$trees: a metric space X is an $\mathbb{R}-$tree if for $x, y \in X$, there is a unique geodesic $[x, y]$ and $[x, y] \cap [y, z] = \{y\}$ implies that $[x, z] = [x, y] \cup [y, z]$; (iii) Classical hyperbolic spaces $\mathbb{H}^n$; (iv) Complete simply connected Riemannian manifolds with nonpositive sectional curvature; (v) Euclidean buildings. These examples illustrate that the class of $CAT(0)$ spaces encompasses both smooth and singular objects.

There are several ways to construct new $CAT(0)$ spaces from known ones. Clearly, a convex subset of a $CAT(0)$ space, endowed with the induced metric, is itself a $CAT(0)$ space. Another key feature of the $CAT(0)$ inequality is its inheritance under Cartesian products, that is, given any two $CAT(0)$ spaces: (X_1, d_1) and (X_2, d_2), the Cartesian product $X = X_1 \times X_2$, endowed with the metric d defined by $d^2 = d_1^2 + d_2^2$, is a $CAT(0)$ space.

Fixed point theorems in $CAT(0)$ spaces (specially in $\mathbb{R}-$trees) are applicable in biology, computer science and graph theory (see e.g., [6, 10]).

Kirk [17] proved the existence of fixed points for nonexpansive mappings in $CAT(0)$ spaces as follows: If C is a bounded, closed and convex subset of a complete $CAT(0)$ space X, then a nonexpansive mapping $T : C \to C$ has a fixed point.

On the other hand, the iterative construction of fixed points of nonlinear mappings is itself a fascinating field of research (see [2]). The fixed point problems for nonexpansive mappings and their generalizations have been studied extensively on linear as well as nonlinear domains (see e.g. [1, 11, 12, 13, 14, 15, 16, 18, 19, 20]).

For a bounded sequence $\{x_n\} \subset X$, we define a functional $r(., \{x_n\}) : X \to \mathbb{R}^+$ by

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n),$$

for all $x \in X$.

The *asymptotic radius* $r_C(\{x_n\})$ of $\{x_n\}$ with respect to a subset $C \subseteq X$ is defined as

$$r_C(\{x_n\}) = \inf_{x \in C} r(x, \{x_n\}),$$

while the *asymptotic center* $A_C(\{x_n\})$ of $\{x_n\}$ with respect to C is, by definition, the set

$$A_C(\{x_n\}) = \{y \in C : r(y, \{x_n\}) = r_C(\{x_n\})\}.$$

A sequence $\{x_n\}$ $\triangle$-converges to $x \in C$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$ (see [9]; in this case, we write x as $\triangle$-limit of $\{x_n\}$, *i.e.*, $\triangle-\lim_{n\to\infty} x_n = x$). A sequence $\{x_n\}$ is said to *polar converge* to a point $x \in X$ (see [8]) if for every $y \in X$ with $y \neq x$, there exists $n_0 \geq 1$ such that $d(x_n, x) < d(x_n, y)$ for all $n \geq n_0$. A sequence $\{x_n\}$ is said to converge $\triangle$-strongly to a point x if the limit $\lim_{n\to\infty} d(x_n, x)$ exists and for any $y \neq x$, $\lim_{n\to\infty} d(x_n, x) \leq \liminf_{n\to\infty} d(x_n, y)$. The notion of polar convergence was introduced by Devillanova et al. [8] who have also discussed the relationship between polar convergence and $\triangle$-convergence in metric spaces.

In the sequel, we shall need the following well known results.

Lemma 1.1. [9] *If X is a $CAT(0)$ space, then*

$$d(z, \alpha x \oplus (1-\alpha)y) \leq \alpha d(z, x) + (1-\alpha) d(z, y)$$

for any $\alpha \in [0,1]$ and $x, y, z \in X$.

Lemma 1.2. [9] *A geodesic space X is a $CAT(0)$ space if and only if*

$$d(z, \alpha x \oplus (1-\alpha)y)^2 \leq \alpha d(z, x)^2 + (1-\alpha) d(z, y)^2 - \alpha(1-\alpha) d(x, y)^2$$

for any $\alpha \in [0,1]$ and $x, y, z \in X$.

Lemma 1.3. [11] *Let C be a nonempty, closed and convex subset of a complete $CAT(0)$ space X. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to C.*

Lemma 1.4. [11] *Let C be a nonempty closed and convex subset of a $CAT(0)$ space X. Let $\{x_n\}$ be a bounded sequence in C such that $A_C(\{x_n\}) = \{x\}$ and $r_C(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in C such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = x$.*

2. Fixed points of Prešić nonexpansive mappings

We recall the definition of a Prešić nonexpansive mapping, first introduced in [4].

Let (X, d) be a metric space, k a positive integer, and let $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_k \in \mathbb{R}_+$ be real numbers such that $\sum_{i=1}^{k} \alpha_i = \alpha \leq 1$. A mapping $f : X^k \to X$ satisfying

$$d(f(x_0, x_1, ..., x_{k-1}), f(x_1, x_2, ..., x_k)) \leq \sum_{i=0}^{k-1} \alpha_{i+1} d(x_i, x_{i+1}),$$

for all $x_0, x_1, ..., x_k \in X$, is called a *Prešić nonexpansive mapping*.

Since, in the definition of Prešić nonexpansive mappings, we have $\alpha \leq 1$, it follows that this class of mappings includes the class of Prešić contractions appearing in Theorem 1.1. In the case $k = 1$, the Prešić nonexpansiveness condition reduces to Banach contractive condition if $\alpha < 1$ and to the usual nonexpansiveness condition if $\alpha = 1$ (see [4]).

Example 2.1. Let $I = [0,1]$ be the unit interval with the usual Euclidean norm and let $f : I^3 \to I$ be given by $f(x, y, z) = \frac{2x+y+2z}{5}$, for all $x, y, z \in I$. Then f is Prešić nonexpansive but is not a Prešić contraction.

Proof. In this case, the Prešić nonexpansive condition reads as follows: there exists $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}_+$ with $\alpha_1 + \alpha_2 + \alpha_3 = \alpha \leq 1$ such that for all $x_0, x_1, x_2, x_3 \in I$, we have

$$|f(x_0, x_1, x_2) - f(x_1, x_2, x_3)| \leq \alpha_1 |x_0 - x_1| + \alpha_2 |x_1 - x_2| + \alpha_3 |x_2 - x_3|.$$

By the definition of f, the above inequality becomes

$$\left|\frac{2}{5}(x_0 - x_1) + \frac{1}{5}(x_1 - x_2) + \frac{2}{5}(x_2 - x_3)\right| \leq \alpha_1 |x_0 - x_1| + \alpha_2 |x_1 - x_2| + \alpha_3 |x_2 - x_3|$$

which obviously holds for $\alpha_1 = \frac{2}{5} = \alpha_3, \alpha_2 = \frac{1}{5}$ by using the triangle inequality.

By the fact that the set of all fixed points of f, $F(f)$, coincides with the whole interval I and by Theorem 1.1, it follows that f is not a Prešić contraction. □

Here is a fixed point theorem for Prešić nonexpansive mappings.

Theorem 2.2. *Let C be a bounded, closed and convex subset of a complete $CAT(0)$ space X, k a positive integer, and let $f : X^k \to X$ be a Prešić nonexpansive mapping. Then f has a fixed point, that is, there exists $x^* \in X$ such that $f(x^*, x^*, ..., x^*) = x^*$.*

Proof. Define $T : C \to C$ by

$$T(z) = f(z, z, ..., z), \quad z \in C.$$

Then, for any $x, y \in C$, we have

$$d(T(x), T(y)) = d(f(x, x, ..., x), f(y, y, ..., y))$$
$$\leq d(f(x, x, ..., x), f(x, ..., x, y)) + d(f(x, ..., x, y), f(x, ..., x, y, y)) + ...$$
$$+d(f(x, y, ..., y), f(y, y, ..., y)) \leq \alpha_k d(x, y) + \alpha_{k-1} d(x, y) + ... + \alpha_1 d(x, y)$$
$$= \sum_{i=1}^{k} \alpha_i d(x, y) = \alpha d(x, y) \leq d(x, y).$$

This proves that T is nonexpansive. By Kirk's fixed point theorem ([17]), there exists $x^* \in C$ such that $T(x^*) = x^* = f(x^*, x^*, ..., x^*)$. □

Next, we study approximation of fixed point of Prešić nonexpansive mapping defined on the product of $CAT(0)$ spaces.

Let C be a convex subset of a $CAT(0)$ space X, k a positive integer, and let $f : C^k \to C$ be a Prešić nonexpansive mapping.

Starting with $x_1 \in C$, we construct the following algorithms.

(i) The Mann iterative algorithm [20] in a $CAT(0)$ space is defined by

$$x_{n+1} = (1 - \alpha_n) x_n \oplus \alpha_n f(x_n, x_n, ..., x_n), \quad n \geq 1, \tag{2.1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

(ii) The Ishikawa iterative algorithm [15] is defined by

$$x_{n+1} = (1 - \alpha_n) x_n \oplus \alpha_n f(y_n, y_n, ..., y_n); y_n = (1 - \beta_n) x_n \oplus \beta_n f(x_n, x_n, ..., x_n), \tag{2.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

Lemma 2.5. *Let C be a closed and convex subset of a $CAT(0)$ space X, k a positive integer, and $f : C^k \to C$, a Prešić nonexpansive mapping. Then for $\{x_n\}$ in (2.2), $\lim_{n\to\infty} d(x_n, x^*)$ exists for each $x^* \in F(f)$.*

Proof. Define $T : C \to C$ by

$$T(z) = f(z, z, ..., z), \quad z \in C.$$

Then by Theorem 2.2, T is nonexpansive with $F(T) = F(f)$.

Applying Lemma 1.1 to the algorithm(2.2) and doing routine calculations, we get that $\lim_{n\to\infty} d(x_n, x^*)$ exists for each $x^* \in F(f)$. □

Lemma 2.6. *Let C be a closed and convex subset of a $CAT(0)$ space X, k a positive integer, and $f : C^k \to C$, a Prešić nonexpansive mapping. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy either of the following two sets of conditions:*
(i) $\sum_{n=1}^{\infty} \alpha_n (1-\alpha_n) = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$.
(ii) $\sum_{n=1}^{\infty} \beta_n (1-\beta_n) = \infty$ and $\liminf_{n\to\infty} \alpha_n > 0$.
If $F(f) \neq \emptyset$, then for $\{x_n\}$ in (2.2), we have that $\liminf\limits_{n\to\infty} d(x_n, f(x_n, x_n, ..., x_n)) = 0$.

Proof. Suppose that conditions (i) holds. Since $\limsup_{n\to\infty} \beta_n < 1$, there exists $b \in (0,1)$ such that $\beta_n < b$ for all $n \geq 1$. Define $T : C \to C$ as in the proof of Lemma 2.5.

For any $x^* \in F(f) = F(T)$, we apply Lemma 1.2 to the algorithm (2.2) to get

$$d(x_{n+1}, x^*)^2 = d((1-\alpha_n)x_n \oplus \alpha_n Ty_n, x^*)^2 \leq (1-\alpha_n) d(x_n, x^*)^2 + \alpha_n d(Ty_n, x^*)^2 -$$
$$\alpha_n (1-\alpha_n) d(x_n, Ty_n)^2 \leq (1-\alpha_n) d(x_n, x^*)^2 + \alpha_n d(y_n, x^*)^2 - \alpha_n (1-\alpha_n) d(x_n, Ty_n)^2$$
$$\leq (1-\alpha_n) d(x_n, x^*)^2 + \alpha_n \left[\left((1-\beta_n) d(x_n, x^*)^2 + \beta_n d(Tx_n, x^*)^2\right)\right] - \alpha_n (1-\alpha_n) \cdot$$
$$d(x_n, Ty_n)^2 \leq (1-\alpha_n) d(x_n, x^*)^2 + \alpha_n \left[\left((1-\beta_n) d(x_n, x^*)^2 + \beta_n d(x_n, x^*)^2\right)\right] -$$
$$\alpha_n (1-\alpha_n) d(x_n, Ty_n)^2 = d(x_n, x^*)^2 - \alpha_n (1-\alpha_n) d(x_n, Ty_n)^2.$$

That is,

$$\alpha_n (1-\alpha_n) d(x_n, Ty_n)^2 \leq d(x_n, x^*)^2 - d(x_{n+1}, x^*)^2,$$

and therefore $\sum_{n=1}^{\infty} \alpha_n (1-\alpha_n) d(x_n, Ty_n)^2 < \infty$.

By the condition $\sum_{n=1}^{\infty} \alpha_n (1-\alpha_n) = \infty$, it follows that $\liminf_{n\to\infty} d(x_n, Ty_n) = 0$.

Now, from

$$d(x_n, Tx_n) \leq d(Tx_n, Ty_n) + d(x_n, Ty_n) \leq d(x_n, y_n) + d(x_n, Ty_n)$$
$$= d(x_n, (1-\beta_n)x_n \oplus \beta_n Tx_n) + d(x_n, Ty_n) \leq \beta_n d(x_n, Tx_n) + d(x_n, Ty_n)$$
$$\leq b d(x_n, Tx_n) + d(x_n, Ty_n),$$

we have that

$$\liminf_{n\to\infty} d(x_n, Tx_n) \leq \frac{1}{1-b} \liminf_{n\to\infty} d(x_n, Ty_n) = 0.$$

Therefore $\liminf_{n\to\infty} d(x_n, Tx_n) = 0$.

Next, assume that condition (ii) is satisfied. From $\liminf_{n\to\infty} \alpha_n > 0$, we may assume that there exists $a > 0$ such that $\alpha_n > a$ for all $n \geq 1$.

Now, let us evaluate

$$d(x_{n+1}, x^*)^2 = d((1-\alpha_n)x_n \oplus \alpha_n Ty_n, x^*)^2 \leq (1-\alpha_n) d(x_n, x^*)^2 + \alpha_n d(Ty_n, x^*)^2$$
$$\leq (1-\alpha_n) d(x_n, x^*)^2 + \alpha_n d(y_n, x^*)^2 = \alpha_n d((1-\beta_n)x_n \oplus \beta_n Tx_n, x^*)^2 + (1-\alpha_n) d(x_n, x^*)^2$$
$$\leq \alpha_n (1-\beta_n) d(x_n, x^*)^2 + \beta_n d(Tx_n, x^*)^2 - \beta_n (1-\beta_n) d(Tx_n, x_n)^2$$
$$+ (1-\alpha_n) d(x_n, x^*)^2 \leq \alpha_n (1-\beta_n) d(x_n, x^*)^2 + \alpha_n \beta_n d(x_n, x^*)^2$$
$$-\alpha_n \beta_n (1-\beta_n) d(Tx_n, x_n)^2 + (1-\alpha_n) d(x_n, x^*)^2 = d(x_n, x^*)^2 - \alpha_n \beta_n (1-\beta_n) d(Tx_n, x_n)^2.$$

This implies

$$\beta_n (1-\beta_n) d(Tx_n, x_n)^2 \leq \frac{d(x_n, x^*)^2 - d(x_{n+1}, x^*)^2}{a}$$

which further implies that $\sum_{n=1}^{\infty} \beta_n (1-\beta_n) d(Tx_n, x_n)^2 < \infty$.

By condition $\sum_{n=1}^{\infty} \beta_n (1-\beta_n) = \infty$, we get that

$$\liminf_{n\to\infty} d(x_n, Tx_n) = 0,$$

that is,

$$\liminf_{n\to\infty} d(x_n, f(x_n, x_n, ..., x_n)) = 0.$$

□

Recall that a mapping $f : C^k \to C$ is semi-compact if every bounded sequence $\{x_n\}$ has a convergent subsequence whenever $d(x_n, f(x_n, x_n, ..., x_n)) \to 0$.

Let $g : [0, \infty) \to [0, \infty)$ be a nondecreasing function with $g(0) = 0$ and $g(t) > 0$ for all $t \in (0, \infty)$. The mapping $f : C^k \to C$, with $F(f) \neq \emptyset$, satisfies Condition (I) if

$$d(x, f(x, x, ..., x)) \geq g(d(x, F(f))), \text{ for } x \in C,$$

where $d(x, F(f)) = \inf_{p \in F(f)} d(x, p)$.

Now we present our $\triangle$-convergence and strong convergence results based on Lemma 2.6.

Theorem 2.3. *Let C be a closed and convex subset of a complete $CAT(0)$ space X, k a positive integer, and $f : C^k \to C$ a Prešić nonexpansive mapping such that $F(f) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy either of the following two sets of conditions:*
(i) $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$ *and* $\limsup_{n\to\infty} \beta_n < 1$;
(ii) $\sum_{n=1}^{\infty} \beta_n (1 - \beta_n) = \infty$ *and* $\liminf_{n\to\infty} \alpha_n > 0$.
Then for $\{x_n\}$ in (2.2), there is a subsequence $\{z_n\}$ of $\{x_n\}$ which $\triangle$-converges to a fixed point of f.

Proof. In Lemma 2.6, we have shown that $\liminf_{n\to\infty} d(x_n, f(x_n, x_n, ..., x_n)) = 0$ for the sequence $\{x_n\}$ in (2.2). Therefore there is a subsequence $\{z_n\}$ of $\{x_n\}$ such that $\lim_{n\to\infty} d(z_n, f(z_n, z_n, ..., z_n)) = 0$. It follows from Lemma 2.5 that $\{z_n\}$ is bounded in C.

By Lemma 1.3, $\{z_n\}$ has a unique asymptotic center, that is, $A_C(\{z_n\}) = \{z\}$. Let $\{u_n\}$ be any subsequence of $\{z_n\}$ such that $A_C(\{u_n\}) = \{u\}$.

By Lemma 2.6, we have

$$\lim_{n\to\infty} d(z_n, f(u_n, u_n, ..., u_n)) = 0.$$

We now show that $u \in F(f)$. Define a sequence $\{z_k\}$ in C by $z_k = f^k(u, u, ..., u) = T^k u$ and observe that

$$d(z_k, u_n) \leq d(T^k u, T^k u_n) + \sum_{j=1}^{k} d(T^j u_n, T^{j-1} u_n) \leq d(u, u_n) + kd(Tu_n, u_n).$$

Therefore, we have

$$r(z_k, \{u_n\}) = \limsup_{n\to\infty} d(z_k, u_n) \leq \limsup_{n\to\infty} d(u, u_n) = r(u, \{u_n\}).$$

This implies that $|r(z_k, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $k \to \infty$. It follows from Lemma 1.4 that $\lim_{k\to\infty} T^k u = u$. Since C is closed, so $\lim_{k\to\infty} T^k u = u \in C$ and $\lim_{k\to\infty} T^{k+1} u = Tu$.That is, $Tu = u$.Therefore $u \in F(f)$. If $z \neq u$, then by the uniqueness of asymptotic centres and the fact that $\lim_{n\to\infty} d(x_n, p)$ exists for each $x^* \in F(f)$, we have

$$\limsup_{n\to\infty} d(u_n, u) < \limsup_{n\to\infty} d(u_n, z) \leq \limsup_{n\to\infty} d(z_n, z) < \limsup_{n\to\infty} d(z_n, u) = \limsup_{n\to\infty} d(u_n, u).$$

This is a contradiction and therefore $z = u$, which proves that, indeed, the sequence $\{z_n\}$ $\triangle$-converges to $z \in F(f)$. □

Remark 2.1. The $CAT(0)$ spaces are rotund metric spaces ([8]). The polar and $\triangle$-convergence coincide in a complete rotund metric space (see Remark 2.8 and Lemma 3.6 in [8]). The reader interested in "rotundity" in a normed space is referred to Wilansky ([28], p. 107-111).

As a consequence of Remark 2.1 and Theorem 2.3, we have:

Theorem 2.4. *Let C be a closed and convex subset of a complete $CAT(0)$ space X, k a positive integer, and $f : C^k \to C$ a Prešić nonexpansive mapping such that $F(f) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy either of the following two sets of conditions:*
(i) $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$ *and* $\limsup_{n\to\infty} \beta_n < 1$.
(ii) $\sum_{n=1}^{\infty} \beta_n (1 - \beta_n) = \infty$ *and* $\liminf_{n\to\infty} \alpha_n > 0$.
Then for $\{x_n\}$ in (2.2), there is a subsequence $\{z_n\}$ of $\{x_n\}$ which polar converges to a fixed point of f.

Finally, we state our strong convergence theorem.

Theorem 2.5. *Let C be a closed and convex subset of a $CAT(0)$ space X, k a positive integer, and $f : C^k \to C$ a Prešić nonexpansive mapping such that $F(f) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy either of the following two sets of conditions:*
(i) $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$ *and* $\limsup_{n\to\infty} \beta_n < 1$.
(ii) $\sum_{n=1}^{\infty} \beta_n (1 - \beta_n) = \infty$ *and* $\liminf_{n\to\infty} \alpha_n > 0$.
Then $\{x_n\}$ in (2.2) strongly converges to $x^* \in F(f)$ if one of the following conditions holds:
(i) C is compact;
(ii) f is semi-compact;
(iii) f satisfies condition (I).

For some other related results that consider nonexpansive type mappings, see the very recent papers [3]-[5] and [21], where bivariate weakly nonexpansive mappings, Prešić nonexpansive mappings and bivariate asymptotically nonexpansive mappings, were considered, respectively.

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