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Dedicated to professor Iulian Coroian on his 70^{th} anniversary

Stabilizing discrete dynamical systems by monotone Krasnoselskij type iterative schemes

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ABSTRACT. In this note monotone approximations of fixed points of real Lipschitz functions are produced by employing a variation controlling mechanism and a growth-rate controlling mechanism, both with generalized Krasnoselskij type iterations and both inspired from discrete dynamical systems.

1. PRELIMINARIES

Discrete dynamical systems are intensively studied due to their applications in various fields. Even if one dimensional, they are able to model many different kind of phenomena.

For $a, b \in \mathbb{R}$, $a < b$ and $f : [a, b] \to [a, b]$ denote

$$[[a, b], f]$$

the discrete dynamical system defined by f. In such systems the trajectory of an element $x_0 \in [a, b]$ is the sequence started with x_0 and generated by the Picard iteration

$$x_{n+1} = f(x_n), n \in \mathbb{N}.$$

A basic problem regarding the discrete dynamical system $[[a, b], f]$ is the study of trajectories for all starting points and the analysis of the dependences on starting points of the trajectories when f satisfies some smoothness conditions.

Denote F_f the set of fixed points of f, $F_f = \{x|\ x \in [a, b], f(x) = x\}$ (possible empty).

If f is continuous, since $f(a) \geq a$ and $f(b) \leq b$, by the intermediate value theorem applied to $f(x) - x$, it results that f possesses at least one fixed point, $F_f \neq \varnothing$; moreover, the set F_f is compact, as it is a bounded and closed subset of $\mathbb{R}$.

In the discrete dynamical system $[[a, b], f]$ a fixed point x^* of f is considered as ([2], [5])

-*attracting* or *stable* if there exists an open interval I which contains x^* such that $f(x) \in I$ for all $x \in I$ and $\lim\limits_{n\to\infty} f^n(x) = x^*$ for all $x \in I$;

-*repelling* or *instable* if there exists an open interval I which contains x^* such that for every $x \in I \setminus \{x^*\}$ there exists $n \in \mathbb{N}^*$ with $f^n(x) \notin I$.

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We call the sequence $(x_n)_{n\in\mathbb{N}}$ *s-increasing* if either $x_n < x_{n+1}$ for all $n \in \mathbb{N}$, or there is a $k \in \mathbb{N}$ such that $x_0 < x_1 < \ldots < x_{k-1} < x_k = x_{k+1} = x_{k+2} = \ldots$. We call the sequence $(x_n)_{n\in\mathbb{N}}$ *s-decreasing* if either $x_n > x_{n+1}$ for all $n \in \mathbb{N}$, or there is a $k \in \mathbb{N}$ such that $x_0 > x_1 > \ldots > x_{k-1} > x_k = x_{k+1} = x_{k+2} = \ldots$.

Slightly differently from [1] where the strict monotony is required, through this paper we consider a fixed point x^* of f as

-*monotonously attracting from below* if there exists $\epsilon > 0$ such that all trajectories starting with $x_0 \in (x^* - \epsilon, x^*)$ are s-increasing and converge to x^*;

-*monotonously attracting from above* if there exists $\epsilon > 0$ such that all trajectories starting with $x_0 \in (x^*, x^* + \epsilon)$ are s-decreasing and converge to x^*;

-*monotonously stable* if it is monotonously attracting both from below and from above.

We associate to f the following two families of functions

$$\overline{f}_\gamma : [a,b] \to \mathbb{R}, \overline{f}_\gamma (x) = x + \gamma (f(x) - x),$$

$$\widetilde{f}_\gamma : [a,b] \to \mathbb{R}, \widetilde{f}_\gamma (x) = x (1 + \gamma (f(x) - x)),$$

where $\gamma \in \mathbb{R}^*$.

The function f and all the functions $\overline{f}_\gamma$ are related to each other by sharing exactly the same fixed points set

$$F_f = F_{\overline{f}_\gamma}, \gamma \in \mathbb{R}^*.$$

If $0 \notin [a,b]$, then also

$$F_f = F_{\widetilde{f}_\gamma}, \gamma \in \mathbb{R}^*.$$

Indeed, if $x \in F_f$, then $f(x) - x = 0$, so $\widetilde{f}_\gamma (x) = x$ and $x \in F_{\widetilde{f}_\gamma}$. Conversely, if $x \in F_{\widetilde{f}_\gamma}$, from $\widetilde{f}_\gamma (x) = x$, since $x \neq 0$ and $\gamma \neq 0$, it results that $f(x) - x = 0$, so $x \in F_f$.

With $\gamma \in \mathbb{R}^*$ on some suitable interval $I \subset [a,b]$ we will consider, associated to $[[a,b], f]$, the discrete dynamical system

$$\left[I, \overline{f}_\gamma\right]$$

and we refer to it as a variation controlled discrete dynamical system with control parameter γ. In $\left[I, \overline{f}_\gamma\right]$ the trajectory of an element $y_0 \in I$ is generated by

$$y_{n+1} = \overline{f}_\gamma(y_n), n \in \mathbb{N},$$

i. e.

$$y_{n+1} = y_n + \gamma(f(y_n) - y_n), n \in \mathbb{N},$$

or

$$y_{n+1} = (1-\gamma)\, y_n + \gamma f(y_n), n \in \mathbb{N}.$$

For $I = [a,b]$, in the system $\left[[a,b], \overline{f}_\gamma\right]$ with $\gamma \in (0,1)$ given, this is exactly a Krasnoselskij iteration applied to f.

In case that $0 \notin [a,b]$, with $\gamma \in \mathbb{R}^*$ on some suitable interval $I \subset [a,b]$ we will also consider, associated to $[[a,b], f]$, the discrete dynamical system

$$\left[I, \widetilde{f}_\gamma\right].$$

In $\left[I, \widetilde{f}_{\gamma}\right]$ the trajectory of an element $z_0 \in I$ is generated by

$$z_{n+1} = \widetilde{f}(z_n), n \in \mathbb{N},$$

i. e.

$$z_{n+1} = z_n \left(1 + \gamma \left(f(z_n) - z_n\right)\right), n \in \mathbb{N},$$

or

$$z_{n+1} = z_n + \gamma z_n \left(f(z_n) - z_n\right),$$

iteration studied by Huang, W. [6] under some conditions on f' at the fixed point of f. Remark that

$$\frac{z_{n+1} - z_n}{z_n} = \gamma \left(f(z_n) - z_n\right),$$

a ground for referring to the system $\left[I, \widetilde{f}_{\gamma}\right]$ as a growth-rate controlled discrete dynamical system with control parameter γ ([6]).

For recent and comprehensive results on Picard and Krasnoselskij iterations, both presented within more general settings, we refer to Berinde's monograph [3].

Through this paper we focus on discrete dynamical systems $[[a, b], f]$ with $f : [a, b] \to [a, b]$ satisfying a Lipschitz condition, i. e.

$$|f(u_1) - f(u_2)| \leq L |u_1 - u_2|, u_1, u_2 \in [a, b],$$

where $L > 0$ is a constant. Such a function is continuous, so it possesses at least one fixed point and the set of its fixed points is compact.

2. Monotone iterations with Lipschitz functions

Let $f : [a, b] \to [a, b]$ satisfying a Lipschitz condition. As it is mentioned in Section 1, f is continuous, $F_f \neq \varnothing$ and F_f is compact. If $f(c_1) - c_1$ and $f(c_2) - c_2$ are of opposite sign, then, by the intermediate value theorem $F_f \cap [c_1, c_2] \neq \varnothing$. In this case $F_f \cap [c_1, c_2]$, as a compact subset of $\mathbb{R}$, possesses a least element and a greatest element.

When f satisfies the Lipschitz condition with $L < 1$, by the contraction principle F_f consists of a unique fixed point of f and all sequences $(x_n)_{n\in\mathbb{N}}$ generated by the Picard iteration $x_0 \in [a, b]$, $x_{n+1} = f(x_n)$, converge to this fixed point.

When a function $f : [a, b] \to [a, b]$ is monotone, the sequences $(x_n)_{n\in\mathbb{N}}$ generated by the Picard iteration are either monotone or compounded from two monotone subsequences $(x_{2k+1})_{k\in\mathbb{N}}$ and $(x_{2k})_{k\in\mathbb{N}}$; if f is also continuous, then these sequences converge to a fixed point of f. For results on Picard iterations with monotone and continuous functions see [7].

Hillam ([4]) prove for $f : [a, b] \to [a, b]$ satisfying the Lipschitz condition with constant $L > 0$ that for any $x_0 \in [a, b]$ the sequence $(x_n)_{n\in\mathbb{N}}$ defined by the Krasnoselskij iteration $x_{n+1} = (1 - \gamma) x_n + \gamma f(x_n)$ with $\gamma = \dfrac{1}{L+1}$ converges monotonically to a fixed point of f.

Hillam's result is remarkable in giving monotone iterations for functions $f : [a, b] \to [a, b]$ that are not necessarily monotone.

Hillam's paper [4] inspires us for our next theorem and proof, dealing with generalized Krasnoselskij type iterations of f, that are, in fact, Picard iterations of the function $\overline{f}_{\gamma}$.

Theorem 2.1. *Let $a, b \in \mathbb{R}$, $a < b$, $f : [a, b] \to [a, b]$ satisfying the Lipschitz condition with $L > 0$, and let $x_0 \in [a, b]$.*

i) If $f(x_0) > x_0$, letting $\gamma \in \left(0, \dfrac{1}{L+1}\right]$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = (1-\gamma)\, x_n + \gamma f(x_n)$, is s-increasing and convergent to $\min\left(F_f \cap [x_0, b]\right)$.

ii) If $f(x_0) > x_0$ and $F_f \cap [a, x_0] \neq \varnothing$, letting $\gamma \in \left[-\dfrac{1}{L+1}, 0\right)$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = (1-\gamma)\, x_n + \gamma f(x_n)$, is s-decreasing and convergent to $\max\left(F_f \cap [a, x_0]\right)$.

iii) If $f(x_0) < x_0$, letting $\gamma \in \left(0, \dfrac{1}{L+1}\right]$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = (1-\gamma)\, x_n + \gamma f(x_n)$, is s-decreasing and convergent to $\max\left(F_f \cap [a, x_0]\right)$.

iv) If $f(x_0) < x_0$ and $F_f \cap [x_0, b] \neq \varnothing$, letting $\gamma \in \left[-\dfrac{1}{L+1}, 0\right)$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = (1-\gamma)\, x_n + \gamma f(x_n)$, is s-increasing and convergent to $\min\left(F_f \cap [x_0, b]\right)$.

Proof. We discuss the case when $f(x_n) \neq x_n$ for all $n \in \mathbb{N}$.

i) Remark that $F_f \cap [x_0, b] \neq \varnothing$ is assured by $f(x_0) > x_0$ and $f(b) \leq b$. Denote $p = \min\left(F_f \cap [x_0, b]\right)$.

We have $x_0 < p$ and $f(x_0) > x_0$. We show that if $x_0 < x_1 < \cdots < x_k < p$ and $f(x_k) > x_k$, then $x_k < x_{k+1} < p$ and $f(x_{k+1}) > x_{k+1}$:

- Having $x_k < p$ and supposing $x_{k+1} > p$, it follows successively

$$\begin{gathered} |p - x_k| < |x_{k+1} - x_k| = \gamma\, |f(x_k) - x_k| = \gamma\, |f(x_k) - f(p) + p - x_k| \leq \\ \gamma \left(|f(x_k) - f(p)| + |p - x_k|\right) \leq \gamma \left(L\, |x_k - p| + |p - x_k|\right) = \\ \gamma\, (L+1)\, |p - x_k| \leq |p - x_k|, \end{gathered}$$

which is a contradiction. Thus $x_{k+1} < p$.

- The inequality $x_k < x_{k+1}$ follows from $x_{k+1} = x_k + \gamma\left(f(x_k) - x_k\right)$ since $\gamma > 0$ and $f(x_k) - x_k > 0$.

- Now, supposing $f(x_{k+1}) < x_{k+1}$, as $f(x_k) > x_k$, it follows that f has a fixed point in (x_k, x_{k+1}), which contradicts $\min\left(F_f \cap [x_0, b]\right) = p > x_{k+1}$. Thus $f(x_{k+1}) > x_{k+1}$.

By induction it follows that $x_n < x_{n+1} < p$ and $f(x_n) > x_n$ for all $n \in \mathbb{N}$.

The sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to an $x^* \in [x_0, p]$, since it is monotone increasing and bounded from above by p. Since f is continuous and since $\gamma \neq 0$, the recurrence $x_{n+1} = x_n + \gamma\left(f(x_n) - x_n\right)$ implies $x^* = f(x^*)$, so $x^* = p$.

ii) Denote $q = \max\left(F_f \cap [a, x_0]\right)$. From $f(x_0) > x_0$ it follows that $q < x_0$.

We have $q < x_0$ and $f(x_0) > x_0$. We show that if $q < x_k < \cdots < x_1 < x_0$ and $f(x_k) > x_k$, then $q < x_{k+1} < x_k$ and $f(x_{k+1}) > x_{k+1}$:

-Having $q < x_k$ and supposing $x_{k+1} < q$, it follows successively

$$\begin{gathered} |q - x_k| < |x_{k+1} - x_k| = |\gamma|\, |f(x_k) - x_k| = |\gamma| \cdot |f(x_k) - f(q) + q - x_k| \leq \\ |\gamma| \left(|f(x_k) - f(q)| + |q - x_k|\right) \leq |\gamma| \left(L\, |x_k - q| + |q - x_k|\right) = \end{gathered}$$

$$|\gamma|\,(L+1)\,|q-x_k| \le |q-x_k|\,,$$

which is a contradiction. Thus $q < x_{k+1}$.

- The inequality $x_{k+1} < x_k$ follows from $x_{k+1} = x_k + \gamma\,(f\,(x_k) - x_k)$ since $\gamma < 0$ and $f\,(x_k) - x_k > 0$.

- Now, supposing $f(x_{k+1}) < x_{k+1}$, as $f(x_k) > x_k$, it follows that f has a fixed point in (x_{k+1}, x_k), which contradicts $\max\,(F_f \cap [a, x_0]) = q < x_{k+1}$. Thus $f(x_{k+1}) > x_{k+1}$.

By induction it follows that $q < x_{n+1} < x_n$ and $f(x_n) > x_n$ for all $n \in \mathbb{N}$.

The sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to an $x^* \in [q, x_0]$, since it is monotone decreasing and bounded from below by q. Since f is continuous and since $\gamma \neq 0$, the recurrence $x_{n+1} = x_n + \gamma\,(f\,(x_n) - x_n)$ implies $x^* = f(x^*)$, so $x^* = q$.

The proofs of iii) and iv) are similar to that of i) and ii) respectively. □

Our next two theorems - inspired by the growth-rate adjustment mechanism studied under some conditions on f' at the fixed point of f by Huang, W. [6] - deal with generalized Krasnoselskij type iterations for f, that are, in fact, Picard iterations of the function $\widetilde{f}_\gamma$. The proofs we formulate here are inspired by the proof in [4].

Theorem 2.2. *Let $a, b \in \mathbb{R}$, $0 < a < b$, $f : [a, b] \to [a, b]$ satisfying the Lipschitz condition with $L > 0$, and let $x_0 \in [a, b]$.*

i) If $f(x_0) > x_0$, consider $p = \min\,(F_f \cap [x_0, b])$. Letting $\gamma \in \left(0, \dfrac{1}{p\,(L+1)}\right]$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = x_n + \gamma x_n\,(f(x_n) - x_n)$, is s-increasing and convergent to p.

ii) If $f(x_0) > x_0$ and $F_f \cap [a, x_0] \neq \varnothing$, consider $q = \max\,(F_f \cap [a, x_0])$. Letting $\gamma \in \left[-\dfrac{1}{x_0\,(L+1)}, 0\right)$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = x_n + \gamma x_n\,(f(x_n) - x_n)$, is s-decreasing and convergent to q.

iii) If $f(x_0) < x_0$, consider $q = \max\,(F_f \cap [a, x_0])$. Letting $\gamma \in \left(0, \dfrac{1}{x_0\,(L+1)}\right]$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = x_n + \gamma x_n\,(f(x_n) - x_n)$, is s-decreasing and convergent to q.

iv) If $f(x_0) < x_0$ and $F_f \cap [x_0, b] \neq \varnothing$, consider $p = \min\,(F_f \cap [x_0, b])$. Letting $\gamma \in \left[-\dfrac{1}{p\,(L+1)}, 0\right)$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = x_n + \gamma x_n\,(f(x_n) - x_n)$, is s-increasing and convergent to p.

Proof. We discuss the case when $f\,(x_n) \neq x_n$ for all $n \in \mathbb{N}$.

i) Remark that $F_f \cap [x_0, b] \neq \varnothing$ is assured by $f(x_0) > x_0$ and $f(b) \le b$.

We have $x_0 < p$ and $f(x_0) > x_0$. We show that if $x_0 < x_1 < \cdots < x_k < p$ and $f(x_k) > x_k$, then $x_k < x_{k+1} < p$ and $f(x_{k+1}) > x_{k+1}$:

- Having $x_k < p$ and supposing $x_{k+1} > p$, it follows successively

$$|p - x_k| < |x_{k+1} - x_k| = \gamma x_k\,|f\,(x_k) - x_k| = \gamma x_k\,|f\,(x_k) - f\,(p) + p - x_k| \le$$
$$\gamma x_k\,(|f\,(x_k) - f\,(p)| + |p - x_k|) \le \gamma x_k\,(L\,|x_k - p| + |p - x_k|) =$$
$$\gamma x_k\,(L+1)\,|p - x_k| \le \gamma p\,(L+1)\,|p - x_k| \le |p - x_k|\,,$$

which is a contradiction. Thus $x_{k+1} < p$.

- The inequality $x_k < x_{k+1}$ follows from $x_{k+1} = x_k + \gamma x_k \left(f\left(x_k\right) - x_k\right)$ since $\gamma > 0$, $x_k > 0$ and $f\left(x_k\right) - x_k > 0$.

- Now, supposing $f(x_{k+1}) < x_{k+1}$, as $f(x_k) > x_k$, it follows that f has a fixed point in (x_k, x_{k+1}), which contradicts $\min F_f \cap [x_0, b] = p > x_{k+1}$. Thus $f(x_{k+1}) > x_{k+1}$.

By induction it follows that $x_n < x_{n+1} < p$ and $f(x_n) > x_n$ for all $n \in \mathbb{N}$.

The sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to an $x^* \in [x_0, p]$, since it is monotone increasing and bounded from above by p. Since f is continuous and since $x^* \neq 0$, $\gamma \neq 0$, the recurrence $x_{n+1} = x_n + \gamma x_n \left(f(x_n) - x_n\right)$ implies $x^* = f(x^*)$, so $x^* = p$.

ii) From $f\left(x_0\right) > x_0$ it follows that $q < x_0$.

We have $q < x_0$ and $f\left(x_0\right) > x_0$. We show that if $q < x_k < \cdots < x_1 < x_0$ and $f\left(x_k\right) > x_k$, then $q < x_{k+1} < x_k$ and $f\left(x_{k+1}\right) > x_{k+1}$:

-Having $q < x_k$ and supposing $x_{k+1} < q$, it follows successively

$$|q - x_k| < |x_{k+1} - x_k| = |\gamma| \cdot x_k \cdot |f\left(x_k\right) - x_k| = |\gamma| \cdot x_k \cdot |f\left(x_k\right) - f\left(q\right) + q - x_k| \leq$$
$$|\gamma|\, x_k \left(|f\left(x_k\right) - f\left(q\right)| + |q - x_k|\right) \leq |\gamma|\, x_k \left(L\,|x_k - q| + |q - x_k|\right) =$$
$$|\gamma|\, x_k \left(L + 1\right)|q - x_k| \leq |\gamma|\, x_0 \left(L + 1\right)|q - x_k| \leq |q - x_k|,$$

which is a contradiction. Thus $q < x_{k+1}$.

- The inequality $x_{k+1} < x_k$ follows from $x_{k+1} = x_k + \gamma x_k \left(f\left(x_k\right) - x_k\right)$ since $\gamma < 0$, $x_k > 0$ and $f\left(x_k\right) - x_k > 0$.

- Now, supposing $f(x_{k+1}) < x_{k+1}$, as $f(x_k) > x_k$, it follows that f has a fixed point in (x_{k+1}, x_k), which contradicts $\max\left(F_f \cap [a, x_0]\right) = q < x_{k+1}$. Thus $f\left(x_{k+1}\right) > x_{k+1}$.

By induction it follows that $q < x_{n+1} < x_n$ and $f(x_n) > x_n$ for all $n \in \mathbb{N}$.

The sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to an $x^* \in [q, x_0]$, since it is monotone decreasing and bounded from below by q. Since f is continuous and since $x^* \neq 0$, $\gamma \neq 0$, the recurrence $x_{n+1} = x_n + \gamma x_n \left(f(x_n) - x_n\right)$ implies $x^* = f(x^*)$, so $x^* = q$.

The proofs of iii) and iv) are similar to that of i) and ii) respectively. □

Theorem 2.3. *Let $a, b \in \mathbb{R}$, $a < b < 0$, $f : [a, b] \to [a, b]$ satisfying the Lipschitz condition with $L > 0$, and let $x_0 \in [a, b]$.*

i) If $f(x_0) > x_0$, consider $p = \min\left(F_f \cap [x_0, b]\right)$. Letting $\gamma \in \left[\dfrac{1}{x_0\left(L+1\right)}, 0\right)$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = x_n + \gamma x_n \left(f(x_n) - x_n\right)$, is s-increasing and convergent to p.

ii) If $f(x_0) > x_0$ and $F_f \cap [a, x_0] \neq \varnothing$, consider $q = \max\left(F_f \cap [a, x_0]\right)$. Letting $\gamma \in \left(0, \dfrac{1}{-q\left(L+1\right)}\right]$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = x_n + \gamma x_n \left(f(x_n) - x_n\right)$, is s-decreasing and convergent to q.

iii) If $f(x_0) < x_0$, consider $q = \max\left(F_f \cap [a, x_0]\right)$. Letting $\gamma \in \left[\dfrac{1}{q\left(L+1\right)}, 0\right)$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = x_n + \gamma x_n \left(f(x_n) - x_n\right)$, is s-decreasing and convergent to q.

iv) If $f(x_0) < x_0$ and $F_f \cap [x_0, b] \neq \varnothing$, consider $p = \min\left(F_f \cap [x_0, b]\right)$. Letting $\gamma \in \left(0, \dfrac{1}{-x_0\left(L+1\right)}\right]$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = x_n + \gamma x_n \left(f(x_n) - x_n\right)$, is s-increasing and convergent to p.

Proof. The proof is similar to that of the previous theorem. □

Remark 2.1. In Theorem 2.2, independently on x_0 and p, the conditions on γ from i) and iii) are satisfied for all $\gamma \in \left(0, \frac{1}{b(L+1)}\right]$, those from ii) and iv) are satisfied for all $\gamma \in \left[-\frac{1}{b(L+1)}, 0\right)$. In Theorem 2.3, independently on x_0 and q, the conditions on γ from i) and iii) are satisfied for all $\gamma \in \left[\frac{1}{a(L+1)}, 0\right)$, those from ii) and iv) are satisfied for all $\gamma \in \left(0, \frac{1}{-a(L+1)}\right]$.

The theorems developed here have concrete usability in searching for fixed points of Lipschitz functions, as well as in the analysis of discrete dynamical systems $[[a,b], f]$ with f satisfying a Lipschitz condition.

3. Numerical experiment

Consider the discrete dynamical system $[[-2,2], f]$, $f(x) = \left|2x^2 - 4\right| - 2$. This function f, $f : [-2,2] \to [-2,2]$, satisfies the Lipschitz condition with $L = 8$, and has the fixed points set $F_f = \left\{-\frac{3}{2}, \frac{-1-\sqrt{17}}{4}, \frac{-1+\sqrt{17}}{4}, 2\right\}$. Remark that f is not differentiable at $x = \pm\sqrt{2}$. Figure 1 depicts the graph of f. Figure 3 depicts the graph of f^3.

The trajectory of $x_0 = -1.45$ in the discrete dynamical system $[[-2,2], f]$ is - only the first two decimal places being listed trough this paper
$\{-1.45, -1.80, 0.44, 1.61, -0.84, 0.58, 1.34, -1.57, -1.04, -0.18, 1.93, 1.47,$
$-1.66, -0.46, 1.58, -1.00, 0.02, 2.00, 2.00, 1.97, 1.74, 0.04, 2.00, 1.97, 1.78, 0.31,$
$1.80, 0.51, 1.47, -1.67, -0.40, 1.68, ...\}$
The trajectory of $x_0 = 0.25$ in $[[-2,2], f]$ is
$\{0.25, 1.88, 1.03, -0.13, 1.97, 1.74, 0.08, 1.99, 1.89, 1.12, -0.49, 1.52, -1.35,$
$-1.65, -0.54, 1.41, -2.00, 1.98, 1.87, 1.01, -0.05, 1.99, 1.96, 1.65, -0.56, 1.37,$
$-1.73, -0.04, 2.00, 1.98, 1.84, 0.75, 0.87, 0.48, 1.53, -1.30, ...\}$

By Theorem 2.1 iv) the sequence $x_0 = -1.45$, $x_{n+1} = \overline{f}_\gamma(x_n) = (1-\gamma)x_n + \gamma f(x_n)$ with $\gamma = -0.1$ is s-increasing and convergent to $\frac{-1-\sqrt{17}}{4}$; the same is true for any $x_0 \in \left(-1.45, \frac{-1-\sqrt{17}}{4}\right)$, so in the discrete dynamical system $\left[\left[-1.45, \frac{-1-\sqrt{17}}{4}\right], \overline{f}_{-0.1}\right]$ the fixed point $\frac{-1-\sqrt{17}}{4}$ is monotonously attracting from below.
The trajectory of $x_0 = -1.45$ in this dynamical system is

$$\{-1.45, -1.42, -1.36, -1.32, -1.31, -1.30, -1.29, -1.29, -1.28,$$

$$-1.28, -1.28, -1.28, -1.28, -1.28, -1.28, ...\}.$$

By Theorem 2.1 ii) the sequence $x_0 = 0.25$, $x_{n+1} = \overline{f}_\gamma(x_n)$ with $\gamma = -0.1$ is s-decreasing and convergent to $\frac{-1-\sqrt{17}}{4}$; the same is true for any $x_0 \in \left(\frac{-1-\sqrt{17}}{4}, 0.25\right)$, so in the discrete dynamical system $\left[\left[\frac{-1-\sqrt{17}}{4}, 0.25\right], \overline{f}_{-0.1}\right]$ the fixed point $\frac{-1-\sqrt{17}}{4}$ is monotonously attracting from above.
The trajectory of $x_0 = 0.25$ in this dynamical system is

$$\{0.25, 0.09, -0.10, -0.31, -0.52, -0.72, -0.89, -1.02, -1.11, -1.18, -1.22,$$
$$-1.24, -1.26, -1.27, -1.27, -1.28, -1.28, -1.28, -1.28, -1.28, -1.28, ...\}.$$

It follows that in the discrete dynamical system $\left[[-1.45, 0.25], \overline{f}_{-0.1}\right]$ the fixed point $\frac{-1-\sqrt{17}}{4}$ is monotonously stable.

By Theorem 2.1 i) the sequence $x_0 = 0.25, x_{n+1} = \overline{f}_\gamma(x_n)$ with $\gamma = 0.1$ is s-increasing and convergent to $\frac{-1+\sqrt{17}}{4}$; the same is true for any $x_0 \in \left(0.25, \frac{-1+\sqrt{17}}{4}\right)$, so in the discrete dynamical system $\left[\left[0.25, \frac{-1+\sqrt{17}}{4}\right], \overline{f}_{0.1}\right]$ the fixed point $\frac{-1+\sqrt{17}}{4}$ is monotonously attracting from below.
The trajectory of $x_0 = 0.25$ in this dynamical system is

$$\{0.25, 0.41, 0.54, 0.63, 0.68, 0.72, 0.75, 0.76, 0.77, 0.77, 0.78,$$
$$0.78, 0.78, 0.78, 0.78, 0.78, 0.78, ...\}.$$

By Theorem 2.3 iii) the sequence $x_0 = -1.45, x_{n+1} = \widetilde{f}_\gamma(x_n) = x_n + \gamma x_n(f(x_n) - x_n)$ with $\gamma = -0.05$ is s-decreasing and convergent to $-\frac{3}{2}$; the same is true for any $x_0 \in \left(-\frac{3}{2}, -1.45\right)$, so in the discrete dynamical system $\left[\left[-\frac{3}{2}, -1.45\right], \widetilde{f}_{-0.05}\right]$ the fixed point $-\frac{3}{2}$ is monotonously attracting from above.
The trajectory of $x_0 = -1.45$ in this dynamical system is

$$\{-1.45, -1.48, -1.49, -1.49, -1.50, -1.50, -1.50, -1.50, -1.50, -1.50, ...\}.$$

By Theorem 2.3 iv) the sequence $x_0 = -1.45$, $x_{n+1} = \widetilde{f}_\gamma(x_n)$ with $\gamma = 0.05$ is s-increasing and convergent to $\frac{-1-\sqrt{17}}{4}$; the same is true for any $x_0 \in \left(-1.45, \frac{-1-\sqrt{17}}{4}\right)$, so in the discrete dynamical system

$\left[\left[-1.45, \dfrac{-1-\sqrt{17}}{4}\right], \widetilde{f}_{0.05}\right]$ the fixed point $\dfrac{-1-\sqrt{17}}{4}$ is monotonously attracting from below.
The trajectory of $x_0 = -1.45$ in this dynamical system is

$$\{-1.45, -1.42, -1.39, -1.36, -1.33, -1.32, -1.31, -1.30, -1.30, -1.29,$$
$$-1.29, -1.29, -1.29, -1.28, -1.28, -1.28, -1.28, -1.28, -1.28, -1.28, ...\}.$$

Figures 3 and 4 show the graphs of $\overline{f}_\gamma$, $\overline{f}^5_\gamma$, for $\gamma = -0.1$ and for $\gamma = 0.1$, respectively. Figures 5 and 6 show the graphs of $\widetilde{f}_\gamma$, $\widetilde{f}^5_\gamma$, for $\gamma = -0.05$ and for $\gamma = 0.05$, respectively.

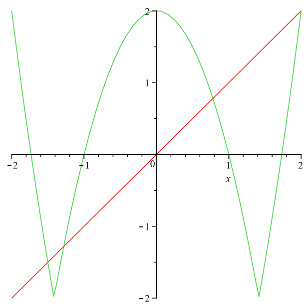

Figure 1. The graph of f

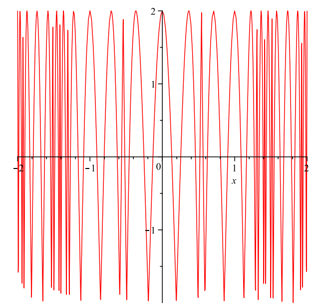

Figure 2. The graph of f^3

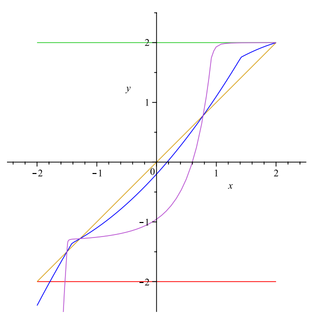

Figure 3. The graphs of $\overline{f}_\gamma$ and $\overline{f}^5_\gamma$, $\gamma = -0.1$

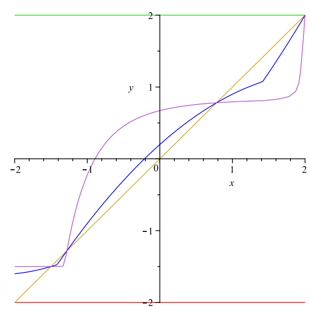

Figure 4. The graphs of $\overline{f}_\gamma$ and $\overline{f}^5_\gamma$, $\gamma = 0.1$

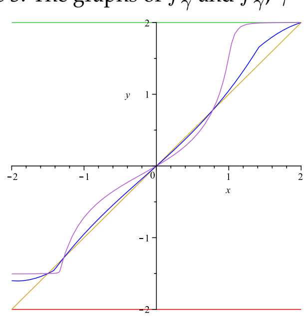

Figure 5. The graphs of $\widetilde{f}_\gamma$ and $\widetilde{f}^5_\gamma$, $\gamma = -0.05$

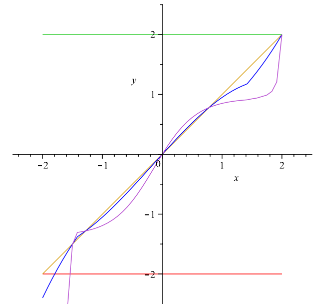

Figure 6. The graphs of $\widetilde{f}_\gamma$ and $\widetilde{f}^5_\gamma$, $\gamma = 0.05$

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