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On the iterative solution of some decomposable nonlinear operator equations

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ABSTRACT. The aim of this note is to extend a recent result concerning the solvability of a nonlinear equation that can be decomposed under the form $Au = Pu$, where one of the two operators A and P has richer properties than the other one.

1. INTRODUCTION

Let (X, d) be a metric space and let $A,\ P : X \to X$ be two nonlinear operators. In order to solve the operator equation

$$Au = Pu \tag{1.1}$$

S. Mukherjee and A. Biswas [16] used a seemingly implicit iterative process given by

$$Au_{n+1} = Pu_n,\ n = 0, 1, \dots \tag{1.2}$$

where $u_0 \in X$ is the initial approximation.

Their main result, obtained basically by a fixed point argument, can be restated as follows.

Theorem 1.1. *Let (X, d) be a complete metric space and let $A,\ P : X \to X$ be two operators, A is surjective, and such that the following conditions are fulfilled*
(i) $d(Au, Av) \geq \alpha d(u, v),\ u, v \in X,\ \alpha > 0;$
(ii) $d(Pu, Pv) \leq \beta \left[d(A^{-1}Pu, u) + d(A^{-1}Pv, v)\right],\ u, v \in X,\ \beta > 0;$
(iii) $2\beta < \alpha.$

Then the sequence of iterations $\{u_n\}$ starting from $u_0 \in X$ is uniquely defined by (1.2) *and converges to the unique solution of* (1.1).

Equations of the form (1.1) do appear in several applicative contexts, see [25], [12], like, for example, in the case of finding the solution of semi-linear elliptic boundary value problems. Papers dealing with applications of fixed point techniques or with iterative methods for approximating fixed points and their applications to solving operator equations abounds in literature, see for example the following recent journal papers: [4]-[14], [16]-[20], which are closely related to the topic of the present paper, as well as the very recent monograph [2].

It is easy to notice that, by (i), it results that A is one to one and so, by the assumed surjectivity, it is bijective, hence invertible. Denoting by A^{-1} its inverse, the equation (1.1) is equivalent to the next fixed point problem

$$u = A^{-1}Pu \tag{1.3}$$

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and so, in order to solve the last one, we can use the Picard iteration associated to (1.3), that is, the explicit fixed point iteration

$$u_{n+1} = A^{-1}Pu_n,\ n = 0, 1, \dots \tag{1.4}$$

which, under the hypotheses of Theorem 1.1, coincides with (1.2).

In order to prove the convergence of the fixed point iterative method defined by (1.2), the authors of [16] used the well known Kannan's fixed point theorem [15].

It is the main aim of this paper to show that assumptions in Theorem 1.1 are too restrictive, on the one hand, and further to prove some more general results of the same kind that extend Theorem 1.1 to larger classes of nonlinear discontinuous operators, on the other hand. Moreover, all these general theorems also provide the rate of convergence for the method (1.2).

2. An example

Let $X = \mathbb{R}$ be the real axis with the usual distance and $A, P : \mathbb{R} \to \mathbb{R}$ be given by the following relations: $Ax = \frac{1}{x}$, for $x \neq 0$ and $A0 = 0$; and $Px = 0$, if $x \leq 2$ and $Px = -2$, if $x > 2$, respectively.

Then A is bijective, $A^{-1} = A$ and A^{-1} is not Lipschitzian, that is, A does not satisfy (i) in Theorem 1.1.

Moreover, A^{-1} is not continuous but the discontinuos function $T = A^{-1}P$, which is given by $Tx = 0$, if $x \leq 2$ and $Tx = -\frac{1}{2}$, if $x > 2$, fulfills the Kannan's contractive condition [15]:

$$d(Tx, Ty) \leq a\left[d(x, Tx) + d(y, Ty)\right],\ x, y \in \mathbb{R},$$

with the contraction condition $a = d\frac{1}{5}$, see [2]-[5] for details and other recent developments.

Condition (ii) in Theorem 1.1 is also satisfied, with $\beta > \frac{4}{5}$ but condition (iii) cannot be checked in this case, because α simply does not exist.

However, the equation $Au = Pu$ has 0 as its unique solution and the iterative method (1.2) converges to that solution, for any initial approximation $u_0 \in \mathbb{R}$.

Indeed, if we take $u_0 \leq 2$, then $u_1 = u_2 = \cdots = 0$, while if $u_0 > 2$, then $u_1 = -\frac{1}{2}$ and then $u_2 = u_3 = \cdots = 0$. In both cases, $\{u_n\}$ converges to the solution.

So, the previous example shows that assumption (i) in Theorem 1.1 is too restrictive, while (iii) is prohibitive. It is then the main aim of the next section to extend Theorem 1.1 by considering several more general contractive type conditions. As expected, each of the next results, contained in Theorems 3.2-4.6 as well as in Corollary 4.1 can be applied to the operator equation in the example above.

3. Main results

First of all we state a direct extension of Theorem 1.1 by removing assumptions (i) and (iii).

Theorem 3.2. *Let (X, d) be a complete metric space and let A, $P : X \to X$ be two operators, such that A is bijective and there exists $b \in [0, \frac{1}{2})$ for which*

$$d(Tu, Tv) \leq b\left[d(Tu, u) + d(Tv, v)\right],\ \forall u, v \in X, \tag{3.5}$$

where we denoted $T = A^{-1}P$.

Then the sequence of iterations $\{u_n\}$ *starting from* $u_0 \in X$ *is uniquely defined by* (1.2) *and converges to the unique solution* u^* *of* (1.1) *with the following error estimate*

$$d(u_{n+i-1}, u^*) \le \frac{\alpha^i}{1-\alpha}\, d(u_n, u_{n-1}), \quad n = 0, 1, 2, \dots ; \; i = 1, 2, \dots \tag{3.6}$$

where $\alpha = b/(1-b)$.

Moreover, the rate of convergence of the Picard iteration is linear, that is,

$$d(u_n, u^*) \le \alpha \cdot d(u_{n-1}, u^*), \quad n = 1, 2, \dots \tag{3.7}$$

Proof. The Kannan's contractive condition (3.5) ensures, see [2], that T has a unique fixed point, say u^*, and that the Picard iteration associated to T, that is, the iteration (1.2), converges to u^*.

Indeed, let $\{u_n\}_{n=0}^{\infty}$ be the Picard iteration defined by (1.2) and starting from $u_0 \in X$, arbitrary. Then by (3.5) we have

$$d(u_n, u_{n+1}) = d(Tu_{n-1}, Tu_n) \le b\big[d(u_{n-1}, u_n) + d(u_n, u_{n+1})\big]$$

which implies

$$d(u_n, u_{n+1}) \le \frac{b}{1-b}\, d(u_{n-1}, u_n), \quad \text{for all } n = 1, 2, \dots \tag{3.8}$$

Denote $\alpha = \dfrac{b}{1-b}$. Since $0 \le b < \dfrac{1}{2}$, it results $0 \le \alpha < 1$.
Using now (3.8) we obtain by induction

$$d(u_{n+k}, u_{n+k-1}) \le \alpha^k d(u_n, u_{n-1}), \quad k \in \mathbb{N}^* \tag{3.9}$$

which, for $p > i$, yields

$$d(u_{n+p}, u_{n+i-1}) \le \frac{\alpha^i(1-\alpha^{p-i+1})}{1-\alpha}\, d(u_n, u_{n-1}), \; n, p, i \in \mathbb{N}^*. \tag{3.10}$$

Now by letting $p \to \infty$ in (3.10) we obtain the desired estimate (3.6).

Again, by (3.5) we have

$$\begin{aligned} d(Tu, Tv) &\le b\big[d(u, Tu) + d(v, Tv)\big] \\ &\le b\big\{d(u, Tu) + \big[d(v, u) + d(u, Tu) + d(Tu, Tv)\big]\big\} \end{aligned}$$

which implies

$$d(Tu, Tv) \le \frac{b}{1-b} \cdot d(u, v) + \frac{2b}{1-b}\, d(u, Tu). \tag{3.11}$$

Take $u := x^*$, $y := u_{n-1}$ in (3.11) to obtain the estimate (3.7), that is,

$$d(u_n, u^*) \le \frac{b}{1-b}\, d(u_{n-1}, u^*).$$

□

Note that the error estimate (3.6), which has been suggested by a result in [19], includes both the classical *a priori* and *a posteriori* error estimates

$$d(u_n, u^*) \le \frac{\alpha^n}{1-\alpha}\, d(u_0, u_1), \quad n = 0, 1, 2, \dots$$

$$d(u_n, u^*) \le \frac{\alpha}{1-\alpha}\, d(u_{n-1}, u_n), \quad n = 1, 2, \dots$$

as particular cases. They are obtained by putting $n = 1$ and then formally taking $i := n$ in (3.6), respectively by taking $i = 1$ in (3.6).

A similar result to that in Theorem 3.2 can be obtained by considering instead of Kannan's contraction condition (3.5), a similar but independent condition, i.e., the Chatterjea's contractive condition [9].

Theorem 3.3. *Let (X, d) be a complete metric space and let A, $P : X \to X$ be two operators, such that A is bijective and there exists $c \in [0, \frac{1}{2})$ for which*

$$d(Tu, Tv) \le c\,[d(Tu, v) + d(Tv, u)]\,,\ \forall u, v \in X, \tag{3.12}$$

where we denoted $T = A^{-1}P$.

Then the sequence of iterations $\{u_n\}$ starting from $u_0 \in X$ is uniquely defined by (1.2) and converges to the unique solution u^* of (1.1) with the following error estimates

$$d(u_{n+i-1}, u^*) \le \frac{\alpha^i}{1-\alpha}\, d(u_n, u_{n-1})\,, \quad n = 0, 1, 2, \dots;\ i = 1, 2, \dots \tag{3.13}$$

where $\beta = c/(1 - c)$.

Moreover, the convergence of the Picard iteration is linear, that is,

$$d(u_n, u^*) \le \beta \cdot d(u_{n-1}, u^*)\,, \quad n = 1, 2, \dots \tag{3.14}$$

Proof. Similarly to the proof of the previous theorem, by taking $u := u_{n-1}$ and $v := u_n$ in (3.12) we have

$$d(u_n, u_{n+1}) = d(Tu_{n-1}, Tu_n) \le c\big[d(u_{n-1}, Tu_n) + d(u_n, Tu_{n-1})\big]$$

which gives

$$d(u_n, u_{n+1}) \le cd(u_{n-1}, Tu_n) \le c\big[d(u_{n-1}, u_n) + d(u_n, u_{n+1})\big]$$

and so we get

$$d(u_n, u_{n+1}) \le \frac{c}{1-c}\, d(u_{n-1}, u_n)\,, \quad \text{for all } n = 1, 2, \dots$$

The rest of the proof is similar to that of the previous theorem. □

Now we can establish a very general convergence theorem which unify both Theorem 3.2 and Theorem 3.3, on the one hand, and also includes the well known contraction mapping principle, on the other hand.

Theorem 3.4. *Let (X, d) be a complete metric space and let A, $P : X \to X$ be two operators, with A is bijective, for which there exist $a \in [0, 1)$, $b, c \in [0, \frac{1}{2})$ such that for all $u, v \in X$, at least one of the following conditions is true, where we denoted $T = A^{-1}P$:*
(i) $d(Tu, Tv) \le ad(u, v)$;
(ii) $d(Tu, Tv) \le b\,[d(Tu, u) + d(Tv, v)]\,;$
(iii) $d(Tu, Tv) \le c\,[d(Tu, v) + d(Tv, u)]\,.$
Then the sequence of iterations $\{u_n\}$ starting from $u_0 \in X$ is uniquely defined by (1.2) and converges to the unique solution u^* of (1.1) with the following error estimates

$$d(u_{n+i-1}, u^*) \le \frac{\delta^i}{1-\delta}\, d(u_n, u_{n-1})\,, \quad n = 0, 1, 2, \dots;\ i = 1, 2, \dots$$

where $\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$.

Moreover, the convergence of the Picard iteration is linear, that is,

$$d(u_n, u^*) \leq \delta \cdot d(u_{n-1}, u^*)\,, \quad n = 1, 2, \ldots \tag{3.15}$$

Proof. We use basically the arguments used to prove the previous two theorems. □

If we are working in a normed linear space, then the equation (1.1) may be replaced by a non-homogeneous equation of the form

$$Au = Pu + v,$$

where $v \in X$ is given, in such a way that all results established in the present section for (1.1) could be correspondingly adapted to the new setting.

Remark 3.1. If condition (i) in Theorem 3.4 holds, for all $u, v \in X$, then we get the classical mapping contraction principle, which can be applied directly to the operator equations in [8], [11], [13] and [18].

If condition (ii) in Theorem 3.4 holds, for all $u, v \in X$, then we get the Kannan's fixed point theorem [15], while if condition (iii) in Theorem 3.4 holds, for all $u, v \in X$, then we get the Chatterjea's fixed point theorem [9], both of these classical results being completed in Theorem 3.4 with the a priori and a posteriori error estimates, taken from the recent paper [5], but here in a new condensed form inspired by a result in [19].

4. Convergence results in the class of almost contractions

All convergence theorems given in the previous section could be further unified in a single and general result, adapted from [4], see also [3]. The proofs are similar to the ones given in the papers [2] and [4].

Definition 4.1. [4] Let (X, d) be a metric space. A map $T : X \to X$ is called *almost contraction* or (δ, L)*-almost contraction* if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx)\,, \quad \text{for all } \ x, y \in X\,. \tag{4.16}$$

Remark 4.1. Due to the symmetry of the distance, the almost contraction condition (4.16) implicitly includes the following dual one

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + L \cdot d(x, Ty)\,, \quad \text{for all } \ x, y \in X\,, \tag{4.17}$$

obtained from (4.16) by formally replacing $d(Tx, Ty)$ and $d(x, y)$ by $d(Ty, Tx)$ and $d(y, x)$, respectively, and then interchanging x and y.

Consequently, in order to check the almost contractiveness of T, it is necessary to check both (4.16) and (4.17).

It is quite easy to show that any mapping satisfying the assumptions in Theorem 3.4 is an almost contraction, see [5] and [2]. For other examples of single-valued almost contractions, see [4] and [1].

We state now the first unifying result of all convergence theorems in the previous section.

Theorem 4.5. *Let (X, d) be a complete metric space and let $A, P : X \to X$ be two operators, such that A is bijective and $T = A^{-1}P$ is a (δ, L)-almost contraction.*

Then the sequence of iterations $\{u_n\}$ starting from $u_0 \in X$ is uniquely defined by (1.2) *and converges to a solution u^* of* (1.1) *with the following error estimate*

$$d(u_{n+i-1}, u^*) \leq \frac{\delta^i}{1-\delta}\, d(u_n, u_{n-1}), \quad n = 0, 1, 2, \ldots; \, i = 1, 2, \ldots$$

Note that, although all the convergence theorems in Section 3 actually forces the uniqueness of the solution and all these results are included in Theorem 4.5, however, under the assumptions of Theorem 4.5 we do not have generally a unique solution, as shown by Example 1 in [4], in terms of fixed points. But, as we have shown there, it is possible to force the uniqueness of the solution, by imposing an additional contractive condition, quite similar to (4.16), as shown by the next theorem.

Theorem 4.6. *Let (X, d) be a complete metric space and let $A, P : X \to X$ be two operators, such that A is bijective and $T = A^{-1}P$ is a (δ, L)-almost contraction for which there exist $\theta \in (0, 1)$ and some $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \theta \cdot d(x, y) + L_1 \cdot d(x, Tx), \quad \text{for all } \; x, y \in X. \tag{4.18}$$

Then the sequence of iterations $\{u_n\}$ starting from $u_0 \in X$ is uniquely defined by (1.2) *and converges to the unique solution u^* of* (1.1) *with the error estimate*

$$d(u_{n+i-1}, u^*) \leq \frac{\delta^i}{1-\delta}\, d(u_n, u_{n-1}), \quad n = 0, 1, 2, \ldots; \, i = 1, 2, \ldots$$

Moreover, the rate of convergence of the Picard iteration is linear, that is,

$$d(u_n, u^*) \leq \delta \cdot d(u_{n-1}, u^*), \quad n = 1, 2, \ldots$$

Another alternative contractive condition that implies uniqueness of fixed points of almost contractions has been given in the very recent paper of Babu et al. [1]. By using it in the actual context we get the next result.

Corollary 4.1. *Let (X, d) be a complete metric space and let $A, P : X \to X$ be two operators, such that A is bijective and $T = A^{-1}P$ is a (δ, L)-strict almost contraction, that is, there exist $\theta \in (0, 1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \theta \cdot d(x, y) + L \cdot \max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \tag{4.19}$$

for all $x, y \in X$. Then the sequence of iterations $\{u_n\}$ starting from $u_0 \in X$ is uniquely defined by (1.2) *and converges to the unique solution u^* of* (1.1) *with the error estimate*

$$d(u_{n+i-1}, u^*) \leq \frac{\theta^i}{1-\theta}\, d(u_n, u_{n-1}), \quad n = 0, 1, 2, \ldots; \, i = 1, 2, \ldots$$

Moreover, the rate of convergence of the Picard iteration is linear, that is,

$$d(u_n, u^*) \leq \theta \cdot d(u_{n-1}, u^*), \quad n = 1, 2, \ldots$$

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REFERENCES

[1] Babu, G. V. R., Sandhya, M. L. and Kameswari, M. V. R., *A note on a fixed point theorem of Berinde on weak contractions*, Carpathian J. Math. **24** (2008), No. 1, 8-12

[2] Berinde, V., *Iterative Approximation of Fixed Points*, 2nd Ed., Springer Verlag, Berlin Heidelberg New York, 2007

[3] Berinde, V., *Error estimates for approximating fixed points of discontinuous quasi-contractions*, General Mathematics **13** (2005), No. 2, 23-34

[4] Berinde, V., *On the approximation of fixed points of weak contractive mappings*, Carpathian J. Math. **19** (2003), No. 1, 7-22

[5] Berinde, V., *Approximating fixed points of weak contractions using Picard iteration*, Nonlinear Analysis Forum **9** (2004), No. 1, 43-53

[6] Budişan, S., *Positive solutions of functional differential equations*, Carpathian J. Math. **22** (2006), no. 1-2, 13-19

[7] Carabineanu, A., *The free-boundary flow past an obstacle. Qualitative and numerical results*, Carpathian J. Math. **23** (2007), no. 1-2, 53-62

[8] Cădariu, L. and Radu, V., *The fixed points method for the stability of some functional equations*, Carpathian J. Math. **23** (2007), no. 1-2, 63-72

[9] Chatterjea, S. K., *Fixed-point theorems*, C.R. Acad. Bulgare Sci. **25** (1972) 727-730

[10] Chiş, Adela, *Fixed point theorems for multivalued generalized contractions on complete gauge spaces*, Carpathian J. Math. **22** (2006), no. 1-2, 33-38

[11] Dobriţoiu, Maria, *Properties of the solution of an integral equation with modified argument*, Carpathian J. Math. **23** (2007), no. 1-2, 77-80

[12] Dugundji, J. and Granas, A., *Fixed Point Theory*, Monografie Matematycne, Warsazawa, 1982

[13] Gabor, R. V., *Successive approximations for the solution of second order advanced differential equations*, Carpathian J. Math. *22* (2006), no. 1-2, 57-64

[14] Guran, Liliana *Fixed points for multivalued operators with respect to a w-distance on metric spaces*, Carpathian J. Math. **23** (2007), no. 1-2, 89-92

[15] Kannan, R., *Some results on fixed points*, Bull. Calcutta Math. Soc. **10** (1968) 71-76

[16] Mukherjee, S. and Biswas, A., *On the iterative of solving nonlinear operator equations in a complete metric space*, Ranchi Univ. Math. J. **31** (2000), 99-103 (1968) 71-76

[17] Mureşan, A. S., *From Maia fixed point theorem to the fixed point theory in a set with two metrics*, Carpathian J. Math. **23** (2007), no. 1-2, 133-140

[18] Mureşan, Viorica, *A boundary value problem for some functional-differential equations, via Picard operators*, Carpathian J. Math. **23** (2007), no. 1-2, 141-148

[19] Neumaier, A., *A better estimate for fixed points of contractions*, Z. Angew. Math. Mech., **62** (1982), 627

[20] Păcurar, Mădălina and Păcurar, R., *Approximate fixed point theorems for weak contractions on metric spaces*, Carpathian J. Math. **23** (2007), no. 1-2, 149-155

[21] Rhoades, B. E., *A biased discussion of fixed point theory*, Carpathian J. Math. **23** (2007), No. 1-2, 11-26

[22] Rus, I. A., *Principles and Applications of the Fixed Point Theory* (in Romanian), Editura Dacia, Cluj-Napoca, 1979

[23] Rus, I. A., *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001

[24] Zamfirescu, T., *Fix point theorems in metric spaces* Arch. Math. (Basel), 23 (1972), 292-298

[25] Zeidler, E., *Nonlinear Functional Analysis and its Applications. Fixed-point theorems*, Springer-Verlag, New York, 1986

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