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A note on a difference inequality used in the iterative approximation of fixed points

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ABSTRACT. In this note we present a more detailed proof of Lemma 2 in [Liu, L.S., *Ishikawa and Mann iteration process with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl., **194** (1995), 114-125], regarding the global asymptotic stability of the solution of a first order difference inequality.

1. A FIRST ORDER DIFFERENCE INEQUALITY

The lemma bellow, concerning the global asymptotic stability of the zero solution of a first order difference inequality, appears to have been first given in [10] as Lemma 2.2 and further used by many authors as the key tool in proving several convergence theorems, see [4], for a very recent and comprehensive bibliography on the papers based on this technique of proof in the field of iterative approximation of fixed points.

Lemma 1.1. *Let* $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{c_n\}_{n=0}^{\infty}$ *be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n, \text{ for all } n \geq 0, \tag{1.1}$$

with $\{t_n\}_{n=0}^{\infty} \subset [0,1]$, $\sum\limits_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$ *and* $\sum\limits_{n=0}^{\infty} c_n < \infty$. *Then*

$$\lim_{n\to\infty} a_n = 0.$$

Lemma 1.1 extends Lemma 1.4 in [18], which is obtained from Lemma 1.1 for $c_n \equiv 0$. Because both Lemma 1.1 and Lemma 1.4 are very useful in the iterative approximation of fixed points (see for example the papers [1], [2], [3], [6], [7], [9], [12], [13], [14], [15], [16], [17], where these lemmas or some of their particular cases are used as auxiliary tools in proving convergence theorems), it is the main aim of this note to offer a different but more detailed proof of Lemma 1.1, based on Cauchy's Lemma.

Proof of Lemma 1. By a straightforward induction, by using (1.1) one obtains

$$0 \leq a_{n+1} \leq a_0 u_n + u_n \sum_{j=0}^{n} \frac{b_j}{u_j} + \sum_{j=0}^{n} c_j \prod_{i=j+1}^{n} (1-t_i), \; \forall n \geq 0, \tag{1.2}$$

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where, for brevity, we denoted

$$u_n = \prod_{j=0}^{n}(1-t_j),\ n \geq 0.$$

In view of the inequality

$$\prod_{j=0}^{\infty}(1-t_i) \leq exp(-\sum_{j=0}^{\infty} t_j)$$

from $\sum\limits_{n=0}^{\infty} t_n = \infty$, we get $\lim\limits_{n\to\infty} u_n = 0$, which shows that the first term in the right hand side of (1.2) converges to zero.

If we apply Lemma 2.1 at the end of this note, with

$$a_{n-k} = \frac{u_n}{u_k} = \prod_{i=k+1}^{n}(1-t_i), \text{ and } b_n = c_n$$

then both conditions (i) and (ii) are satisfied and so it follows

$$\lim_{n\to\infty} \sum_{j=0}^{n} c_j \prod_{i=j+1}^{n}(1-t_i) = 0,$$

which shows that the third term in the right hand side of (1.2) converges to zero, too. As we need to compute

$$\lim_{n\to\infty} u_n \sum_{j=0}^{n} \frac{b_j}{u_j} = \lim_{n\to\infty} (\sum_{j=0}^{n} \frac{b_j}{u_j})/(\frac{1}{u_n}),$$

let us denote

$$\alpha_n = \sum_{j=0}^{n} \frac{b_j}{u_j} \text{ and } \beta_n = \frac{1}{u_n}.$$

Since $\lim\limits_{n\to\infty} u_n = 0$ implies $\lim\limits_{n\to\infty} \beta_n = \infty$, we can apply Stolz-Césaro theorem: if there exists the limit

$$\lim_{n\to\infty} \frac{\beta_{n+1}-\beta_n}{\alpha_{n+1}-\alpha_n} = l, \text{ then the limit } \lim_{n\to\infty} \frac{\beta_n}{\alpha_n},$$

also exists and equals l. But

$$\frac{\beta_{n+1}-\beta_n}{\alpha_{n+1}-\alpha_n} = \frac{b_{n+1}}{u_{n+1}} \cdot \frac{u_{n+1}u_n}{u_n-u_{n+1}} = \frac{b_{n+1}u_n}{u_n t_{n+1}} = \frac{b_{n+1}}{t_{n+1}},$$

and since, by hypothesis, $b_n = o(t_n)$, that is,

$$\lim_{n\to\infty} \frac{b_{n+1}}{t_{n+1}} = 0,$$

we finally get

$$\lim_{n\to\infty} u_n \sum_{j=0}^{n} \frac{b_j}{u_j} = 0$$

which completes the proof of Lemma 1.1. □

2. CAUCHY'S LEMMA

For the sake of completeness, we present in the end a direct proof (which does not make use of Toeplitz theorem) of the result we used before, which is generally known as Cauchy's lemma after a result in [5], see also [8], Application 9 (b), page 78.

Lemma 2.1. (Cauchy) *Let $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers satisfying*

$$(i) \lim_{n\to\infty} a_n = 0; \qquad (ii) \sum_{k=0}^{\infty} b_k < \infty.$$

Then

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_k b_{n-k} = 0.$$

Proof. By (ii) it follows that the sequence of partial sums, $\{B_n\}_{n=0}^{\infty}$, given by $B_n = b_0 + \cdots + b_n,\ n \geq 0$ converges to some $B \geq 0$ and hence it is bounded. Let $M > 0$ be such that

$$B_n \leq M, \text{ for all } n \geq 0.$$

Now, by (i), we have that for any $\epsilon > 0$, there exists an integer m such that

$$a_n < \frac{\epsilon}{2M}, \text{ for all } n \geq m+2.$$

For $n \geq m+2$, we can write

$$\sum_{k=0}^{n} a_k b_{n-k} = (a_n b_0 + \cdots + a_{m+2} b_{n-m-2}) + (a_{m+1} b_{n-m-1} + \cdots + a_0 b_n).$$

Then

$$a_n b_0 + \cdots + a_{m+2} b_{n-m-2} < \frac{\epsilon}{2M} \cdot (b_0 + \cdots + b_{n-m-2}) \leq \frac{\epsilon}{2}, \text{ for all } n \geq m+2.$$

On the other hand, if we denote $A = \max\{a_0, \dots, a_{m+1}\}$, then we have

$$a_{m+1} b_{n-m-1} + \cdots + a_0 b_n \leq A \cdot (b_{n-m+1} + \cdots + b_n) = A \cdot (B_n - B_{n-m}).$$

As m is fixed, $\lim\limits_{n\to\infty} B_n = \lim\limits_{n\to\infty} B_{n-m} = B$, which shows that there exists a positive integer k such that

$$a_{m+1} b_{n-m-1} + \cdots + a_0 b_n < \frac{\epsilon}{2}, \text{ for all } n \geq k.$$

Now we put $N = \max\{k, m+2\}$ to get

$$a_n b_0 + \cdots + a_0 b_n < \epsilon, \text{ for all } n \geq N.$$

□

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