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A new generalization of Euler's constant

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ABSTRACT. A new generalization of Euler's constant, $c = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right)$, this time as a function $\gamma(a,b)$ of two positive real variables, $a\in(0,\infty)$, $b\in\left[0,\frac{1}{2a}\right]$, is given such that, in particular, we have $c=\gamma(1,0)$.

1. INTRODUCTION

Let $\{H_n\}$ be the sequence of partial sums of the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$, that is,

$$H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n},\ n=1,2,\ldots$$

It is well known that unless $\{H_n\}$ diverges to $+\infty$, the sequence $\{\gamma_n\}$ given by $\gamma_n = H_n - \ln n,\ n=1,2,\ldots$ is convergent. Its limit

$$c = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right) \tag{1.1}$$

is called the Euler's constant or Euler-Mascheroni constant and has the approximate value $c = 0.577215\ldots$.

A vast literature has been devoted to this topic, see [1]-[4] and [6]-[17], for a selective list of papers published in the last twenty years.

Very recently, A. Sîntămărian, in a series of papers [8]-[11] and then in the monograph [12] studied a very interesting generalization of Euler's constant, which is actually a function $\gamma:(0,\infty)\to\mathbb{R}$, defined by

$$\gamma(a) = \lim_{n\to\infty}\left(\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}\right). \tag{1.2}$$

Clearly, the Euler's constant is obtained for $a=1$, that is, $c=\gamma(1)$.

The first main result in [12] (Theorem 1.2.1) is stated below in a simplified form.

Theorem 1.1. *Let $a\in(0,+\infty)$. Consider the sequences $\{x_n\}$, $\{y_n\}$ defined for each $n\geq 1$ by*

$$x_n = \frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n}{a}, \tag{1.3}$$

and

$$y_n = \frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}, \tag{1.4}$$

respectively. Then

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(i) the sequences $\{x_n\}$, $\{y_n\}$ *are convergent to the same limit, denoted by* $\gamma(a)$, *and satisfy*

$$x_n < x_{n+1} < \gamma(a) < y_{n+1} < y_n,\ \forall n = 1, 2, \dots \tag{1.5}$$

(ii) $0 < \dfrac{1}{a} - \ln(1 + \dfrac{1}{a}) < \gamma(a) < \dfrac{1}{a}$;

(iii) For each $n = 1, 2, \dots$ *and* $a \in (0, +\infty)$, *we have the estimates*

$$\frac{1}{2(a+n)} < \gamma(a) - x_n < \frac{1}{2(a+n-1)}, \tag{1.6}$$

$$\frac{1}{2(a+n)} < y_n - \gamma(a) < \frac{1}{2(a+n-1)}. \tag{1.7}$$

Note that by (1.6) and (1.7) we deduce a result which expresses the *linear* order of convergence of the sequences $\{x_n\}$ and $\{y_n\}$ to $\gamma(a)$:

$$\lim_{n\to\infty} n\left(\gamma(a) - x_n\right) = \frac{1}{2} \text{ and } \lim_{n\to\infty} n\left(y_n - \gamma(a)\right) = \frac{1}{2}. \tag{1.8}$$

The main aim of this note is to further generalize the Euler's constant, by defining it as the limit of a sequence of the form (1.3)-(1.4), which will depend this time of two parameters a and b. The single variable case $\gamma(a)$, contained in Theorem 1.1, will be thus recovered for $b = 0$ in Theorem 2.2.

2. Main result

Our main result in this paper, given by the next theorem, corresponds to $(i)-(ii)$ in Theorem 1.1.

Theorem 2.2. *Let* $a \in (0, +\infty)$ *and* $b \in \left[0, \dfrac{1}{2a}\right]$. *Consider the sequences* $\{x_n\}$, $\{y_n\}$ *defined for each* $n \geq 1$ *by*

$$x_n = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln\left(\frac{a+n}{a} + b\right), \tag{2.1}$$

and

$$y_n = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln\left(\frac{a+n-1}{a} + b\right), \tag{2.2}$$

respectively. Then

(i) the sequences $\{x_n\}$, $\{y_n\}$ *are convergent to the same limit, denoted by* $\gamma(a, b)$, *and satisfy*

$$x_n < x_{n+1} < \gamma(a, b) < y_{n+1} < y_n,\ \forall n = 1, 2, \dots \tag{2.3}$$

(ii) $0 < \dfrac{1}{a} - \ln(1 + \dfrac{1}{a} + b) < \gamma(a, b) < \dfrac{1}{a} - \ln b.$

Proof. (i) We have

$$x_{n+1} - x_n = \frac{1}{a+n} - \ln\left(1 + \frac{1}{a+n+ab}\right),\ \forall n = 1, 2, \dots \tag{2.4}$$

and by using the inequality

$$\ln(1+t) < t,\ t \in (0, \infty) \tag{2.5}$$

we get

$$\ln\left(1+\frac{1}{a+n+ab}\right)<\frac{1}{a+n+ab}\leq\frac{1}{a+n}$$

which, by (2.4), shows that the sequence $\{x_n\}$ is strictly increasing. We consider now the Neper's inequality

$$\frac{1}{n+1}<\ln(n+1)-\ln n<\frac{1}{n},\ n=0,1,2,\ldots \tag{2.6}$$

which can be proved in various ways, e.g. by using the mean-value theorem for the function $f(x)=\ln x$ defined on the interval $[n,n+1]$. By means of (2.6) it is now easy to prove that, for $n\geq 1$,

$$1+\frac{1}{2}+\cdots+\frac{1}{n-1}-\ln(n-1)<1.$$

As, for $n\geq 2$, we have

$$x_n<\frac{1}{a}+\left[1+\frac{1}{2}+\cdots+\frac{1}{n-1}-\ln(n-1)\right]+\ln\frac{a(n-1)}{a+n+ab},$$

and $\ln\dfrac{a(n-1)}{a+n+ab}<\ln a$, we deduce that $\{x_n\}$ is upper bounded, hence convergent. Similarly,

$$y_n-y_{n+1}=\ln\frac{a+n+ab}{a+n+ab+1}-\frac{1}{a+n},\ \forall n=1,2,\ldots \tag{2.7}$$

which suggest us to consider the function $f:[1,\infty)\to\mathbb{R}$, given by

$$f(x)=\ln\frac{a+x+ab}{a+x+ab-1}-\frac{1}{a+x},\ x\in[1,\infty),$$

whose first derivative can be expressed in closed form as

$$f'(x)=\frac{(2ab-1)x+2a^2b+a^2b^2-ab-a}{(a+x+ab)(a+x+ab-1)(a+x)^2}.$$

By hypothesis, $0\leq 2ab\leq 1$. Hence we have $(2ab-1)x<0$, for all $x\in[1,\infty)$ and $2a^2b+a^2b^2-ab-a=a(2ab-1)+a^2b^2-ab\leq a^2b^2-ab<0$, which shows that

$$f'(x)<0,\ \text{ for all } x\in[1,\infty),$$

that is, f is strictly decreasing on $[1,\infty)$. As $\lim\limits_{x\to\infty}f(x)=0$, this means that

$$f(x)>0,\ \text{ for all } x\in[1,\infty).$$

In view of the inequality above, (2.7) implies that the sequence $\{y_n\}$ is strictly decreasing. In a similar manner to the case of $\{x_n\}$, we prove that $\{y_n\}$ is lower bounded, and hence convergent.

As

$$x_n-y_n=\ln\frac{a+n+ab-1}{a+n+ab},\ \forall n=1,2,\ldots$$

it immediately follows that $\{x_n\}$ and $\{y_n\}$ have the same limit. This proves (i), while (ii) follows by (2.3). □

Remark 2.1. For $b = 0$, by Theorem 2.2 we obtain the first part of Theorem 1.1. Note that the assumption $b \in \left[0, \frac{1}{2a}\right]$ was used only for proving the convergence of the sequence $\{y_n\}$ which, in the case $a = 1$ and $b = 0$, reduces to the Euler's original sequence $\{\gamma_n\}$.

For $a = 0$ and $b = \frac{1}{2}$, the sequence $\{y_n\}$ in Theorem 2.2 reduces to the sequence $\{R_n\}$ considered by De Temple [3]:

$$R_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right),$$

which still converges to the Euler's constant c but with a *quadratic* rate of convergence, which is guaranteed by the estimates [3]:

$$\frac{1}{24(n+1)^2} < R_n - c < \frac{1}{24n^2}. \tag{2.8}$$

By means of some elementary inequalities, L. Tóth proved in [13] a very interesting result which is directly related to our topic and which can be briefly stated as follows. Consider the sequence $\{\alpha_n\}$ defined by the implicit relation

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n + \alpha_n) = c,\ n = 1, 2, \ldots \tag{2.9}$$

where c is the Euler's constant. Then $\{\alpha_n\}$ is convergent and $\lim_{n\to\infty} \alpha_n = \frac{1}{2}$. Note that $\{\alpha_n\}$ can explicitly be given by

$$\alpha_n = \exp\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - c\right) - n,\ n \geq 1.$$

The proof of $\lim_{n\to\infty} \alpha_n = \frac{1}{2}$ is essentially based on the double inequality

$$\frac{1}{2} + \frac{1}{24(n+1)} < \alpha_n < \frac{1}{2} + \frac{1}{24n} \tag{2.10}$$

established in [13] and valid for all $n \geq 1$. The inequalities (2.10) suggested to Negoi [6] to improve De Temple's result by considering instead of $\{R_n\}$ the sequence $\{T_n\}$ given by

$$T_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right),$$

which converges to the Euler's constant c but with a *cubic* rate of convergence demonstrated by the estimates [6]

$$\frac{1}{48(n+1)^3} < c - T_n < \frac{1}{48n^3}. \tag{2.11}$$

The first part of Theorem 3.1 in [9], which is also given as Theorem 3.1.1 in [12], can be obtained as a particular case of Theorem 2.2, for $b = \frac{1}{2a}$.

Corollary 2.1. *Let $a \in (0, +\infty)$ and consider the sequence $\{\alpha_n\}$ defined for each $n \geq 1$ by*

$$\alpha_n = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln\left(\frac{a+n-1}{a} + \frac{1}{2a}\right), \tag{2.12}$$

Then $\{\alpha_n\}$ converges to $\gamma(a)$ and satisfies

$$\gamma(a) < \alpha_{n+1} < \alpha_n, \; \forall n = 1, 2, \ldots \tag{2.13}$$

3. Conclusions and open problems

Motivated by the previous results regarding the convergence order of the various sequences that converges to Euler's constant, on the one hand, and by the results established in [1]-[17] and other related papers, on the other hand, we formulate the following problems.

Problem 3.1. *Establish results regarding the convergence order of the sequences $\{x_n\}$ and / or $\{y_n\}$ defined by (2.1) and (2.2), respectively, similar to the estimates (1.6), (1.7) and (2.8).*

Problem 3.2. *Consider the case when the two sequences $\{x_n\}$, $\{y_n\}$ that are involved in Theorem 2.2 are introduced by means of a constant $a \in (0, \infty)$ and a sequence $\{b_n\} \subset \left[0, \frac{1}{2a}\right]$, satisfying appropriate conditions, instead of the constant b, e.g., $\{x_n\}$ is given by*

$$x_n = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln\left(\frac{a+n}{a} + b_n\right). \tag{3.1}$$

Establish similar results to the ones given in Theorems 1.1 and 2.2.

Problem 3.3. *Theorem 4.1.1 in* [12] *expresses the relationship between the generalization $\gamma(a)$ of Euler's constant and the logarithmic derivative of the Euler Γ function. Establish similar results for the case when the generalization $\gamma(a, b)$ of Euler's constant is considered instead of $\gamma(a)$.*

Problem 3.4. *More general sequences of Euler type have been considered so far, for example, the ones in* [1] *and* [2]*. The sequence considered in* [1] *is of the form*

$$\alpha_n = \sum_{k=n_0}^{n} f(k) - \int_{n_0}^{n} f(x)dx,$$

where $f : [a, \infty) \to \mathbb{R}$ is a strictly decreasing function, $a > 0$ and $n_0 = [a] + 1$. Clearly, the original Euler's sequence is obtained in the case $f(x) = \frac{1}{x}$ and $n_0 = 1$. Establish similar results to Theorem 1.1 and Theorem 2.2 for these types of sequences.

Problem 3.5. *Study the properties of the function $\gamma(a, b)$ in connection with other functions, cf.* [5], [12] *and* [14].

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