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Dedicated to Professor Emeritus Ioan A. Rus on the occasion of his 80^{th} anniversary

Fixed point theorems for nonself Kannan type contractions in Banach spaces endowed with a graph

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ABSTRACT. Let K be a non-empty closed subset of a Banach space X endowed with a graph G. The main result of this paper is a fixed point theorem for nonself Kannan G-contractions $T : K \to X$ that satisfy Rothe's boundary condition, i.e., T maps ∂K (the boundary of K) into K. Our new results are extensions of recent fixed point theorems for self mappings on metric spaces endowed with a partial order and also of various fixed point theorems for self and nonself mappings on Banach spaces or convex metric spaces.

1. INTRODUCTION

Let (X, d) be a metric space. Denote by $Fix\,(T)$ the set of fixed points of a mapping $T : X \to X$, i.e.,

$$Fix\,(T) = \{x \in X : Tx = x\}.$$

In a recent paper [19], the second author and M. Păcurar established two fixed point theorems for non self contractions defined on Banach spaces endowed with a graph. The aim of the present paper is to obtain a fixed point theorem for non-self Kannan type contractions $T : X \to X$ on Banach spaces endowed with a graph, by considering, instead of the classical Banach contraction condition

$$d(Tx, Ty) \le a\big[d(x, y)\big], \quad \text{for all } x, y \in X, \tag{1.1}$$

where $0 \le a < 1$, an alternative and independent contraction condition due to Kannan [41]: there exists a constant $b \in \left[0, \dfrac{1}{2}\right)$ such that

$$d(Tx, Ty) \le b\big[d(x, Tx) + d(y, Ty)\big], \quad \text{for all } x, y \in X\,. \tag{1.2}$$

Since, any mapping satisfying (1.1) is continuous but this is not the case with mappings satisfying (1.2), our new results established in the present paper are very general and include as particular cases many existing results in literature.

In order to do so, we present in the next section a few preliminary notions and results regarding fixed point theorems for mappings defined on metric spaces endowed with a graph and next, present some basic results related to fixed point theorems for non self contractions in Banach spaces or convex metric spaces, basically taken from [19].

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2. METRIC SPACES ENDOWED WITH A GRAPH

Let (X,d) be a metric space and let Δ denote the diagonal of the Cartesian product $X \times X$. Consider now a directed simple graph $G = (V(G), E(G))$ such that the set of its vertices, $V(G)$, coincides with X and $E(G)$, the set of its edges, contains all loops, i.e., $\Delta \subset E(G)$.

By G^{-1} we denote the *converse graph* of G, i.e., the graph obtained by G by reversing its edges, i.e.,

$$E(G^{-1}) = \{(y,x) \in X \times X : (x,y) \in E(G)\}.$$

If x, y are vertices in the graph G, then a *path* from x to y of length N is a sequence $\{x_i\}_{i=1}^N$ of $N+1$ vertices of G such that

$$x_0 = x,\ x_N = y \text{ and } (x_{i-1}, x_i) \in E(G),\ i = 1, 2, \ldots, N.$$

A graph G is said to be connected if there is at least a path between any two vertices. If $G = (V(G),\ E(G))$ is a graph and $H \subset V(G)$, then the graph $(H, E(H))$ with $E(H) = E(G) \cap (H \times H)$ is called the *subgraph of G determined by* H. Denote it by G_H.

If $\tilde{G} = (X, E(\tilde{G}))$ is the symmetric graph obtained by putting together the vertices of both G and G^{-1}, i.e.,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}),$$

then G is called *weakly connected* if $\tilde{G}$ is connected.

A mapping $T : X \to X$ is said to be (well) defined on a metric space endowed with a graph G if it has the property

$$\forall x, y \in X,\ (x,y) \in E(G) \text{ implies } (Tx, Ty) \in E(G). \tag{2.3}$$

According to [40], a mapping $T : X \to X$, which is well defined on a metric space endowed with a graph G, is called a *G-contraction* if there exists a constant $\alpha \in (0,1)$ such that for all $x, y \in X$ with $(x,y) \in E(G)$ we have

$$d(Tx, Ty) \leq \alpha \cdot d(x,y). \tag{2.4}$$

Example 2.1. If G_0 is the complete graph on X, that is, $E(G_0) = X \times X$, then a G_0-contraction is a usual contraction in the sense of Banach, i.e., it satisfies condition (1.1), while a G_0-Kannan contraction is a usual Kannan contraction, i.e., it satisfies condition (1.2).

Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a non-self mapping. If $x \in K$ is such that $Tx \notin K$, then we can always choose an $y \in \partial K$ (the boundary of K) such that $y = (1-\lambda)x + \lambda Tx\ (0 < \lambda < 1)$, which actually expresses the fact that

$$d(x, Tx) = d(x,y) + d(y, Tx),\ y \in \partial K, \tag{2.5}$$

where we denoted $d(x,y) = \|x - y\|$.

In general, the set Y of points y satisfying condition (2.5) above may contain more than one element. We suppose Y is always nonempty.

In this context we shall need the following important concept first introduced and used in [18].

Definition 2.1. Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a non-self mapping. Let $x \in K$ with $Tx \notin K$ and let $y \in \partial K$ be the corresponding elements given by (2.5). If, for any such elements x, we have

$$d(y, Ty) \leq d(x, Tx), \tag{2.6}$$

for all corresponding $y \in Y$, then we say that T has property (M).

Note that the non-self mapping T in the next example has property (M).

Example 2.2. ([19], Example 4) Let $X = [0,1] \cup \{3\}$ be endowed with the usual norm and let $K = \{0,1,3\}$. Consider the function $T : K \to X$, defined by $Tx = 0$, for $x \in \{0,1\}$ and $T3 = 0.5$. As the only value $x \in K$ with $Tx \notin K$ is $x = 3$ and to it corresponds the set $Y = \{1\}$, and since

$$d(y,Ty) = d(1,T1) = |1-0| < |3-0.5| = d(3,T3) = d(xTx),$$

property (M) obviously holds.

A condition quite similar to (2.6), called inward condition, has been used by Caristi [27] to obtain a generalization of contraction mapping principle for non self mappings. The inward condition is more general since it does not require y in (2.5) to belong to ∂K, see also [36] (this has been communicated to us by Professor Rus [68]).

Note also that, in general, the set Y of points y satisfying condition (2.5) above may contain more than one element.

For a non self mapping $T : K \to X$ we shall say that it is (well) defined on the Banach space X endowed with the graph G if it has this property for the subgraph of G induced by K, that is,

$$(x,y) \in E(G) \text{ with } Tx, Ty \in K \text{ implies } (Tx,Ty) \in E(G) \cap (K \times K), \tag{2.7}$$

for all $x, y \in K$.

The next theorem establishes a fixed point theorem for non self Kannan contractions defined on a Banach space endowed with a graph.

Theorem 2.1. *Let (X,d,G) be a Banach space endowed with a simple directed and weakly connected graph G such that the property (L) holds, i.e., for any sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, there exists a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$ satisfying*

$$(x_{k_n}, x) \in E(G),\ \forall n \in \mathbb{N}. \tag{2.8}$$

Let K be a nonempty closed subset of X and $T : K \to X$ be a mapping having property (M) for which there exists a constant $a \in [0,1/2)$ such that

$$d(Tx,Ty) \leq a \cdot [d(x,Tx) + d(y,Ty)], \text{ for all } \ (x,y) \in E(G_K), \tag{2.9}$$

where G_K is the subgraph of G determined by K.

If $K_T := \{x \in \partial K : (x,Tx) \in E(G)\} \neq \emptyset$ and T satisfies Rothe's boundary condition

$$T(\partial K) \subset K, \tag{2.10}$$

then

(i) $Fix\,(T) = \{x^*\}$;

(ii) Picard iteration $\{x_n = T^n x_0\}_{n=1}^{\infty}$ converges to x^*, for all $x_0 \in K_T$, and the following estimate holds

$$d(x_n, x^*) \leq \frac{\delta^{[n/2]}}{1-\delta} \max\{d(x_0,x_1), d(x_1,x_2)\}, \quad n = 0,1,2,\dots \tag{2.11}$$

where $\delta = \dfrac{a}{1-a}$.

Proof. If $T(K) \subset K$, then T is actually a self mapping of the closed set K and the conclusion follows by Kannan fixed point theorem [41] with $X = K$. Therefore, in the following we consider only the case $T(K) \not\subset K$. Let $x_0 \in K_T$. This means that $(x_0, Tx_0) \in E(G)$ and in view of (2.3), we have

$$(T^n x_0, T^{n+1} x_0) \in E(G),\ \forall n \in \mathbb{N}. \tag{2.12}$$

Denote $y_n := T^n x_0$, for all $n \in \mathbb{N}$.

By (2.10) it also follows that $Tx_0 \in K$.

Denote $x_1 := y_1 = Tx_0$. Now, if $Tx_1 \in K$, set $x_2 := y_2 = Tx_1$. If $Tx_1 \notin K$, we can choose an element x_2 on the segment $[x_1, Tx_1]$ which also belong to ∂K, that is,

$$x_2 = (1-\lambda)x_1 + \lambda Tx_1 \, (0 < \lambda < 1).$$

Continuing in this way we obtain two sequences $\{x_n\}$ and $\{y_n\}$ whose terms satisfy one of the following properties:

i) $x_n := y_n = Tx_{n-1}$, if $Tx_{n-1} \in K$;

ii) $x_n = (1-\lambda)x_{n-1} + \lambda Tx_{n-1} \in \partial K \, (0 < \lambda < 1)$, if $Tx_{n-1} \notin K$.

To simplify the argumentation in the proof, let us denote

$$P = \{x_k \in \{x_n\} : \, x_k = y_k = Tx_{k-1}\}$$

and

$$Q = \{x_k \in \{x_n\} : \, x_k \neq Tx_{k-1}\}.$$

Note that $\{x_n\} \subset K$ for all $n \in \mathbb{N}$ and that, if $x_k \in Q$, then both x_{k-1} and x_{k+1} belong to the set P.

Moreover, by virtue of (2.10), we cannot have two consecutive terms of $\{x_n\}$ in the set Q (but we can have two consecutive terms of $\{x_n\}$ in the set P) .

We claim that $\{x_n\}$ is a Cauchy sequence.

To prove this, we must discuss three different cases: Case I. $x_n, \, x_{n+1} \in P$; Case II. $x_n \in P, \, x_{n+1} \in Q$; Case III. $x_n \in Q, \, x_{n+1} \in P$;

Case I. $x_n, \, x_{n+1} \in P$.

In this case we have $x_n = y_n = Tx_{n-1}$, $x_{n+1} = y_{n+1} = Tx_n$, hence

$$d(x_{n+1}, x_n) = d(y_{n+1}, y_n) = d(Tx_n, Tx_{n-1}).$$

Since $\{x_n\} \subset K$ for all $n \in \mathbb{N}$, by (2.12) $(x_n, x_{n-1}) \in E(G_K)$, and hence by the contraction condition (2.9), we have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq a[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)],$$

which means

$$d(x_{n+1}, x_n) \leq a[d(x_{n-1}, x_n) + d(x_{n+1}, x_n)],$$

and this leads to

$$d(x_{n+1}, x_n) \leq \delta d(x_n, x_{n-1}), \tag{2.13}$$

with $\delta = \dfrac{a}{1-a}$.

Case II. $x_n \in P, \, x_{n+1} \in Q$.

In this case we have $x_n = y_n = Tx_{n-1}$, but $x_{n+1} \neq y_{n+1} = Tx_n$ and

$$d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, Tx_n).$$

Thus $d(x_{n+1}, Tx_n) \neq 0$ and hence

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) - d(x_{n+1}, Tx_n) < d(x_n, Tx_n).$$

Now, with a similar argument to that in Case I, $(x_n, x_{n-1}) \in E(G_K)$ and hence by the contraction condition (2.9) we get

$$d(x_n, Tx_n) = d(Tx_{n-1}, Tx_n) \leq a[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)],$$

which yields

$$d(x_n, Tx_n) \leq \delta d(x_n, x_{n-1}).$$

So,

$$d(x_n, x_{n+1}) < d(x_n, Tx_n) \leq \delta d(x_n, x_{n-1}).$$

and thus we obtain again inequality (2.13).

Case III. $x_n \in Q,\ x_{n+1} \in P$. In this case we have $x_{n+1} = Tx_n$, $x_n \neq y_n = Tx_{n-1}$ and

$$d(x_{n-1}, x_n) + d(x_n, Tx_{n-1}) = d(x_{n-1}, Tx_{n-1}). \tag{2.14}$$

Hence, by property (M) we get

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) \leq d(x_{n-1}, Tx_{n-1}) = d(Tx_{n-2}, Tx_{n-1}).$$

(since $x_n \in Q \Longrightarrow x_{n-1} \in P$). Thus,

$$d(x_n, x_{n+1}) \leq d(Tx_{n-2}, Tx_{n-1}).$$

Since, by (2.12), $(y_{n-1}, y_n) \in E(G)$, by the contraction condition (2.9) with $x := x_{n-2}$ and $y := x_{n-1}$ we obtain

$$\begin{aligned} d(Tx_{n-2}, Tx_{n-1}) &\leq a[d(x_{n-2}, Tx_{n-2}) + d(x_{n-1}, Tx_{n-1})] \\ &= a[d(x_{n-2}, x_{n-1}) + d(x_{n-1}, Tx_{n-1})], \end{aligned}$$

and so,

$$d(Tx_{n-2}, Tx_{n-1}) = d(x_{n-1}, Tx_{n-1}) \leq a[d(x_{n-2}, x_{n-1}) + d(x_{n-1}, Tx_{n-1})],$$

which yields

$$d(x_{n-1}, Tx_{n-1}) \leq \frac{a}{1-a} d(x_{n-2}, x_{n-1}),\ n \geq 2.$$

Therefore,

$$d(x_n, x_{n+1}) \leq \delta d(x_{n-2}, x_{n-1}). \tag{2.15}$$

Now, by summarizing all three cases and using (2.13) and (2.15), it follows that the sequence $\{d(x_n, x_{n-1})\}$ satisfies the inequality

$$d(x_n, x_{n+1}) \leq \delta \max\{d(x_{n-2}, x_{n-1}),\ d(x_{n-1}, x_n)\}, \tag{2.16}$$

for all $n \geq 2$. Now, by induction for $n \geq 2$, from (2.16) one obtains

$$d(x_n, x_{n+1}) \leq \delta^{[n/2]} \max\{d(x_0, x_1),\ d(x_1, x_2)\}, \tag{2.17}$$

where $[n/2]$ denotes the greatest integer not exceeding $n/2$.

Further, for $m > n > N$,

$$d(x_n, x_m) \leq \sum_{i=N}^{\infty} d(x_i, x_{i-1}) \leq 2\frac{\delta^{[N/2]}}{1-\delta} \max\{d(x_0, x_1),\ d(x_1, x_2)\},$$

which shows that $\{x_n\}$ is a Cauchy sequence.

Since $\{x_n\} \subset K$ and K is closed, $\{x_n\}$ converges to some point x^* in K, i.e.,

$$x^* = \lim_{n\to\infty} x_n. \tag{2.18}$$

By property (L), there exists a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ satisfying

$$(x_{k_n}, x^*) \in E(G),\ \forall n \in \mathbb{N}.$$

and hence, by the contraction condition (2.9),

$$d(x_{k_n+1}, Tx^*) = d(Tx_{k_n}, Tx^*) \leq a[d(x_{k_n}, Tx_{k_n}) + d(x^*, Tx^*)]. \tag{2.19}$$

Therefore,

$$d(x^*, Tx^*) \leq d(x^*, x_{k_n+1}) + d(x_{k_n+1}, Tx^*) = d(x_{k_n+1}, x^*) + d(Tx_{k_n}, Tx^*)\,.$$

which, by (2.19) yields

$$d(x^*, Tx^*) \leq \frac{1}{1-a} d(x^*, x_{k_n+1}) + \delta \cdot d(x_{k_n}, Tx_{k_n})\,, \tag{2.20}$$

for all $n \geq 1$. But, by means of (2.17),

$$d(x_{k_n}, Tx_{k_n}) = d(x_{k_n}, x_{k_n+1} \leq \delta^{[k_n/2]} \max\{d(x_{k_0}, x_{k_1}),\ d(x_{k_1}, x_{k_2})\},$$

and letting now $n \to \infty$ in (2.20), we obtain

$$d(x^*, Tx^*) = 0,$$

which shows that x^* is a fixed point of T.

The uniqueness of x^* immediately follows by the contraction condition (2.9).

In the end, by using the estimate (2.17) and triangle inequality we obtain for any $n, p \in \mathbb{N}^*$

$$d(x_n, x_{n+p}) \leq \delta^{[n/2]} \frac{1 - \delta^{[(p+1)/2]}}{1 - \delta} \max\{d(x_0, x_1),\ d(x_1, x_2)\},$$

from which, by letting $p \to \infty$, we get exactly the error estimate (2.11). $\square$

A weaker form of Theorem 2.1 can be stated as follows.

Theorem 2.2. *Let (X, d, G) be a Banach space endowed with a simple directed and weakly connected graph G. Let K be a nonempty closed subset of X and $T : K \to X$ be a G-Kannan contraction on K.*

If $K_T := \{x \in \partial K : (x, Tx) \in E(G)\} \neq \emptyset$, T is orbitally G-continuous and T satisfies Rothe's boundary condition

$$T(\partial K) \subset K,$$

then the conclusion of Theorem 2.1 remains valid.

3. Conclusions and further study

The Kannan-type contractive condition (1.2) (or (2.9) in the non self mapping case) is independent of the Banach type contraction condition considered in [19], see the next example.

Example 3.3. ([52], Example 1.3.1) Let $X = [0, 1]$ with the usual norm and $T : [0, 1] \to [0, 1]$ be defined by

$$T(x) = \begin{cases} \dfrac{2}{5}, & x \in \left[0, \dfrac{2}{3}\right) \\[2ex] \dfrac{1}{5}, & x \in \left[\dfrac{2}{3}, 1\right]. \end{cases}$$

Then T is a discontinuous Kannan operator with constant $k = \dfrac{3}{7}$.

This shows that the Theorems 2.1 and 2.2 established in the present are important alternative fixed point theorems for non self mappings in Banach spaces endowed with a graph. They provide effective generalisations and extensions of similar results in literature and subsume several important results in the fixed point theory of self and nonself mappings.

Both Theorem 2.1 and Theorem 2.2 were established in Banach spaces endowed with a graph for the sake of simplicity of exposition but they can be transposed in more general settings, like convex metric spaces or $CAT\,(0)$ spaces without any major technical difficulty.

By working on Banach spaces endowed with a graph, our results are valid not only for mappings that satisfy the contraction condition (2.9) for all pairs (x, y) of the space $X \times X$, but only for the pairs (x, y) which are vertices of a simple directed and weakly connected graph $G = (X, E(G))$, with $E(G) \subset X \times X$.

Amongst the most important particular cases of Theorem 2.1 and Theorem 2.2, we mention in the following just the following ones:

1. If G is the graph G_0 in Example 2.1, then by Theorem 2.1 we obtain an extension of Kannan fixed point theorem [41] for non self mappings, restricted here for the reasons mentioned above to Banach spaces instead of usual complete metric spaces.

2. If $K = X$, then Theorem 2.1 we obtain the main result in [24].

For further developments, we have in view considering nonself single-valued as well as multi-valued mappings by starting from the corresponding case of self mappings, see [1]-[4], [5], [20], [21], [24], [29], [31], [32], [37], [38], [39], [42], [43], [50]-[58], [70]-[72], [73]-[76] etc.

For example, starting from the fact that the following contractive condition, due to Chatterjea [29]: there exists a constant $c \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big], \quad \text{for all } x, y \in X, \tag{3.21}$$

is a kind of dual condition of (1.2), it is our aim to obtain similar results to the ones established in the present paper, on the basis that (1.2) and (3.21) are independent contractive conditions, see Example 1.3.4 in [52].

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