CARPATHIAN J. MATH. Online version at http://carpathian.ubm.ro
33 (2017), No. 3, 275 - 286 Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

A general concept of multiple fixed point for mappings defined on spaces with a distance

MITROFAN M. CHOBAN[1] and VASILE BERINDE[2,3]

ABSTRACT. Our main aim in this paper is to introduce a general concept of multidimensional fixed point of a mapping in spaces with distance and establish various multidimensional fixed point results. This new concept simplifies the similar notion from [A. Roldan, J. Martinez-Moreno, C. Roldan, *Multidimensional fixed point theorems in partially ordered complete metric spaces*, J. Math. Anal. Appl. 396 (2012), 536–545]. The obtained multiple fixed point theorems extend, generalise and unify many related results in literature.

1. INTRODUCTION

The notion of *multidimensional fixed point* emerged naturally from the rich literature devoted to the study of coupled fixed points in the last four decades. The concept of *coupled fixed point* itself has been first introduced and studied by V. I. Opoitsev, in a series of papers published in the period 1975-1986, see [58]-[62], for the case of mixed monotone nonlinear operators satisfying a nonexpansive type condition.

Later, in 1987, Guo and Lakshmikantham [41], studied coupled fixed points in connection with coupled quasisolutions of an initial value problem for ordinary differential equations (see also [39]). In 1991, Chen [30] obtained coupled fixed point results of $\frac{1}{2}$-α-condensing and mixed monotone operators, where α denotes the Kuratowski's measure of non compactness, thus extending some previous results from [41] and [77]. In the same year, Chang and Ma [29] discussed some existence results and iterative approximation of coupled fixed points for mixed monotone condensing set-valued operators. Next, Chang, Cho and Huang [28] obtained coupled fixed point results of $\frac{1}{2}$-α-contractive and generalized condensing mixed monotone operators.

More recently, Gnana Bhaskar and Lakshmikantham in [37] established coupled fixed point results for mixed monotone operators in partially ordered metric spaces in the presence of a Bancah contraction type condition. Essentialy, the results by Bhaskar and Lakshmikantham in [37] combine, in the context of coupled fixed point theory, the main fixed point results previously obtained by Nieto and Rodriguez-Lopez in [55] and [56]. The last two papers are, in turn, in continuation to a very important fixed point theorem established in the seminal paper of Ran and Reurings [63], which has the merit to combine a metrical fixed point theorem (the contraction mapping principle) and an order theoretic fixed point result (Tarski's fixed point theorem).

Various applications of the theoretical results in the previous mentioned papers were also given by several authors to: a) Uryson integral equations [60]; b) a system of Volterra integral equations [30], [28]; c) a class of functional equations arising in dynamic programming [29]; d) initial value problems for first order differential equations with discontinuous right hand side [41]; e) (two point) periodic boundary value problems [17],

Received: 21.05.2017 . In revised form: 21.07.2017. Accepted: 23.07.2017
2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. *Distance space, coincidence point, fixed point.*
Corresponding author: Berinde Vasile; vasile.berinde@gmail.com

[37], [33], [80]; f) integral equations and systems of integral equations [3], [6], [9], [24], [38], [42], [76], [78], [83]; g) nonlinear elliptic problems and delayed hematopoesis models [82]; h) nonlinear Hammerstein integral equations [74]; i) nonlinear matrix and nonlinear quadratic equations [4], [24]; j) initial value problems for ODE [8], [73] etc. For a very recent account on the developments of coupled fixed point theory, we also refer to [22].

In 2010, Samet and Vetro [72] considered a concept of *f*ixed point of m-order as a natural extension of the notion of coupled fixed point. One year later, Berinde and Borcut [18] introduced the concept of *triple fixed point* and proved triple fixed-point theorems using mixed monotone mappings, while, in 2012, Karapinar and Berinde [47], have studied quadruple fixed points of nonlinear contractions in partially ordered metric spaces.

After these papers, a substantial number of articles were dedicated to the study of triple fixed point and quadruple fixed point theory. Next, J. Roldan, Martinez-Moreno and C. Roldan [64] introduced a new concept of *fixed point of m-order*, which is also called by various authors "a multidimensional fixed point", or "an m-tuplet fixed point", or "an m-tuple fixed point". For some other very recent results on this topic we refer to [1], [2], [7], [25], [26], [27], [35], [46], [48], [49], [43], [44], [45], [50], [51], [57], [64]-[69], [79], [81], [84].

In the present paper, our main aim is to introduce and study a general concept of multidimensional fixed point in the setting of ordered spaces with distance. This concept simplifies the similar notion from [64] and allows us to obtain general multiple fixed point theorems that include as particular cases several related results in literature.

This point of view allows us to reduce the multidimensional case of fixed points and coincidence points to the one-dimensional case. Note that, the first author who reduced the problem of finding a coupled fixed point of mixed monotone operators to the problem of finding a fixed point of an increasing operator was Opoitsev, see for example [60]. For a more recent similar approach we refer to [14].

2. Preliminaries

By a space we understand a topological T_0-space. We use the terminology from [36, 40, 70, 31].

Let X be a non-empty set and $d : X \times X \to \mathbb{R}$ be a mapping such that:

(i_m) $d(x, y) \geq 0$, for all $x, y \in X$;

(ii_m) $d(x, y) + d(y, x) = 0$ if and only if $x = y$.

Then d is called a *distance* on X, while (X, d) is called a *distance space*.

Let d be a distance on X and

$$B(x, d, r) = \{y \in X : d(x, y) < r\}$$

be the *ball* with the center x and radius $r > 0$. The set $U \subset X$ is called *d-open* if for any $x \in U$ there exists $r > 0$ such that $B(x, d, r) \subset U$. The family $\mathcal{T}(d)$ of all d-open subsets is the topology on X generated by d. A distance space is a *sequential space*, i.e., a space for which a set $B \subseteq X$ is closed if and only if together with any sequence it contains all its limits [36].

Let (X, d) be a distance space, $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$. We say that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is:

1) *convergent to x* if and only if

$$\lim_{n\to\infty} d(x, x_n) = 0.$$

We denote this by $x_n \to x$ or

$$x = \lim_{n\to\infty} x_n.$$

Actually, we might denote better

$$x \in \lim_{n\to\infty} x_n.$$

2) *convergent* if it converges to some point x in X;
3) *Cauchy* or *fundamental* if

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

A distance space (X, d) is called *complete* if every Cauchy sequence in X converges to some point x in X.

Let X be a non-empty set and d be a distance on X. Then:

- (X, d) is called a *symmetric space* and d is called a *symmetric* on X if
  (iii_m) $d(x, y) = d(y, x)$, for all $x, y \in X$;
- (X, d) is called a *quasimetric space* and d is called a *quasimetric* on X if
  (iv_m) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$;
- (X, d) is called a *metric space* and d is called a *metric* if d is a symmetric and a quasimetric, simultaneously.

Let X be a non-empty set and $d(x, y)$ be a distance on X with the following property:
(N) for each point $x \in X$ and any $\varepsilon > 0$ there exists $\delta = \delta(x, \varepsilon) > 0$ such that from $d(x, y) \leq \delta$ and $d(y, z) \leq \delta$ it follows $d(x, z) \leq \varepsilon$.
Then (X, d) is called an *N-distance space* and d is called an *N-distance* on X. If d is a symmetric, then we say that d is a N-symmetric.

Spaces with N-distances were studied by V. Niemyzki [53, 54] and by S. I. Nedev [52]. Clearly, any (quasi) metric space is a N-distance space. If d satisfies uniformly the N-distance condition, that is,

(F) for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that from $d(x, y) \leq \delta$ and $d(y, z) \leq \delta$ it follows $d(x, z) \leq \varepsilon$, then d is called a *F-distance* or a *Fréchet distance*, while (X, d) is called an *F-distance space*.

Obviously, any F-distance d is an N-distance, too, but the reverse is not true, in general, see Examples 1.1 and 1.2 in [31]. If d is a symmetric and a F-distance on a space X, then we say that d is a F-symmetric.

Remark 2.1. If (X, d) is an F-symmetric space, then any convergent sequence is a Cauchy sequence. For N-symmetric spaces and for quasimetric spaces this assertion is not more true.

If $s > 0$ and

$$d(x, y) \leq s[d(x, z) + d(z, y)]$$

for all points $x, y, z \in X$, then we say that d is an *s-distance*. Any s-distance is an F-distance.

A distance space (X, d) is called an *H-distance space* if, for any two distinct points $x, y \in X$, there exists $\delta = \delta(x, y) > 0$ such that

$$B(x, d, \delta) \cap B(y, d, \delta) = \varnothing.$$

Remark 2.2. A distance space (X, d) is an H-distance space if and only if any convergent sequence in X has a *unique* limit point.

We say that (X, d) is a *C-distance space* or a *Cauchy distance space* if any convergent Cauchy sequence has a unique limit point.

Fix a mapping $\varphi : X \longrightarrow X$. For any point $x \in X$ we put

$$\varphi^0(x) = x, \varphi^1(x) = \varphi(x), ..., \varphi^n(x) = \varphi(\varphi^{n-1}(x)), \ldots.$$

The sequence

$$O(\varphi, x) = \{x_n = \varphi^n(x) : n \in \mathbb{N}\}$$

is called the *orbit of* φ at the point x or the *Picard sequence* at the point x.

Let (X, d) be a distance space and $\varphi : X \longrightarrow X$ a mapping. We say that the mapping φ is:

- *contractive* if

$$d(\varphi(x), \varphi(y)) < d(x, y), \text{ provided } d(x, y) > 0;$$

- a *contraction* if there exists $\lambda \in [0, 1)$ such that

$$d(\varphi(x), \varphi(y)) \leq \lambda d(x, y), \text{ for all } x, y \in X;$$

- *strongly asymptotically regular* if

$$\lim_{n\to\infty}(d(\varphi^n(x), \varphi^{n+1}(x) + d(\varphi^{n+1}(x), \varphi^n(x)))) = 0, \text{ for any } x \in X.$$

Now, let (X, d) be a distance space and $m \in \mathbb{N} = \{1, 2, ...\}$. On the set X^m we consider the distances

$$d^m((x_1, ..., x_m), (y_1, ..., y_m)) = sup\{d(x_i, y_i) : i \leq m\}$$

and

$$\bar{d}^m((x_1, ..., x_m), (y_1, ..., y_m)) = \sum_{i=1}^{m} d(x_i, y_i).$$

Obviously, (X^m, d^m) and $(X^m, \bar{d}^m)$ are distance spaces, too.

Proposition 2.3. *Let (X, d) be a distance space. Then:*

1. If d is a symmetric, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are symmetric spaces, too.

2. If d is a quasimetric, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are quasimetric spaces, too.

3. If d is a metric, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are metric spaces, too.

4. If d is an F-distance space, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are F-distance spaces, too.

5. If d is an N-distance space, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are N-distance spaces, too.

6. If d is an H-distance space, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are H-distance spaces, too.

7. If (X, d) is a C-distance space, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are C-distance spaces, too.

8. If (X, d) is a complete distance space, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are complete distance spaces, too.

9. If d is an s-distance space, then (X^m, d^m) and $(X^m, \bar{d}^m)$ are s-distance spaces, too.

10. The spaces (X^m, d^m) and $(X^m, \bar{d}^m)$ share the same convergent sequences and the same Cauchy sequences. Moreover, the distances d^m and $\bar{d}^m$ are uniformly equivalent, i.e., for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that:

- from $d^m(x, y) \leq \delta$ it follows $\bar{d}^m(x, y) \leq \varepsilon$;

- from $\bar{d}^m(x, y) \leq \delta$ it follows $d^m(x, y) \leq \varepsilon$.

Proof. It is well known. □

3. Multiple fixed point principles

Fix $m \in \mathbb{N}$ and denote by $\lambda = (\lambda_1, ..., \lambda_m)$ a collection of mappings

$$\{\lambda_i : \{1, 2, ..., m\} \longrightarrow \{1, 2, ..., m\} : 1 \leq i \leq m\}.$$

Let (X, d) be a distance space and $F : X^m \longrightarrow X$ be an operator. The operator F and the mappings λ generate the operator $\lambda F : X^m \longrightarrow X^m$, where

$$\lambda F(x_1, ..., x_m) = (y_1, ..., y_m) \text{ and } y_i = F(x_{\lambda_i(1)}, ..., x_{\lambda_i(m)}),$$

for any point $(x_1, ..., x_m) \in X^m$ and any index $i \in \{1, 2, ..., m\}$.

A point $a = (a_1, ..., a_m) \in X^m$ is called a λ-*multiple fixed point* of the operator F if

$$a = \lambda F(a), \text{ i.e., } a_i = F(a_{\lambda_i(1)}, ..., a_{\lambda_i(m)}), \text{ for any } i \in \{1, 2, ..., m\}.$$

We say that the operator F is:

- λ-*contractive* if
$$d^m(\lambda F(x), \lambda F(y)) < d^m(x, y), \text{ for all } x, y \in X^m \text{ with } d^m(x, y) > 0;$$
- a λ-*contraction* if there exist a number $k \in [0, 1)$ such that
$$d(F(x_1, ..., x_m), F(y_1, ..., y_m)) \leq k \sup\{d(x_i, y_i) : i \leq m\},$$
for all $(x_1, ..., x_m), (y_1, ..., y_m) \in X^m$.
- $\bar{\lambda}$-*contractive* if
$$\bar{d}^m(\lambda F(x), \lambda F(y)) < \bar{d}^m(x, y), \text{ for all } x, y \in X^m \text{ with } \bar{d}^m(x, y) > 0;$$
- a $\bar{\lambda}$-*contraction* if there exists a number $k \in [0, 1)$ such that
$$d(F(x_1, ..., x_m), F(y_1, ..., y_m)) \leq \frac{k}{m} \cdot \sum_{i=1}^{m} d(x_i, y_i),$$
for all $(x_1, ..., x_m), (y_1, ..., y_m) \in X^m$.

Proposition 3.1. *Let (X, d) be a distance space, $m \in \mathbb{N}$, $F : X^m \to X$ be an operator,*

$$\lambda = \{\lambda_i : \{1, 2, ..., m\} \longrightarrow \{1, 2, ..., m\} : 1 \leq i \leq m\}$$

be a collection of mappings, $k \geq 0$, $a = (a_1, ..., a_m) \in X^m$, $b = (b_1, ..., b_m) \in X^m$ such that

$$d(F(a_{\lambda_i(1)}, ..., a_{\lambda_i(m)}), F(b_{\lambda_i(1)}, ..., b_{\lambda_i(m)})) \leq k \sup\{d(a_i, b_i) : i \leq m\},$$

for each $1 \leq i \leq m$.

Then

$$d^m(\lambda F(a), \lambda F(b)) \leq k d^m(a, b).$$

Proof. Let

$$u_i = F(a_{\lambda_i(1)}, ..., a_{\lambda_i(m)})$$

and

$$v_i = F(b_{\lambda_i(1)}, ..., b_{\lambda_i(m)}),$$

for any $i \leq m$. Then

$$\lambda F(a) = u = (u_1, ..., u_m)$$

and

$$\lambda F(b) = v = (v_1, ..., v_m).$$

We have

$$d^m(\lambda F(a), \lambda F(b)) = d^m(u, v) = \sup\{d(u_i, v_i) : i \leq m\}$$
$$= \sup\{d(F(a_{\lambda_i(1)}, ..., a_{\lambda_i(m)}), F(b_{\lambda_i(1)}, ..., b_{\lambda_i(m)})) : i \leq m\}$$
$$\leq \sup\{k \sup\{d(a_{\lambda_i(j)}, b_{\lambda_i(j)}) : j \leq m\} : i \leq m\}$$
$$\leq k \sup\{d(a_i, b_i) : i \leq m\} = k d^m(a, b).$$

□

Corollary 3.2. *Let (X, d) be a distance space $m \in \mathbb{N}$ and $F : X^m \to X$ be an operator. If F is a λ-contraction, then λF is a contraction on the distance space (X^m, d^m).*

Proposition 3.3. *Let (X,d) be a distance space, $m \in \mathbb{N}$ and $F : X^m \to X$ be an operator,*

$$\{\lambda_i : \{1,2,...,m\} \longrightarrow \{1,2,...,m\} : 1 \leq i \leq m\}$$

be a collection of mappings, $k \geq 0$, $a = (a_1, ..., a_m) \in X^m$, $b = (b_1, ..., b_m) \in X^m$ such that

$$d(F(a_{\lambda_i(1)}, ..., a_{\lambda_i(m)}), F(b_{\lambda_i(1)}, ..., b_{\lambda_i(m)})) \leq k/m \sum_{i=1}^{m} d(a_i, b_i),$$

for each $i \in \{1,2,...,m\}$. If the mapping λ_i is a surjection or, more generally, if

$$| \cup \{\lambda_i^{-1}(j) : j \leq m\}| = m, \text{ for each } i \in \{1,2,...,m\},$$

then

$$\bar{d}^m(\lambda F(a), \lambda F(b)) \leq k\bar{d}^m(a,b).$$

Proof. We put $u = (u_1, ..., u_m) = \lambda F(a)$ and $v = (v_1, ..., v_m) = \lambda F(b)$. Then

$$\bar{d}^m(\lambda F(a), \lambda F(b)) = \sum_{i=1}^{m} d(u_i, v_i) =$$

$$= \sum_{i=1}^{m} d(F(a_{\lambda_i(1)}, ..., a_{\lambda_i(m)}), F(b_{\lambda_i(1)}, ..., b_{\lambda_i(m)})) \leq$$

$$\leq \sum_{i=1}^{m} k/m \sum_{j=1}^{m} d(a_{\lambda_i(j)}, b_{\lambda_i(j)}) \leq k \sum_{i=1}^{m} d(a_i, b_i) = k\bar{d}^m(a,b).$$

□

Corollary 3.4. *Let (X,d) be a distance space, $m \in \mathbb{N}$ and $F : X^m \to X$ be an operator. If F is a $\bar{\lambda}$-contraction and for any $i \in \{1,2,...,m\}$ the mapping λ_i is a surjection or, more generally, if*

$$| \cup \{\lambda_i^{-1}(j) : j \leq m\}| = m, \text{ for each } i \in \{1,2,...,m\},$$

then λF is a contraction on the distance space $(X^m, \bar{d}^m)$.

4. Multiple fixed points of general operators

Fix $m \in \mathbb{N}$, a distance space (X,d), an operator $\varphi : X^m \to X$ and the mappings

$$\lambda = \{\lambda_i : \{1,...,m\} \to \{1,...,m : i \leq m\}.$$

For any point $a = (a_1, ..., a_n) \in X^m$ we put

$$a(1) = \lambda F(a) \text{ and } a(n+1) = \lambda F(a(n)),$$

for each $n \in \mathbb{N}$. The sequence

$$O(F, \lambda, a) = \{a(n) : n \in \mathbb{N}\}$$

is the Picard sequence at the point a relatively to the operator λF. The orbit $O(F, \lambda, a)$ is called (F, λ)-*bounded* if

$$\sup\{d^m(a, a(n)) + d^m(a(n), a) : n \in \mathbb{N}\} < \infty.$$

(this is equivalent to

$$\sup\{\bar{d}^m(a, a(n)) + \bar{d}^m(a(n), a) : n \in \mathbb{N}\} < \infty.)$$

The space (X,d) is called (F, λ)-*bounded* if any Picard sequence $O(F, \lambda, a)$ is (F, λ)-bounded.

Proposition 4.1. *Let (X, d) be a C-distance space. Then:*

1. $d(x, y) = 0$ if and only if $x = y$.

2. If, for $a \in X^m$, the Picard sequence $O(F, \lambda, a) = \{a(n) : n \in \mathbb{N}\}$ is a convergent Cauchy sequence and

$$\lim_{n\to\infty} a_n = b = (b_1, ..., b_m),$$

then b is a multidimensional fixed point of the operator F with respect to the mappings λ, i.e.,

$$b_i = F(b_{\lambda_i(1)}, ..., b_{\lambda_i(m)}), \text{ for each } i \in \{1, 2, ..., m\}.$$

Proof. Assume that x, y are two distinct points of X and $d(x, y) = 0$. Then the points x, y are both limits of the Cauchy sequence $\{y_n = y : n \in \mathbb{N}\}$, a contradiction. So, assertion 1 is proved.

In the conditions of assertion 2, we have $\lambda F(b) = b$. □

Corollary 4.2. *Let (X, d) be a complete C-distance space, $\rho \in \{d^m, \bar{d}^m\}$, $k > 0$ and $F : X^m \longrightarrow X$ be an operator with the following properties:*

(1) *there exists $k > 0$ such that*

$$d(F(x), F(y)) < k\rho(x, y), \text{ for all distinct points } x, y \in X^m;$$

(2) *if $x \in X^m$, then the Picard sequence $\{x_n \in X : n \in \mathbb{N}\}$ of F at the point x is a Cauchy sequence.*

Then

1. The operators F and λF are continuous.

2. The set $Fix(F)$ of the multidimensional fixed points of F is closed in X^m and non-empty.

3. If $k \leq 1$, then F has a unique multidimensional fixed point.

Theorem 4.3. *Let (X, d) be a (F, λ)-bounded complete C-distance space and $F : X^m \longrightarrow X$ be an operator.*

1. If F is a λ-contraction, then any Picard sequence of the operator λF is a convergent Cauchy sequence and F has a unique multidimensional fixed point.

2. If F is a $\bar{\lambda}$-contraction and for any $i \in \{1, 2, ..., m\}$ the mapping λ_i is a surjection or, more generally, if

$$| \cup \{\lambda_i^{-1}(j) : j \leq n\}| = m, \text{ for each } i \in \{1, 2, ..., m\},$$

then any Picard sequence of the operator λF is a convergent Cauchy sequence and F has a unique multidimensional fixed point.

Proof. Let $\rho = d^m$ in the conditions of Assertion 1 and $\rho = \bar{d^m}$ in the conditions of Assertion 2. From Propositions 3.1 and 3.3, respectively, it follows that λF is a contraction on the complete C-distance space (X^m, ρ). Proposition 3.4 from [31] ensures that the operator λF has a unique fixed point which is a multidimensional fixed point of F. □

Theorem 4.4. *Let (X, d) be an N-symmetric space and $F : X^m \longrightarrow X$ be an operator.*

1. If F is a λ-contractive operator and for each point $x \in X^m$ the Picard sequence $O(F, \lambda, x) = \{x_n : n \in \mathbb{N}\}$ has an accumulation point and $\lim_{n\to\infty} d^m(x_n, x_{n+1}) = 0$, then any Picard sequence of the operator λF is a convergent Cauchy sequence and F has a unique multidimensional fixed point.

2. If F is a $\bar{\lambda}$-contractive operator and, for any $i \in \{1, 2, ..., m\}$, the mapping λ_i is a surjection or, more generally, $| \cup \{\lambda_i^{-1}(j) : j \leq m\}| = m$ for each $i \in \{1, 2, ..., m\}$ and for each point $x \in X^m$ the Picard sequence $O(F, \lambda, x) = \{x_n : n \in \mathbb{N}\}$ has an accumulation point and $\lim_{n\to\infty} \bar{d^m}(x_n, x_{n+1}) = 0$, then any Picard sequence of the operator λF is a convergent Cauchy sequence and F has a unique multidimensional fixed point.

3. d is an H-distance and any Picard sequence has a unique accumulation point.

Proof. Assertion 3 follows immediately from Theorem 4.1 from [31]. Let ρ be the symmetric constructed in the proof of Theorem 4.3. Then λF is a strongly asymptotically regular contractive mapping on the N-symmetric space (X^m, ρ) and, for each point $x \in X^m$, the Picard sequence $O(F, \lambda, x)$ has an accumulation point. Now, Theorem 4.1 from [31] completes the proof. □

Corollary 4.5. *Let (X, d) be an N-symmetric compact space and $F : X^m \longrightarrow X$ be an operator.*

1. If F is a λ-contraction, then any Picard sequence of the operator λF is a convergent Cauchy sequence and F has a unique multidimensional fixed point.

2. If F is a $\bar{\lambda}$-contraction and for any $i \leq m$ the mapping λ_i is a surjection or, more generally, $| \cup \{\lambda_i^{-1}(j) : j \leq m\}| = m$, for each $i \leq m$, then any Picard sequence of the operator λF is a convergent Cauchy sequence and F has a unique multidimensional fixed point.

The problem of the existence of fixed points for contracting mappings on F-symmetric spaces was first studied in [20]. The following statement improves the fixed point theorems of S. Czerwik [34] and I. A. Bakhtin [10] (see also [70]).

Theorem 4.6. *Let (X, d) be a complete s-distance symmetric space and $F : X^m \longrightarrow X$ be an operator.*

1. If F is a λ-contraction, then any Picard sequence of the operator λF is a convergent Cauchy sequence and F has a unique multidimensional fixed point.

2. If F is a $\bar{\lambda}$-contraction and, for any $i \leq m$, the mapping λ_i is a surjection or, more generally, $| \cup \{\lambda_i^{-1}(j) : j \leq m\}| = m$, for each $i \leq m$, then any Picard sequence of the operator λF is a convergent Cauchy sequence and F has a unique multidimensional fixed point.

Proof. Let ρ be the symmetric constructed in the proof of Theorem 4.3. By virtue of Proposition 2.3, ρ is a symmetric s-distance. Then λF is a contractive mapping of the s-symmetric space (X^m, ρ). Now, Theorem 4.2 from [31] completes the proof. □

5. Some particular cases and conclusions

If we take concrete values of $m \in \mathbb{N}$ and consider various particular functions $\lambda = \{\lambda_i : \{1, ..., m\} \to \{1, ..., m\} : 1 \leq i \leq m\}$ then, most of the results in literature dedicated to coupled, triple, quadruple,... fixed point theory, are obtained as particular cases of the multiple fixed point theorems established in the present paper.

For example, if $m = 2$, $\lambda_1(1) = 1$, $\lambda_1(2) = 2$; $\lambda_2(1) = 2$, $\lambda_2(2) = 1$, by our main results we obtain the coupled fixed point theorems in [37] and in various subsequent papers, see especially the singular paper [71], where the setting is a (cone) metric space without any order relation.

If $m = 3$, $\lambda_1(1) = 1$, $\lambda_1(2) = 2$, $\lambda_1(3) = 3$; $\lambda_2(1) = 2$, $\lambda_2(2) = 1$, $\lambda_2(3) = 2$; $\lambda_3(1) = 3$, $\lambda_3(2) = 2$, $\lambda_3(3) = 1$, then the concept of multiple fixed point studied in the present paper reduces to that of triple fixed point, first introduced in [18] and intensively studied in many other research works emerging from it.

We note that, as pointed out in [75], the notion of tripled fixed point due to Berinde and Borcut [18] is different from the one defined by Samet and Vetro [72] for $m = 3$, since in the case of ordered metric spaces in order to keep the mixed monotone property working, it was necessary to take $\lambda_2(3) = 2$ and not $\lambda_2(3) = 3$.

We mention one more important particular case, i.e., the one of fixed point of N-order introduced and studied in [72], which is obtained as particular case of our concept introduced in the present paper, by taking $m = N$, λ_1 = the identity permutation of $\{1, 2, \ldots, N\}$ and, for $i \geq 2$, λ_i is the cyclical permutation of $\{1, 2, \ldots, N\}$ that starts with $\lambda_i(1) = i$, i.e., for example, $\lambda_2(1) = 2$, $\lambda_2(2) = 3$,…,$\lambda_2(N-1) = N$, $\lambda_2(N) = 1$. Note that in this case the family of mappings $\lambda = \{\lambda_i : \{1, ..., N\} \to \{1, ..., N : i \leq N\}$ satisfies both alternative conditions imposed in Theorems 4.3, 4.4, 4.6, Proposition 3.3 and Corollaries 3.4, 4.5, i.e., λ_i is a surjection and $| \cup \{\lambda_i^{-1}(j) : 1 \leq j \leq N\}| = N$, for each $i \leq N$.

For other concepts of multiple fixed points considered in literature the condition " λ_i is a surjection, for each $i \leq m$" is no more valid, see for example [18] and the research papers emerging from it, while the second condition, $| \cup \{\lambda_i^{-1}(j) : 1 \leq j \leq m\}| = m$, for each $i \leq m$, is satisfied.

As the great majority of the papers dealing with coupled, triple, quadruple,..., multiple fixed points were established in ordered metric spaces or generalised order metric spaces, we shall study them separately in a forthcoming paper [32], where the basic setting will be an ordered distance space.

We point out the fact that the main idea of this paper was to obtain general multiple fixed point theorems by reducing this problem to a unidimensional fixed point problem and by simultaneously working in a more general and very reliable setting, that of distance spaces. Many other related and relevant results could be obtained in the same way, by reducing the multidimensional fixed point problem to many other independent unidimensional fixed point principles, like the ones established in [5], [11], [12], [13], [15], [16], [19], [21], [23] etc.

Acknowledgements. This paper has been finalised during the second author's visit to Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia, in the period December 2016-January 2017. He would like to acknowledge the support provided by the Deanship of Scientific Research at King Fahd University of Petroleum and Minerals for funding this work through the projects IN151014 and IN141047.

REFERENCES

[1] Agarwal, R., Karapinar, E. and Roldán-López-de-Hierro, A.-F., *Some remarks on 'Multidimensional fixed point theorems for isotone mappings in partially ordered metric spaces'*, Fixed Point Theory Appl., 2014, 2014:245, 13 pp.

[2] Agarwal, R., Karapinar, E. and Roldán-López-de-Hierro, A.-F., *Fixed point theorems in quasi-metric spaces and applications to multidimensional fixed point theorems on G-metric spaces*, J. Nonlinear Convex Anal., **16** (2015), No. 9, 1787–1816

[3] Aghajani, A., Abbas, M. and Kallehbasti, E. P., *Coupled fixed point theorems in partially ordered metric spaces and application*, Math. Commun., **17** (2002), No. 2, 497–509

[4] Aghajani, A. and Arab, R., *Fixed points of (ψ, φ, θ)-contractivemappings in partially ordered b-metric spaces and application to quadratic integral equations*, Fixed Point Theory Appl., **2013**, 2013:245 doi:10.1186/1687-1812-2013-245

[5] Alghamdi, M. A., Berinde, V. and Shahzad, N., *Fixed Points of Multivalued Nonself Almost Contractions*, J. Appl. Math., **2013**, 2013: 621614

[6] Alghamdi, M. A., Hussain, N. and Salimi, P., *Fixed point and coupled fixed point theorems on b-metric-like spaces*, J. Ineq. Appl., **2013**, 2013:402 doi:10.1186/1029-242X-2013-402

[7] Al-Mezel, S. A., Alsulami, H. H., Karapinar, E. and Lopez-de-Hierro, A.-F. R., *Discussion on "Multidimensional Coincidence Points" via recent publications*, Abstr. Appl. Anal., 2014, Art. ID 287492, 13 pp.

[8] Amini-Harandi, A., *Coupled and tripled fixed point theory in partially ordered metric spaces with application to initial value problem*, Math. Comput. Model., **57** (2013), No. 9–10, 2343–2348

[9] Aydi, H., Samet, B. and Vetro, C. *Coupled fixed point results in cone metric spaces for $\tilde{w}$-compatible mappings*, Fixed Point Theory Appl., **2011**, 2011:27 doi:10.1186/1687-1812-2011-27

[10] Bakhtin, I. A., *The contraction mapping principle in almost metric spaces* (in Russian), Funct. Anal., Ulianovskii Gosud. Pedag. Inst., **30** (1989), 26–37

[11] Berinde, V., *A common fixed point theorem for compatible quasi contractive self mappings in metric spaces*, Appl. Math. Comput.,**213** (2009), No. 2, 348–354

[12] Berinde, V., *Approximating common fixed points of noncommuting discontinuous weakly contractive mappings in metric spaces*, Carpathian J. Math., **25** (2009), No. 1, 13–22

[13] Berinde, V., *Common fixed points of noncommuting discontinuous weakly contractive mappings in cone metric spaces*, Taiwanese J. Math., **14** (2010), No. 5, 1763–1776

[14] Berinde, V., *Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal., **74** (2011) 7347–7355

[15] Berinde, V., *Stability of Picard iteration for contractive mappings satisfying an implicit relation*, Carpathian J. Math., **27** (2011), No. 1, 13–23

[16] Berinde, V., *Coupled coincidence point theorems for mixed monotone nonlinear operators*. Comput. Math. Appl., **64** (2012), No. 6, 1770–1777

[17] Berinde, V., *Coupled fixed point theorems for Φ-contractive mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal, **75** (2012), No. 6, 3218–3228

[18] Berinde, V. and Borcut, M., *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*, Nonlinear Anal., **74** (2011), No. 15, 4889–4897

[19] Berinde, V. and Choban, M. M., *Remarks on some completeness conditions involved in several common fixed point theorems*, Creat. Math. Inform., **19** (2010), No. 1, 1–10

[20] Berinde, V. and Choban, M. M., *Generalized distances and their associate metrics. Impact on fixed point theory*, Creat. Math. Inform., **22** (2013), No. 1, 23–32

[21] Berinde, V. and Păcurar, M., *Coupled fixed point theorems for generalized symmetric Meir-Keeler contractions in ordered metric spaces*, Fixed Point Theory Appl., 2012, 2012:115, 11 pp.

[22] Berinde, V. and Păcurar, M., *Coupled and triple fixed points theorems for mixed monotone almost contractive mappings in partially ordered metric spaces* (submitted)

[23] Berinde, V. and Păcurar, M., *A constructive approach to coupled fixed point theorems in metric spaces*, Carpathian J. Math., **31** (2015), No. 3, 269–275

[24] Berzig, M. and Samet, B., *An extension of coupled fixed points concept in higher dimension and applications*, Comput. Math. Appl., **63** (2012), 1319–1334

[25] Borcut, M., *Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces*, Appl. Math. Comput., **218** (2012), 7339–7346

[26] Borcut, M., *Puncte triple fixe pentru operatori definiţi pe spaţii metrice parţial ordonate*, Risoprint, Cluj-Napoca, 2016

[27] Borcut, M. and Berinde, V., *Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces*, Appl. Math. Comput., **218** (2012), 5929–5936

[28] Chang, S.-S., Cho, Y. J. and Huang, N. J., *Coupled fixed point theorems with applications*, J. Korean Math. Soc., **33** (1996), No. 3, 575–585

[29] Chang, S.-S. and Ma, Y. H., *Coupled fixed points for mixed monotone condensing operators and an existence theorem of the solutions for a class of functional equations arising in dynamic programming*, J. Math. Anal. Appl., **160** (1991), No. 2, 468–479

[30] Chen, Y. Z., *Existence theorems of coupled fixed points*, J. Math. Anal. Appl., **154** (1991), No. 1, 142–150

[31] Choban, M., *Fixed points of mappings defined on spaces with distance*, Carpathian J. Math., **32** (2016), No. 2, 173–188

[32] Choban, M. and Berinde, V., *A general concept of multiple fixed point for mappings defined on ordered spaces with a distance* (in preparation)

[33] Ćirić, L., Damjanović, B. , Jleli, M. and Samet, B., *Coupled fixed point theorems for generalized Mizoguchi-Takahashi contractions with applications*, Fixed Point Theory Appl., 2012, 2012:51 doi:10.1186/1687-1812-2012-51

[34] Czerwik, S., *Fixed Points Theorems and Special Solutions of Functional Equations*, Katowice, 1980

[35] Dalal, S., Khan, L. A., Masmali, I. and Radenovic, S., *Some remarks on multidimensional fixed point theorems in partially ordered metric spaces*, J. Adv. Math., **7** (2014), No. 1, 1084–1094

[36] Engelking, R., General topology. Translated from the Polish by the author. Second edition. Sigma Series in Pure Mathematics, 6. Heldermann Verlag, Berlin, 1989.

[37] Gnana Bhaskar, T., Lakshmikantham, V., *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65** (2006), No. 7, 1379–1393

[38] Gu, F. and Yin, Y., *A new common coupled fixed point theorem in generalized metric space and applications to integral equations*, Fixed Point Theory Appl., **2013**, 2013:266 doi:10.1186/1687-1812-2013-266

[39] Guo, D. J., *Fixed points of mixed monotone operators with applications*, Appl. Anal., **31** (1988), No. 3, 215–224

[40] Granas, A. and Dugundji, J., *Fixed point theory*, Springer, Berlin, 2003

[41] Guo, D. J. and Lakshmikantham, V., *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal, **11** (1987), No. 5, 623–632

[42] Hussain, N., Salimi, P. and Al-Mezel, S., *Coupled fixed point results on quasi-Banach spaces with application to a system of integral equations*, Fixed Point Theory Appl., **2013**, 2013:261 doi:10.1186/1687-1812-2013-261

[43] Imdad, M., Soliman, A. H., Choudhury, B. S. and Das, P., *On n- tupled coincidence and common fixed points results in metric spaces*, J. Oper. 2013, Article ID 532867, 9 pp.

[44] Imdad, M., Sharma, A., Rao, K. P. R., *n-tupled coincidence and common fixed point results for weakly contractive mappings in complete metric spaces*, Bull. Math. Anal. Appl., **5** (2013), No. 4, 19–39

[45] Imdad, M., Alam, A. and Soliman, A. H., *Remarks on a recent gen- eral even-tupled coincidence theorem*, J. Adv. Math., **9** (2014), No. 1, 1787–1805

[46] Karapinar, E., *Quartet fixed points theorems for nonlinear contractions in partially ordered metric space*, arXiv:1106.5472v1 [math.GN] 27 Jun 2011, 10 pp.

[47] Karapinar, E. and Berinde, V., *Quadruple fixed points theorems for nonlinear contractions in partially ordered spaces*, Banach J. Math. Anal., **6** (2012), No. 1, 74–89

[48] Karapinar, E. and Roldán, A., *A note on 'n-tuplet fixed point theorems for contractive type mappings in partially ordered metric spaces'*, J. Inequal. Appl., 2013, 2013:567, 7 pp.

[49] Karapinar, E., Roldán, A., Martínez-Moreno, J. and Roldán, C., *Meir-Keeler type multidimensional fixed point theorems in partially ordered metric spaces*, Abstr. Appl. Anal., 2013, Art. ID 406026, 9 pp.

[50] Lee, H. and Kim, S., *Multivariate coupled fixed point theorems on ordered partial metric spaces*. J. Korean Math. Soc. 51 (2014), No. 6, 1189–1207

[51] Mutlu, A. and Gürdal, U., *An infinite dimensional fixed point theorem on function spaces of ordered metric spaces*, Kuwait J. Sci., **42** (2015), No. 3, 36–49

[52] Nedev, S. I., *o-metrizable spaces*, Trudy Moskov. Mat. Ob-va, **24** (1971), 201–236 (English translation: Trans. Moscow Math. Soc., **24** (1974), 213–247)

[53] Niemytzki, V., *On the third axiom of metric spaces*, Trans Amer. Math. Soc., **29** (1927), 507–513

[54] Niemytzki, V., *Über die Axiome des metrischen Raumes*, Math. Ann., **104** (1931), 666–671

[55] Nieto, J. J. and Rodriguez-Lopez, R., *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order **22** (2005), No. 3, 223–239 (2006)

[56] Nieto, J. J. and Rodriguez-Lopez, R., *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta. Math. Sin., (Engl. Ser.) **23** (2007), No. 12, 2205–2212

[57] Olaoluwa, H. and Olaleru, J., *Multipled fixed point theorems in cone metric spaces*, Fixed Point Theory Appl., 2014, 2014:43, 15 pp.

[58] Opoitsev, V. I., *Geterogennîie i kombinirovano-vognitîie operatorî*, Syb. Matem. Jurn., **16** (1975), No. 4, 781–792

[59] Opoitsev, V. I., *Dynamics of collective behavior. III. Heterogenic systems*, translated from Avtomat. i Telemeh. 1975, No. 1, 124–138 Automat. Remote Control **36** (1975), No. 1, 111–124

[60] Opoitsev, V. I., *Generalization of the theory of monotone and concave operators*, Tr. Mosk. Mat. Obs., **36** (1978), 237–273

[61] Opoitsev, V. I. and Khurodze, T. A., *Nelineinye operatory v prostranstvakh s konusom. Nonlinear operators in spaces with a cone*, Tbilis. Gos. Univ., Tbilisi, 1984. 271 pp.

[62] Opoitsev, V. I., *Nelineinaya sistemostatika. Nonlinear systemostatics* Ekonomiko-Matematicheskaya Biblioteka Library of Mathematical Economics, **31**, Nauka, Moscow, 1986

[63] Ran, A. C. M. and Reurings, M. C. B., *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132** (2004), No. 5, 1435–1443

[64] Roldán, A., Martínez-Moreno, J. and Roldán, C., *Multidimensional fixed point theorems in partially ordered complete metricspaces*, J. Math. Anal. Appl., **396** (2012), 536–545

[65] Roldán, A., Martínez-Moreno, J., Roldán C. and Karapinar, E., *Multidimensional Fixed-Point Theorems in Partially Ordered Complete Partial Metric Spaces under* (ψ, φ)*-Contractivity Conditions*, Abstr. Appl. Anal., 2013, Art. ID 634371, 12 pp.

[66] Roldán, A., Martínez-Moreno, J., Roldán, C. and Karapinar, E., *Meir-Keeler type multidimensional fixed point theorems inpartially ordered metric spaces*, Abstr. Appl. Anal., 2013, Art. ID 406026, 9 pp.

[67] Roldán, A., Martínez-Moreno, J., Roldán, C., Karapinar, E., *Some remarks on multidimensional fixed point theorems*, Fixed Point Theory **15** (2014), No. 2, 545558

[68] Roldán, A., Martínez-Moreno, J., Roldán, C. and Cho, Y. J., *Multidimensional fixed point theorems under* (ψ, ϕ)*-contractive conditions in partially ordered complete metric spaces*, J. Comput. Appl. Math., **273** (2015), 76–87

[69] Rus, M.-D., *The fixed point problem for systems of coordinate-wise uniformly monotone operators and applications*, Mediterr. J. Math., **11** (2014), No. 1, 109122

[70] Rus, I. A., Petruşel, A. and Petruşel, G., *Fixed Point Theory*, Cluj University Press, Cluj-Napoca, 2008

[71] Sabetghadam, F., Masiha, H. P. and Sanatpour, A. H., *Some coupled fixed point theorems in cone metric spaces*, Fixed Point Theory Appl., 2009, Art. ID 125426, 8 pp.

[72] Samet, B. and Vetro, C., *Coupled fixed point, f-invariant set and fixed point of N-order*, Ann. Funct. Anal., **1** (2010), 46–56

[73] Samet, B., Vetro, C. and Vetro, P., *Fixed point theorems for alpha-psi-contractive type mappings*, Nonlinear Anal., **75** (2012), No. 4, 2154–2165

[74] Sang, Y., *A class of* φ*-concave operators and applications*, Fixed Point Theory Appl., **2013**, 2013:274 doi:10.1186/1687-1812-2013-274

[75] Sharma, A., Imdad, M. and Alam, A., *Shorter proofs of some recent even-tupled coincidence theorems for weak contractions in ordered metric spaces*, Math. Sci. (Springer), **8** (2014), No. 4, 131–138

[76] Shatanawi, W., Samet, B. and Abbas, M., *Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces*, Math. Comput. Model., **55** (2012), No. 3-4, 680–687

[77] Shendge, G. R. and Dasare, V. N., Existence of maximal and minimal quasifixed points of mixed monotone operators by iterative technique in *Methods of functional analysis in approximation theory (Bombay, 1985)*, pp. 401–410, Internat. Schriftenreihe Numer. Math., 76, Birkhäuser, Basel, 1986

[78] Sintunavarat, W., Kumam, P. and Cho, Y. J., *Coupled fixed point theorems for nonlinear contractions without mixed monotone property*, Fixed Point Theory Appl., **2012**, 2012:170 doi:10.1186/1687-1812-2012-170

[79] Soleimani R. G., Shukla, S. and Rahimi, H., *Some relations between n-tuple fixed point and fixed point results*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math., RACSAM **109** (2015), No. 2, 471–481.

[80] Urs, C., *Coupled fixed point theorems and applications to periodic boundary value problems*, Miskolc Math. Notes, **14** (2013), No. 1, 323–333

[81] Wang, S., *Multidimensional fixed point theorems for isotone mappings in partially ordered metric spaces*, Fixed Point Theory Appl., 2014, 2014:137, 13 pp.

[82] Wu, J. and Liu, Y., *Fixed point theorems for monotone operators and applications to nonlinear elliptic problems*, Fixed Point Theory Appl., **2013**, 2013:134 doi:10.1186/1687-1812-2013-134

[83] Xiao, J.-Z., Zhu, X.-H. and Shen, Z.-M., *Common coupled fixed point results for hybrid nonlinear contractions in metric spaces*, Fixed Point Theory, **14** (2013), No. 1, 235–249

[84] Zhu, L., Zhu, C.-X., Chen, C.-F. and Stojanović, Ž., *Multidimensional fixed points for generalized ψ-quasi-contractions in quasi-metric-like spaces*, J. Inequal. Appl. 2014, 2014:27, 15 pp.

[1] DEPARTMENT OF PHYSICS, MATHEMATICS AND INFORMATION TECHNOLOGIES
TIRASPOL STATE UNIVERSITY
GH. IABLOCIKIN 5., MD2069 CHIŞINĂU, REPUBLIC OF MOLDOVA
E-mail address: mmchoban@gmail.com

[2] DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
NORTH UNIVERSITY OF BAIA MARE
VICTORIEI 76, 430122 BAIA MARE, ROMANIA
E-mail address: vberinde@ubm.ro

[3] DEPARTMENT OF MATHEMATICS AND STATISTICS
KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DHAHRAM, KINGDOM OF SAUDI ARABIA
E-mail address: vasile.berinde@gmail.com