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On a family of first order difference inequalities used in the iterative approximation of fixed points

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ABSTRACT. Several first order difference inequalities which are intensively used in proving convergence theorems for various fixed point iteration procedures are unified and detailedly demonstrated. In terms of the difference equations terminology, the obtained results basically show that the zero solution of these difference inequalities is globally asymptotically stable. Some new more general conditions under which the zero solution of these difference inequalities is globally asymptotically stable, which extends and includes corresponding related results obtained in recent literature, are also given.

1. INTRODUCTION

An important tool for proving several convergence theorems in the iterative approximation of fixed points is based on various stability results concerning various first order difference inequalities. This type of inequalities occurs in the following way.

Let (X, d) be a metric space, $T : X \to X$ be a given self-mapping and let $x^* \in X$ be a fixed point of T. The classical way to construct x^* is to use a certain iterative method, i.e., to consider a sequence $\{x_n\} \subset X$ generated by a certain fixed point iterative scheme, see [2] and the very recent introductory monograph [9] and references therein.

For those classes of weak contractive type operators for which we are not able to prove directly that

$$x_n \to x^*, \text{ as } n \to \infty, \tag{1.1}$$

a feasible alternative way is to consider the sequence of positive numbers $\{\delta_n\}_{n\geq 0}$ given by

$$\delta_n = d(x_n, x^*),\ n = 0, 1, \dots,$$

provided we can deduce, from the properties of the operator T and those of the ambient space, a certain difference inequality satisfied by $\{\delta_n\}_{n\geq 0}$ which should be strong enough to ensure that

$$\delta_n \to 0, \text{ as } n \to \infty. \tag{1.2}$$

Clearly, (1.2) implies (1.1) and the convergence theorem is thus proved.

If we interpret (1.2) from the point of view of difference equations, then we are naturally lead to the study of the global stability of solutions of the difference inequalities thus obtained.

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Although there is a current intensive work on the study of stability of solutions of *difference equations*, see for example [18], [3], [19], [20],[21], [27] and the introductory monograph [16], there is little or no systematic research work on the same topic for all kind of *difference inequalities*, see the monograph [26] and the bibliography therein. Moreover, the stability results regarding the difference inequalities which are used in the iterative approximation of fixed points are not explicitly formulated in the appropriate terms of stability of their solutions, see for instance the very recent monograph [9].

Therefore, our main aim in this paper is to unify and improve several results regarding the asymptotic stability of a family of difference inequalities which are intensively used in the iterative approximation of fixed points. Secondly, we would like to draw attention on the fact that several stability results of the solutions of difference inequalities are very important from the point of view of fixed point theory and that there are also some other fields of research where such kind of results are extremely useful, see [31], for example.

We shall be mainly interested of some difference inequalities used in the iterative approximation of fixed points that have been collected in [9], at the end of Section 1.1, as Lemmas 1.2-1.7, but given there without proofs. Some other advanced difference inequalities, accompanied by more or less detailed proofs, are given in several works by Alber et al., see [2] and the papers cited there. As it often happened to the present author to be asked by young researchers working in the field for more complete and detailed proofs of such kind of results, which are missing in literature, this note will subsidiary offer such detailed proofs for most of the difference inequalities we can find in the iterative approximation of fixed points, as a starting point for future insights.

2. A SIMPLE DIFFERENCE INEQUALITY

We start by proving the following result, regarding a simple difference equation which appears in the monograph [9] as Lemma 1.6. It is originating in a convergence test for series of positive numbers that generalizes the well-known ratio test or D'Alembert test and has been first published in 1991 [4], see also [7] for other developments and some sample applications to fixed point theory.

Theorem 2.1. *Let* $\{x_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ *be sequences of nonnegative numbers for which there exists* $0 \leq q < 1$, *so that*

$$x_{n+1} \leq q x_n + b_n, \ \forall n \geq 0. \tag{2.1}$$

(i) If $\lim\limits_{n\to\infty} b_n = 0$, *then* $\lim\limits_{n\to\infty} x_n = 0$.

(ii) If $\sum\limits_{n=0}^{\infty} b_n < \infty$, *then*

$$\sum_{n=0}^{\infty} x_n < \infty.$$

Proof. We divide side by side the inequality (2.1) by $q^{n+1} > 0$, to get:

$$\frac{x_{n+1}}{q^{n+1}} \leq \frac{x_n}{q^n} + \frac{b_n}{q^{n+1}}, \ n \geq 0.$$

By summing up all the inequalities obtained from the previous one by taking $n := 0, 1, \ldots, n-1$, we get

$$\frac{x_n}{q^n} \le x_0 + \frac{b_0}{q} + \frac{b_1}{q^2} + \ldots + \frac{b_{n-1}}{q^n},$$

and hence

$$x_n \le x_0 q^n + q^n \sum_{i=1}^{n} \frac{b_{i-1}}{q^i},\ n \ge 1. \tag{2.2}$$

Since $x_n \ge 0$ and $q^n \to 0$ as $n \to \infty$, in order to prove (i) it suffices to show that

$$\lim_{n\to\infty} q^n \sum_{i=1}^{n} \frac{b_{i-1}}{q^i} = 0.$$

Now we need the Stolz-Césaro theorem, which claims that if $\{u_n\}_{n\ge 0}$, $\{v_n\}_{n\ge 0}$ are two sequences of real numbers, $|v_n| \to \infty$, as $n \to \infty$, and such that there exists the limit

$$l = \lim_{n\to\infty} \frac{u_{n+1} - u_n}{v_{n+1} - v_n},$$

then we also have

$$\lim_{n\to\infty} \frac{u_n}{v_n} = l.$$

Denoting $u_n = \sum\limits_{i=1}^{n} \dfrac{b_{i-1}}{q^i}$ and $v_n = \dfrac{1}{q^n}$, and using the Stolz-Césaro theorem we have:

$$\frac{u_{n+1} - u_n}{v_{n+1} - v_n} = \frac{\frac{b_n}{q^{n+1}}}{\frac{1}{q^{n+1}} - \frac{1}{q^n}} = \frac{b_n}{1-q}.$$

As $\lim\limits_{n\to\infty} b_n = 0$, it follows that

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{u_{n+1} - u_n}{v_{n+1} - v_n} = 0, \quad \text{that is,} \quad \lim_{n\to\infty} q^n \sum_{i=1}^{n} \frac{b_{i-1}}{q^i} = 0.$$

Now by (2.2) immediately follows conclusion (i): $\lim\limits_{n\to\infty} x_n = 0$.

To prove the second part of the theorem we follow the technique of proof used for the generalized ratio test in the paper [5]. Another shorter proof can also be find in [6].

By (2.2) we get

$$\sum_{k=1}^{n} x_k \le x_0 \sum_{k=1}^{n} q^k + \sum_{k=1}^{n} (\sum_{i=1}^{k} q^{k-i} b_{i-1}),\ n \ge 1. \tag{2.3}$$

Denote by S the double sum in (2.3) and evaluate it in the following way:

$$\begin{aligned} S &= b_0 + (qb_0 + b_1) + (q^2 b_0 + qb_1 + b_2) + \cdots + (q^{n-1} b_0 + \cdots + qb_{n-2} + b_{n-1}) \\ &= b_0(1 + q + \cdots + q^{n-1}) + b_1(1 + q + \cdots + q^{n-2}) + \cdots + b_{n-2}(1+q) + b_{n-1} \\ &\le (1 + q + \cdots + q^{n-1})(b_0 + b_1 + \cdots + b_{n-1}) = \frac{1-q^n}{1-q}(b_0 + b_1 + \cdots + b_{n-1}). \end{aligned}$$

Using the fact that $0 \leq q < 1$, it follows that $\frac{1-q^n}{1-q} \leq \frac{1}{1-q}$ and therefore

$$S \leq \frac{1}{1-q}(b_0 + b_1 + \cdots + b_{n-1}) = \frac{1}{1-q} B_{n-1}.$$

Now, in (2.3) the first term is just the partial sum of a geometric series, which is convergent, while the second term, in view of the inequality above, is also less than the partial sum of a convergent series.

The comparison test now proves the second part of the theorem. □

Using the terminology from difference equations [16], on the one hand, and adopting the concept of *summable asymptotical stability* [8] from fixed point iteration procedures to difference equations and difference inequalities, we can restate Theorem 2.1 as a result of independent interest.

Corollary 2.1. *Under the assumptions of Theorem 2.1, (i), the zero solution of the difference inequality (2.1) is globally asymptotically stable, while under the assumptions of Theorem 2.1, (ii), the zero solution of the difference inequality (2.1) is summable asymptotically stable.*

3. More advanced difference inequalities

A more general result than the one given in the first part of Theorem 2.1 could be now proved essentially by the same technique. To this end we need the following result, generally known as Cauchy's lemma ([11], see also [17], Application 9 (b), page 78), for which we present a direct proof which does not make use of Toeplitz theorem.

Lemma 3.1. (Cauchy) *Let* $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ *be sequences of nonnegative numbers satisfying*

$$(i) \lim_{n\to\infty} a_n = 0; \qquad (ii) \sum_{k=0}^{\infty} b_k < \infty.$$

Then

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_k b_{n-k} = 0.$$

Proof. By (ii) it follows that the sequence of partial sums, $\{B_n\}_{n=0}^{\infty}$, given by $B_n = b_0 + \cdots + b_n,\ n \geq 0$ converges to some $B \geq 0$ and hence it is bounded. Let $M > 0$ be such that

$$B_n \leq M, \text{ for all } n \geq 0.$$

Now, by (i), we have that for any $\epsilon > 0$, there exists an integer m such that

$$a_n < \frac{\epsilon}{2M}, \text{ for all } n \geq m+2.$$

For $n \geq m+2$, we can write

$$\sum_{k=0}^{n} a_k b_{n-k} = (a_n b_0 + \cdots + a_{m+2} b_{n-m-2}) + (a_{m+1} b_{n-m-1} + \cdots + a_0 b_n).$$

Then

$$a_n b_0 + \cdots + a_{m+2} b_{n-m-2} < \frac{\epsilon}{2M} \cdot (b_0 + \cdots + b_{n-m-2}) \le \frac{\epsilon}{2}, \text{ for all } n \ge m+2.$$

On the other hand, if we denote $A = \max\{a_0, \ldots, a_{m+1}\}$, then we have

$$a_{m+1} b_{n-m-1} + \cdots + a_0 b_n \le A \cdot (b_{n-m+1} + \cdots + b_n) = A \cdot (B_n - B_{n-m}).$$

As m is fixed, $\lim\limits_{n\to\infty} B_n = \lim\limits_{n\to\infty} B_{n-m} = B$, which shows that there exists a positive integer k such that

$$a_{m+1} b_{n-m-1} + \cdots + a_0 b_n < \frac{\epsilon}{2}, \text{ for all } n \ge k.$$

Now put $N = \max\{k, m+2\}$ to obtain

$$a_n b_0 + \cdots + a_0 b_n < \epsilon, \text{ for all } n \ge N,$$

which concludes the Lemma. □

Theorem 3.2. *Let $\{x_n\}_{n=0}^{\infty}, \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers satisfying*

$$x_{n+1} \le (1 - a_n) x_n + b_n, \forall n \ge 0, \tag{3.1}$$

and such that $\{a_n\}_{n=0}^{\infty} \subset [0,1]$ satisfies

$$\sum_{n=0}^{\infty} a_n = \infty.$$

If either

(i) $\lim\limits_{n\to\infty} \dfrac{b_n}{a_n} = 0$ *or* *(ii)* $\sum\limits_{n=0}^{\infty} b_n < \infty$,

then

$$\lim_{n\to\infty} x_n = 0.$$

Proof. By (3.1) we obtain inductively

$$x_{n+1} \le x_0 u_n + u_n \sum_{k=0}^{n} \frac{b_k}{u_k}, \forall n \ge 0, \tag{3.2}$$

where we denoted

$$u_n = \prod_{k=0}^{n} (1 - a_k), \; n \ge 0.$$

First of all, let us prove a well known and useful fact about the relationship between the convergence of infinite series and products of positive numbers, i.e., that if $\{a_n\}_{n=0}^{\infty} \subset [0,1]$ then

$$\sum_{n=0}^{\infty} a_n = \infty \Rightarrow \prod_{n=0}^{\infty} (1 - a_n) = 0. \tag{3.3}$$

To this end we only need to know the elementary inequality

$$e^x \ge 1 + x, \text{ for all } x \in \mathbb{R},$$

which can be proved by using derivatives and from which we deduce, by formally changing x by $-x$ the following inequality

$$1-x \le e^{-x}, \text{ for all } x \in \mathbb{R}. \tag{3.4}$$

Now, by taking $x=a_n$ in (3.4), we get

$$0 \le \prod_{n=0}^{\infty}(1-a_n) \le e^{-\sum\limits_{n=0}^{\infty} a_n}$$

and so the implication (3.3) is proved. This shows that

$$\lim_{n\to\infty} u_n = 0 \tag{3.5}$$

and so the first term in the right hand side of (3.2) tends to zero.

Let us now prove that the second term in the right hand side of (3.2) tends to zero, too. In order to compute this limit, i.e.,

$$\lim_{n\to\infty} u_n \sum_{k=0}^{n} \frac{b_k}{u_k} = \lim_{n\to\infty} (\sum_{k=0}^{n} \frac{b_k}{u_k})/(\frac{1}{u_n})$$

let us denote

$$\alpha_n = \sum_{k=0}^{n} \frac{b_k}{u_k} \text{ and } \beta_n = \frac{1}{u_n}.$$

Since $\lim\limits_{n\to\infty} u_n = 0$, it results $\lim\limits_{n\to\infty} \beta_n = \infty$, and so by Stolz-Césaro theorem, if there exists the limit

$$\lim_{n\to\infty} \frac{\alpha_{n+1}-\alpha_n}{\beta_{n+1}-\beta_n} = l, \text{ then the limit h } \lim_{n\to\infty} \frac{\alpha_n}{\beta_n},$$

also exists and equals l. But

$$\frac{\alpha_{n+1}-\alpha_n}{\beta_{n+1}-\beta_n} = \frac{b_{n+1}}{u_{n+1}} \cdot \frac{u_{n+1}u_n}{u_n - u_{n+1}} = \frac{b_{n+1}u_n}{u_n a_{n+1}} = \frac{b_{n+1}}{a_{n+1}}.$$

If condition (i) holds, this shows that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}-\alpha_n}{\beta_{n+1}-\beta_n} = 0, \text{ that is, } \lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0,$$

and therefore

$$\lim_{n\to\infty} u_n \sum_{k=0}^{n} \frac{b_k}{u_k} = 0.$$

Since x_n is nonnegative, by (3.2) it follows now immediately that

$$\lim_{n\to\infty} x_n = 0,$$

as required.

If (ii) holds, then the conclusion follows in the following way. We write the second term in (3.2) in the form

$$u_n \sum_{k=0}^{n} \frac{b_k}{u_k} = \sum_{k=0}^{n} b_k \frac{u_n}{u_k} = \sum_{k=0}^{n} b_k \prod_{i=k+1}^{n} (1-a_i). \tag{3.6}$$

If we take now

$$a_{n-k} = \frac{u_n}{u_k} = \prod_{i=k+1}^{n} (1 - a_i),$$

in Lemma 3.1, then condition (i) in that lemma means

$$\lim_{n\to\infty} \frac{u_n}{1 - a_0} = 0,$$

which is obviously satisfied, by (3.5).

Because condition (ii) in Lemma 3.1 coincides with (ii) in the present theorem, by Lemma 3.1 we get

$$\lim_{n\to\infty} u_n \sum_{k=0}^{n} \frac{b_k}{u_k} = 0,$$

which completes the proof. □

The result given by Theorem 3.2 can be re-stated as a result of independent interest.

Corollary 3.1. *Under the assumptions in Theorem 3.2, the zero solution of the difference inequality (3.2) is globally asymptotically stable.*

Remark 3.1. 1) If we take now $a_n \equiv 1 - q < 1$ in Theorem 3.2, we get exactly the first part of Theorem 2.1.

2) When the sequence $\{a_n\}_{n=0}^{\infty}$ converges to zero, what is required in most convergence theorems that are using Theorem 3.2, we actually have instead of (3.3) the equivalence:

$$\sum_{n=0}^{\infty} a_n = \infty \Leftrightarrow \prod_{n=0}^{\infty} (1 - a_k) = 0, \tag{3.7}$$

which follows by the following inequality

$$e^{-2x} \leq 1 - x$$

valid for any $x \in [0, 1/2]$.

3) Note that assumptions (i) and (ii) in Theorem 3.2 are independent, as shown by the second part in Proposition 3.1.

Proposition 3.1. *Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers such that*

$$(a) \sum_{k=0}^{\infty} a_k = \infty; \qquad (b) \sum_{k=0}^{\infty} b_k < \infty.$$

1) If the sequence $\left\{\frac{b_n}{a_n}\right\}$ has a limit, then this limit is necessarily equal to zero;

2) There exists sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ satisfying (a) and (b), for which the limit

$$\lim_{n\to\infty} \frac{b_n}{a_n}$$

does not exist.

Proof. 1) Indeed, if we denote

$$u_n = \sum_{k=0}^{n} b_k,\ v_n = \sum_{k=0}^{n} a_k,\ n = 0, 1, \dots$$

then by (a) and (b) we get

$$\lim_{n\to\infty} u_n = L \text{ (finite) and } \lim_{n\to\infty} v_n = \infty,$$

which yields

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 0.$$

On the other hand, by Stolz-Césaro theorem, if there exists

$$\lim_{n\to\infty} \frac{u_{n+1} - u_n}{v_{n+1} - v_n} = l, \text{ then the limit } \lim_{n\to\infty} \frac{u_n}{v_n},$$

also exists and equals l. But, it is easy to see that

$$\lim_{n\to\infty} \frac{u_{n+1} - u_n}{v_{n+1} - v_n} = \lim_{n\to\infty} \frac{b_n}{a_n},$$

and since, by hypothesis, the last limit exists, the conclusion follows.

2) If we take $a_n = \dfrac{1}{n+1}$, $n = 0, 1, \dots$, then (a) is satisfied. We can construct a subsequence of positive integer numbers

$$n_1 < n_2 < \cdots < n_k < \dots$$

Now construct $\{b_n\}$ in the following way: put $b_n = \dfrac{1}{2^k}$, for $n = n_k$, $k = 1, 2, \dots$ and $a_n = 0$, elsewhere. Then

$$\sum_{n=0}^{\infty} b_n < \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty$$

but the limit

$$\lim_{n\to\infty} \frac{b_n}{a_n}$$

does not exist. □

Note that, if (a) and (b) in Proposition 3.1 are satisfied, then we actually have $\liminf\limits_{n\to\infty} \dfrac{b_n}{a_n} = 0$ and therefore, if $\limsup\limits_{n\to\infty} \dfrac{b_n}{a_n} = 0$, then $\lim\limits_{n\to\infty} \dfrac{b_n}{a_n} = 0$.

Remark 3.2. By comparing Theorem 2.1 and Theorem 3.2, the following question arises: is still valid conclusion (ii) in Theorem 2.1 under the assumptions in Theorem 3.2, that is, do we still have

$$\sum_{n=0}^{\infty} x_n < \infty \ ?$$

In order to answer this question we need another fundamental result due to Cauchy, regarding the multiplication of convergent series.

Lemma 3.2. (Cauchy) *Let* $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ *be sequences of nonnegative numbers such that*

$$(i) \sum_{k=0}^{\infty} a_k < \infty; \qquad (ii) \sum_{k=0}^{\infty} b_k < \infty.$$

If we denote

$$u_n = \sum_{k=0}^{n} a_k b_{n-k},\; n \geq 0,$$

then we also have

$$\sum_{n=0}^{\infty} u_n < \infty.$$

Proof. Denote by S_n the partial sum of the series $\sum_{n=0}^{\infty} u_n$, that is,

$$S_n = \sum_{k=0}^{n} \sum_{i=0}^{k} a_i b_{k-i}.$$

Then, similarly to the proof of the second part of Theorem 2.1, we have

$$\sum_{k=0}^{m} a_k \cdot \sum_{k=0}^{m} b_k < S_n < \sum_{k=0}^{n} a_k \cdot \sum_{k=0}^{n} b_k, \tag{3.8}$$

where m is the integer part of $\frac{n-1}{2}$, i.e., $\frac{n-1}{2}$, if n is odd and $\frac{n-2}{2}$, if n is even. As $m \to \infty$, when $n \to \infty$, by (3.8), we get the conclusion. Moreover, we clearly have

$$\sum_{n=0}^{\infty} u_n = \sum_{k=0}^{\infty} a_k \cdot \sum_{k=0}^{\infty} b_k.$$

□

Theorem 3.3. *Let* $\{x_n\}_{n=0}^{\infty}, \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ *be sequences of nonnegative numbers satisfying*

$$x_{n+1} \leq (1 - a_n)x_n + b_n,\; \forall n \geq 0, \tag{3.9}$$

and such that $\{a_n\}_{n=0}^{\infty} \subset [0, 1]$ *satisfies*

$$\sum_{n=0}^{\infty} a_n = \infty.$$

Denote

$$u_n = \prod_{k=0}^{n} (1 - a_k),\; n \geq 0.$$

If

(i) $\sum_{n=0}^{\infty} u_n < \infty,$ *and* *(ii)* $\sum_{n=0}^{\infty} b_n < \infty,$

then

$$\sum_{n=0}^{\infty} x_n < \infty.$$

Proof. By (3.9) we have similarly to the proof of Theorem 3.2,

$$x_{n+1} \le x_0 u_n + \sum_{k=0}^{n} b_k \frac{u_n}{u_k} = x_0 u_n + \sum_{k=0}^{n} b_k \prod_{i=k+1}^{n} (1-a_i),\ \forall n \ge 0. \tag{3.10}$$

Now apply Lemma 3.2 to the second term in the right hand side of (3.10) to get the conclusion. □

In order to show that assumptions in Theorem 3.3 are much more restrictive than the corresponding ones in Theorem 3.2, simply take $a_n = \dfrac{n+1}{n+2}$, $n = 0, 1, \dots$.

We can have simultaneously both assumptions (i) and (ii) in Theorem 3.2, but imposed to different sequences, as in the next theorem, which is fundamental in proving many convergence theorems regarding the iterative approximation of fixed points, and which first appeared in [22], where a different proof is given.

Theorem 3.4. *Let $\{x_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{c_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers satisfying*

$$x_{n+1} \le (1-a_n)x_n + b_n + c_n, \text{ for all } n \ge 0, \tag{3.11}$$

where $\{a_n\}_{n=0}^{\infty} \subset [0,1]$. If

$$(i) \sum_{n=0}^{\infty} a_n = \infty; \quad (ii)\, b_n = o(a_n) \quad \text{and} \quad (iii) \sum_{n=0}^{\infty} c_n < \infty,$$

then

$$\lim_{n\to\infty} x_n = 0.$$

Proof. Similarly to the proof of Theorem 3.2, by (3.11) we obtain

$$x_{n+1} \le x_0 u_n + u_n \sum_{k=0}^{n} \frac{b_k}{u_k} + u_n \sum_{k=0}^{n} \frac{c_k}{u_k},\ \forall n \ge 0, \tag{3.12}$$

where we used the same notation, i.e.,

$$u_n = \prod_{k=0}^{n} (1-a_k),\ n \ge 0.$$

We write the third term in the right hand side of (3.12) in a similar form to that in (3.6) and so (3.12) will become

$$x_{n+1} \le x_0 u_n + u_n \sum_{k=0}^{n} \frac{b_k}{u_k} + \sum_{k=0}^{n} c_k \prod_{i=k+1}^{n} (1-a_i),\ \forall n \ge 0. \tag{3.13}$$

Now, similarly to the proof of Theorem 3.2, we have

$$\lim_{n\to\infty} x_0 u_n = 0.$$

For the second term in the right hand side of (3.13), we are in the case of Theorem 3.2, condition (i), while for the third term in the right hand side of (3.13), we are in the case of Theorem 3.2, condition (ii). □

Using the terminology from difference equations, the first part of Theorem 3.4 can also be re-stated as a result of independent interest.

Corollary 3.2. *Under the assumptions of Theorem 3.4, the zero solution of the difference inequality (3.11) is globally asymptotically stable.*

The second part of the next theorem, which appeared in several papers, see [13] where it is Lemma 2.2, is an easy consequence of Theorem 3.2, too. We leave the reader to prove its first part.

Theorem 3.5. *Let* $\{x_n\}_{n=0}^{\infty}$ *be a sequence of nonnegative real numbers and let* $\{a_n\}_{n=0}^{\infty}$ *be a real sequence in* $[0,1]$ *such that*

$$\sum_{n=0}^{\infty} a_n = \infty.$$

(i) If for a given $\epsilon > 0$ *there exists a positive integer* n_0 *such that*

$$x_{n+1} \leq (1-a_n)x_n + \epsilon \cdot a_n, \text{for all } n \geq n_0,$$

then we have $0 \leq \limsup_{n\to\infty} x_n \leq \epsilon$.

(ii) If there exists a positive integer n_1 *such that*

$$x_{n+1} \leq (1-a_n)x_n + a_n b_n, \text{ for all } n \geq n_1,$$

where $b_n \geq 0$ *for all* $n = 0, 1, 2, ...$ *and* $b_n \to 0$ *as* $n \to \infty$, *then we have*

$$\lim_{n\to\infty} x_n = 0.$$

4. Two special difference inequalities

By taking advantage of the various tools used in proving all previous results, it is now an easy task to prove the following theorems regarding the asymptotic stability of the zero solution of two special difference inequations.

Theorem 4.6. *Let* $\{x_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ *be sequences of nonnegative numbers satisfying*

$$x_{n+1} \leq x_n + b_n, \text{ for all } n \geq 0,$$

with $\sum\limits_{n=0}^{\infty} b_n < \infty$. *Then*

(i) $\lim\limits_{n\to\infty} x_n$ *exists.*

(ii) If, additionally, $\{x_n\}_{n=0}^{\infty}$ *has a subsequence converging to zero, then*

$$\lim_{n\to\infty} x_n = 0.$$

Theorem 4.7. *Let* $\{x_n\}_{n=0}^{\infty}, \{a_n\}_{n=0}^{\infty}$ *be sequences of nonnegative numbers satisfying*

$$x_{n+1} \leq (1+a_n)x_n, \text{ for all } n \geq 0,$$

where $\sum\limits_{n=0}^{\infty} a_n < \infty$. *Then*

1) $\lim\limits_{n\to\infty} x_n$ *exists; 2) If* $\liminf\limits_{n\to\infty} x_n = 0$ *then*

$$\lim_{n\to\infty} x_n = 0.$$

5. Conclusions and Future Directions

The family of difference inequalities studied in this paper are frequently used in the iterative approximation of fixed points and have been collected in [9] from several sources, at the end of Section 1.1, as Lemmas 1.2-1.7, without proofs. Another monograph devoted to this topic is [12]. Our results here improve, extend and unify the most useful ones. Theorem 3.5, which is Lemma 1.2 in [9], appears in many papers. In the form given here it corresponds to Lemma 2 in [28]. Theorem 3.4, which is Lemma 1.3 in [9], appears to have been first considered in [22]. Theorem 4.6, which is Lemma 1.7 in [9], has been given in [29] (part (i)) and [15] (part (ii)), respectively, while Theorem 4.7 is taken from [24]. Some more advanced difference inequalities of the more general form

$$x_{n+1} \leq \psi(x_n) + \sigma(a_n, b_n), \text{ for all } n \geq 0,$$

utilized in several papers on approximation of fixed points (see for example [1], [2], [25], [34] and the papers cited there), and given as Lemmas 1.4 and 1.5 in [9], will be considered in detail in a future work.

The similar multidimensional difference inequalities, like those in [30], as well as the difference inequalities used in stochastic approximation, see [31], have to be approached from the same point of view of asymptotic stability, too.

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