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Two elementary applications of some Prešić type fixed point theorems

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ABSTRACT. We introduce and illustrate by suitable examples the use of a unified fixed point method for studying the convergence of nonlinear recurrence sequences and for solving cyclic nonlinear systems of equations. Our technique is essentially based on some Prešić type fixed point theorems.

1. INTRODUCTION

Fixed point theorems, common fixed point theorems and coincidence theorems are all known to be very powerful tools in solving nonlinear functional equations: differential equations, integral equations, integro-differential equations and so on, see for example [9], section 1.3, *Fixed point formulation of typical functional equations*.

Based on some fixed point theorems of Prešić type, our main aim in this paper is to introduce a unified method that has, amongst others, two elementary applications: 1) to the study of the convergence of nonlinear recurrent sequences and, 2) to the solution of nonlinear cyclic systems. The technique presented here is in fact a generalization of the method developed by the first author in [3], [7] and [8], for the case of cyclic systems, see also the recent paper [13].

To simplify the presentation, we restrict ourselves to the case of the real line, but we have to stress on the fact that all results are valid in the more general setting of a complete metric space, see [9], [14], [16], [18]-[22], [25], [26], where the basic notions and results are presented in their full generality.

2. ELEMENTARY FIXED POINT THEORY ON THE REAL LINE

In order to build our method for studying nonlinear recurrent sequences and cyclic systems of equations, we shall need some lemmas and two main theorems.

We start by stating the following result, regarding a simple difference inequality, which appears in the monograph [9] as Lemma 1.6. It is originating in a convergence test for series of positive numbers that generalizes the well-known ratio test or D'Alembert test and has been first published in 1991 [1], see also [2], [4], [5] and [6] for other developments and some sample applications to fixed point theory. For a complete proof of Lemma 2.1 see for example [11].

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Lemma 2.1. *Let* $\{x_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ *be sequences of nonnegative numbers for which there exists a constant* $0 \le q < 1$, *such that*

$$x_{n+1} \le q x_n + b_n, \; \forall n \ge 0. \tag{2.1}$$

(i) If $\lim_{n\to\infty} b_n = 0$, *then* $\lim_{n\to\infty} x_n = 0$.

(ii) If $\sum_{n=0}^{\infty} b_n < \infty$, *then*

$$\sum_{n=0}^{\infty} x_n < \infty.$$

The next Lemma is due to Prešić [25]. For the sake of completeness, we give here the Prešić's original beautiful proof.

Lemma 2.2. *Let* k *be a positive integer and* $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}_+$ *such that* $\sum_{i=1}^{k} \alpha_i = \alpha < 1$ *(*$\alpha_1 \ne 0$*). If* $\{\Delta_n\}_{n\ge 1}$ *is a sequence of positive numbers satisfying*

$$\Delta_{n+k} \le \alpha_1 \Delta_n + \alpha_2 \Delta_{n+1} + \ldots + \alpha_k \Delta_{n+k-1}, n \ge 1, \tag{2.2}$$

then there exist $L > 0$ *and* $\theta \in (0,1)$ *such that*

$$\Delta_n \le L \cdot \theta^n, \text{ for all } n \ge 1. \tag{2.3}$$

Proof. Let g be the polynomial function given by

$$g(x) := \alpha_1 + \alpha_2 x + \ldots + \alpha_k x^{k-1} - x^k, \; x \in \mathbb{R}.$$

Since $g(0) = \alpha_1 > 0$, $g(1) = \alpha_1 + \alpha_2 + \ldots + \alpha_k - 1 < 0$ and g is continuous, there exists $\theta \in (0,1)$ such that $g(\theta) = 0$, that is,

$$\theta^k = \alpha_1 + \alpha_2 \theta + \cdots + \alpha_k \theta^{k-1}. \tag{2.4}$$

Now let us denote

$$L := \max\left\{\frac{\Delta_1}{\theta}, \frac{\Delta_2}{\theta^2}, \ldots, \frac{\Delta_k}{\theta^k}\right\} > 0.$$

Then (2.3) is true for $n := 1, 2, \ldots, k$. Assume (2.3) is true for k successive values of n, say $m, m+1, \ldots, m+k-1$ and prove that (2.3) is true for $n = m+k$, too. Indeed, by (2.2) and in view of (2.4) we have

$$\Delta_{m+k} \le \alpha_1 \Delta_m + \alpha_2 \Delta_{m+1} + \ldots + \alpha_k \Delta_{m+k-1} \le$$
$$\le \alpha_1 L\theta^m + \alpha_2 L\theta^{m+1} + \ldots + \alpha_k L\theta^{m+k-1} =$$
$$= L\theta^m \left(\alpha_1 + \alpha_2 \theta^1 + \ldots + \alpha_k \theta^{k-1}\right) = L\theta^{m+k},$$

which completes the proof. □

Remark 2.1. Throughout this paper, if $f : E^k \to E$ is a function of k variables ($k \ge 1$), then the iterate f^n is defined on the diagonal of E^k only, by

$$f^n(a, \ldots, a) = f^{n-1}(f(a, \ldots, a), \ldots, f(a, \ldots, a)), \; \forall n > 1. \tag{2.5}$$

An element $a \in E$ is called a *fixed point* of $f : E^k \to E$ if $f(a, \ldots, a) = a$ and is called a *periodic point of period* m ($m \ge 1$) of f if $f^m(a, \ldots, a) = a$.

Denote by $Fix\,(f)$ the set of all fixed points of f.

Clearly, $Fix\,(f) \subset Fix\,(f^n)$, $n \geq 1$, that is, any fixed point of f is a periodic point of f but the reverse is not generally true, since f can have periodic points of period $m \geq 2$ which are not fixed points of f, as shown by the next Example.

Example 2.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x_1, x_2) = -\dfrac{x_1 + x_2}{2}$. Then $Fix\,(f) = \{0\}$, since $f(a,a) = -a$, while $Fix\,(f^2) = \mathbb{R}$, as $f^2(a,a) = f(f(a,a), f(a,a)) = a$.*

It is therefore our main aim in this paper to find sufficient conditions to ensure the equality

$$Fix\,(f) = Fix\,(f^n),\, n \geq 1,$$

that is, to ensure that f has no other periodic points except for its fixed points.

One of the most important and interesting results of this kind is a generalization of Banach's contraction mapping principle that has been obtained in 1965 by S. Prešić [25]. We state here a version of Prešić's fixed point theorem, in the particular case of the real line, in view of the two elementary applications that will be presented in the next sections.

Theorem 2.1. (Presić's fixed point theorem in $\mathbb{R}$)

Let $E \subset \mathbb{R}$ be closed, k be a positive integer and $f : E^k \to E$ a mapping for which there exist $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}_+$, $\sum\limits_{i=1}^{k} \alpha_i = \alpha < 1$ satisfying

$$|f(x_0, \ldots, x_{k-1}) - f(x_1, \ldots, x_k)| \leq \alpha_1 |x_0 - x_1| + \cdots + \alpha_k |x_{k-1} - x_k|, \qquad (2.6)$$

for all $x_0, \ldots, x_k \in E$.

Then:

1) *f has a unique fixed point $\overline{x}$, that is, there exists a unique $\overline{x} \in E$ such that $f(\overline{x}, \ldots, \overline{x}) = \overline{x}$ and, moreover, $\overline{x}$ is the unique fixed point of any iterate f^n of f;*
2) *the Picard iteration $\{x_n\}_{n\geq 0}$,*
$$x_{n+1} = f(x_n, x_n, \ldots, x_n), n \geq 0, \qquad (2.7)$$
converges to $\overline{x}$, for any initial approximation $x_0 \in E$;
3) *the k-step sequence $\{y_n\}_{n\geq 0}$ given by*
$$y_n = f(y_{n-k}, y_{n-k+1}, \ldots, y_{n-1}), n \geq k, \qquad (2.8)$$
also converges to $\overline{x}$, for any $y_0, \ldots, y_{k-1} \in E$.

Proof. $1) + 2)$

Consider the function $F : E \to E$, defined by $F(x) = f(x, \ldots, x)$, for any $x \in E$. We have:

$$\begin{aligned}
&|F(x) - F(y)| = |f(x, x, \ldots, x) - f(y, y, \ldots, y)| \leq \\
&\leq |f(x, \ldots, x) - f(x, \ldots, x, y)| + |f(x, \ldots, x, y) - f(x, \ldots, x, y, y)| + \\
&\ldots\ldots \\
&+ |f(x, x, y, \ldots, y) - f(x, y, \ldots, y)| + |f(x, y, \ldots, y) - f(y, \ldots, y)|.
\end{aligned}$$

Then by (2.6) one obtains:

$$|F(x) - F(y)| \leq$$

$$\leq [\alpha_1 |x-x| + \alpha_2 |x-x| + \ldots + \alpha_{k-1} |x-x| + \alpha_k |x-y|] +$$
$$+ [\alpha_1 |x-x| + \alpha_2 |x-x| + \ldots + \alpha_{k-1} |x-y| + \alpha_k |y-y|] +$$
$$\ldots\ldots$$
$$+ [\alpha_1 |x-y| + \alpha_2 |y-y| + \ldots + \alpha_{k-1} |y-y| + \alpha_k |y-y|],$$

and so, for any $x, y \in E$ we have

$$|F(x) - F(y)| \leq \sum_{i=1}^{k} \alpha_i |x-y| = \alpha |x-y|. \tag{2.9}$$

Take now $x_0 \in E$ and let $\{x_n\}_{n\geq 0}$ be the sequence of successive approximations defined by F and x_0, that is,

$$x_{n+1} = F(x_n),\ n \geq 0. \tag{2.10}$$

Take $x := x_{n-1}$ and $y := x_n$ in (2.9) to get

$$|x_n - x_{n+1}| \leq \alpha |x_{n-1} - x_n|,\ n \geq 1. \tag{2.11}$$

By induction from (2.11) we obtain

$$|x_n - x_{n+1}| \leq \alpha^n |x_0 - x_1|,\ n \geq 1.$$

and then

$$|x_n - x_{n+p}| \leq \left(\alpha^n + \cdots + \alpha^{n+p-1}\right) |x_0 - x_1|,\ n \geq 1,\ p \geq 1. \tag{2.12}$$

As $0 \leq \alpha < 1$, we have $\lim\limits_{n\to\infty} \alpha_n = 0$ and hence in view of

$$\alpha^n + \cdots + \alpha^{n+p-1} = \frac{\alpha^n (1-\alpha^p)}{1-\alpha} < \frac{\alpha^n}{1-\alpha},$$

by (2.12) it follows that for any $\varepsilon > 0$, there exists $r = r(\varepsilon)$ such that

$$|x_n - x_{n+p}| < \varepsilon,\ \forall n \geq r,\ \forall p \in \mathbb{N},$$

which shows that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence. But any Cauchy sequence in $\mathbb{R}$ is convergent, so there exists $\overline{x} \in \mathbb{R}$ such that

$$\overline{x} = \lim_{n\to\infty} x_n.$$

Since E is closed, it follows that $\overline{x} \in E$. By (2.11) it follows that F is continuous on E. Now letting $n \to \infty$ in (2.10) we obtain in view of the continuity of F

$$\overline{x} = F(\overline{x}) \Leftrightarrow \overline{x} = f(\overline{x}, \ldots, \overline{x}),$$

which shows that $\overline{x}$ is a fixed point of F.

Let us show that $\overline{x}$ is the unique fixed point of F. Seeking for contradiction, we assume there exists $\overline{y} \in E$, $\overline{x} \neq \overline{y}$ such that $\overline{y} = F(\overline{y})$. Then by (2.9) we get

$$|\overline{x} - \overline{y}| \leq \alpha |\overline{x} - \overline{y}|,$$

which leads to the contradiction $|\overline{x} - \overline{y}| \leq 0$.

We note that by (2.9) one has

$$|f^n(x, x, \ldots, x) - f^n(y, y, \ldots, y)| \leq \alpha^n |x-y|, n \geq 1,\ x, y \in E,$$

and so conclusions 1) and 2) are proven.

3) In the following we shall prove the convergence of the multi-step iterative method $\{y_n\}_{n\geq 0}$ defined by (2.8) to the unique fixed point $\overline{x}$.

For $n \geq 0$ we have:

$$|y_n - y_{n+1}| = |f(y_{n-k} - y_{n-k+1}, \dots, y_{n-1}), f(y_{n-k+1}, y_{n-k+2}, \dots, y_n))| \leq$$
$$\leq \alpha_1 |y_{n-k} - y_{n-k+1}| + \alpha_2 |y_{n-k+1} - y_{n-k+2}| + \dots + \alpha_k |y_{n-1} - y_n| . \qquad (2.13)$$

Denoting

$$\Delta_n = |y_{n-1} - y_n| , n \geq 1,$$

then, by (2.13), it follows that the sequence $\{\Delta_n\}_{n\geq 1}$ satisfies:

$$\Delta_{n+1} \leq \alpha_1 \Delta_{n-k+1} + \alpha_2 \Delta_{n-k+2} + \dots + \alpha_k \Delta_n, n \geq 1,$$

where $\alpha_1, \alpha_2, \dots, \alpha_k > 0$ and $\sum\limits_{i=1}^{k} \alpha_i = \alpha < 1$.

By Lemma 2.2, there exist $L > 0$ and $\theta \in (0, 1)$ such that $\Delta_n \leq L\theta^n, n \geq 1$, that is,

$$|y_{n-1} - y_n| \leq L\theta^n, n \geq 1. \qquad (2.14)$$

For $n \geq 0$ and $p \geq 1$, in view of (2.14) we obtain

$$\begin{aligned} |y_n - y_{n+p}| &\leq |y_n - y_{n+1}| + |y_{n+1} - y_{n+2}| + \dots + |y_{n+p-1} - y_{n+p}| \leq \\ &\leq L\theta^{n+1} + L\theta^{n+2} + \dots + L\theta^{n+p} = L\theta^{n+1} \left(1 + \theta + \theta^2 + \dots + \theta^{p-1}\right), \end{aligned}$$

and so

$$|y_n - y_{n+p}| \leq L\theta^{n+1} \frac{1 - \theta^p}{1 - \theta}, n \geq 1, p \geq 1.$$

Since $\theta \in (0, 1)$, it follows that $\{y_n\}_{n\geq 0}$ is a Cauchy sequence, hence convergent.

We shall prove that $\{y_n\}_{n\geq 0}$ converges to $\overline{x}$. For $n \geq 0$ we have:

$$|y_{n+1} - \overline{x}| \leq |f(y_{n-k+1}, y_{n-k+2}, \dots, y_n) - f(\overline{x}, \overline{x}, \dots, \overline{x})| \leq$$
$$\leq |f(y_{n-k+1}, y_{n-k+2}, \dots, y_n) - f(y_{n-k+2}, y_{n-k+3}, \dots, y_n, \overline{x})| +$$
$$+ |f(y_{n-k+2}, y_{n-k+3}, \dots, y_n, \overline{x}) - f(y_{n-k+3}, \dots, y_n, \overline{x}, \overline{x})| +$$
$$+ \dots + |f(y_n, \overline{x}, \dots, \overline{x}) - f(\overline{x}, \overline{x}, \dots, \overline{x})| ,$$

so by (2.6) we obtain:

$$|y_{n+1} - \overline{x}| \leq [\alpha_1 |y_{n-k+1} - y_{n-k+2}| + \dots + \alpha_{k-1} |y_{n-1} - y_n| + \alpha_k |y_n - \overline{x}|] +$$
$$+ [\alpha_1 |y_{n-k+2} - y_{n-k+3}| + \dots + \alpha_{k-1} |y_n - \overline{x}| + \alpha_k |\overline{x} - \overline{x}|] +$$
$$+ \dots + [\alpha_1 |y_n - \overline{x}| + \alpha_2 |\overline{x} - \overline{x}| + \dots + \alpha_k |\overline{x} - \overline{x}|].$$

Now using (2.14) it follows that:

$$|y_{n+1} - \overline{x}| \leq [\alpha_1 L\theta^{n-k+2} + \alpha_2 L\theta^{n-k+3} + \dots + \alpha_{k-1} L\theta^n + \alpha_k |y_n - \overline{x}|] +$$
$$+ [\alpha_1 L\theta^{n-k+3} + \alpha_2 L\theta^{n-k+4} + \dots + \alpha_{k-2} L\theta^n + \alpha_{k-1} |y_n - \overline{x}| + \alpha_k \cdot 0] +$$
$$+ \dots$$
$$+ [\alpha_1 |y_n - \overline{x}| + 0] =$$
$$= \alpha_1 L\theta^{n-k+2} + (\alpha_1 + \alpha_2) L\theta^{n-k+3} + \dots + (\alpha_1 + \alpha_2 + \dots + \alpha_{k-1}) L\theta^n +$$
$$+ (\alpha_1 + \alpha_2 + \dots + \alpha_k) |y_n - \overline{x}| .$$

Finally we obtain that:

$$\begin{aligned} |y_{n+1} - \overline{x}| &\leq \alpha \left|y_n - \overline{\overline{x}}\right| + L\theta^n [\alpha_1 \theta^{2-k} + (\alpha_1 + \alpha_2) \theta^{3-k} + \dots + \\ &+ (\alpha_1 + \alpha_2 + \dots + \alpha_{k-2}) \theta + (\alpha_1 + \alpha_2 + \dots + \alpha_{k-1})], n \geq 1, \end{aligned}$$

where $\theta \in (0,1)$. Observe that

$$M = L[\alpha_1\theta^{2-k}+(\alpha_1+\alpha_2)\theta^{3-k}+\ldots+(\alpha_1+\alpha_2+\ldots+\alpha_{k-2})\theta+(\alpha_1+\alpha_2+\ldots+\alpha_{k-1})]$$

is a fixed positive number (since k is fixed).

Now, if we denote $a_n = |y_n - \overline{x}|$, $q = \alpha \in [0,1)$, $b_n = M\theta^n, n \geq 1$, and use Lemma 2.1 one obtains $|y_n - \overline{x}| \to 0$, as $n \to \infty$, that is, $\{y_n\}_{n\geq 0}$ converges to $\overline{x}$, the unique fixed point of f. □

Remark 2.2. Note that in the particular case $k = 1$, from Theorem 2.1 we get exactly the well-known Banach contraction mapping principle, as condition (2.6) reduces in this case to (2.15). Note that, in this particular case, the two sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ in Theorem 2.1 coincide.

Theorem 2.2. (Banach's contraction mapping principle in $\mathbb{R}$)

Let $E \subset \mathbb{R}$ be closed and $f : E \to E$ a function satisfying

$$|f(x) - f(y)| \leq \alpha\ |x - y|\ , \quad \textit{for all } x, y \in E\,, \tag{2.15}$$

where $0 \leq \alpha < 1$ is constant. Then:

1) f has a unique fixed point $\overline{x}$ in E which is the unique fixed point of any iterate f^n $(n > 1)$ of f;

2) The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = f(x_n)\,, \quad n = 0, 1, 2, \ldots \tag{2.16}$$

converges to $\overline{x}$, for any $x_0 \in E$.

Example 2.2. Let $E = [0, +\infty)$ and $f : A \to E$ be given by

$$f(x_1, x_2) = \sqrt{x_1 + 45} - \sqrt{x_2 + 5},\ (x_1, x_2) \in A,$$

where $A = \left\{(x_1, x_2) \in E^2 : x_1 - x_2 + 40 \geq 0\right\}$. Then, for all $(x_0, x_1), (x_1, x_2) \in A$ we have

$$|f(x_0, x_1) - f(x_1, x_2)| \leq \alpha_1 |x_0 - x_1| + \alpha_2 |x_1 - x_2|\,, \tag{2.17}$$

where $\alpha_1 = \dfrac{1}{6\sqrt{5}}$, $\alpha_2 = \dfrac{1}{2\sqrt{5}}$ and $\alpha_1 + \alpha_2 = \dfrac{2}{3\sqrt{5}} < 1$. which shows that the generalized contraction condition (2.6) is satisfied.

By Theorem 2.1 it follows that f has a unique fixed point in $E = [0, +\infty)$, i.e., $f(4, 4) = 4$.

It is possible to weaken the contractive condition (2.6), like in the next theorem, which is a particular variant of the main result in [16].

Theorem 2.3. *Let $E \subset \mathbb{R}$ be closed, k be a positive integer and $f : E^k \to E$ a function for which there exists $\lambda \in [0,1)$ satisfying*

$$|f(x_0, \ldots, x_{k-1}) - f(x_1, \ldots, x_k)| \leq \lambda \max\{|x_0 - x_1|, \ldots, |x_{k-1} - x_k|\}, \tag{2.18}$$

for all $x_0, \ldots, x_k \in E$.

Then there exists $\overline{x}$ in E such that $f(\overline{x}, \ldots, \overline{x}) = \overline{x}$ and the sequence $\{x_n\}_{n\geq 0}$ with $x_0, \ldots, x_{k-1} \in E$ arbitrary and

$$x_n = f(x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}), n \geq k, \tag{2.19}$$

converges to $\overline{x}$.

If, in addition, we suppose that the following condition

$$|f(x,\dots,x)-f(y,\dots,y)|<|x-y| \tag{2.20}$$

holds for all $x,y\in E$ with $x\neq y$, then $\overline{x}$ is the unique point in E with $f(\overline{x},\dots,\overline{x})=\overline{x}$ and, moreover, $\overline{x}$ is the unique fixed point of any iterate f^n of f:

$$f^n(\overline{x},\dots,\overline{x})=\overline{x},\ \forall n>1. \tag{2.21}$$

Example 2.3. ([16]) Let $E=[0,1]\cup[2,3]$ and $f:E^2\to E$ defined by: $f(x,y)=\dfrac{x+y}{4}$, if $(x,y)\in[0,1]\times[0,1]$, $f(x,y)=1+\dfrac{x+y}{4}$, if $(x,y)\in[2,3]\times[2,3]$ and $f(x,y)=\dfrac{x+y}{4}-\dfrac{1}{2}$, if $(x,y)\in[0,1]\times[2,3]$ or $(x,y)\in[2,3]\times[0,1]$. Then f satisfies (2.18) with $\lambda=\dfrac{1}{2}$ but does not satisfy (2.6), which follows by simply taking $x_0=0,x_1=2,x_2=2$. Indeed, if we admit that (2.6) would be satisfied, then we obtain:

$$2=|f(0,0)-f(2,2)|\leq|f(0,0)-f(0,2)|+|f(0,2)-f(2,2)|\leq$$

$$\leq\alpha_1\cdot0+\alpha_2\cdot2+\alpha_1\cdot2+\alpha_2\cdot0=(\alpha_1+\alpha_2)\cdot2<2,$$

a contradiction. Hence f does not satisfy (2.6).

Note also that, for $x=0$ and $y=2$, f in this example does not satisfy (2.20), and so it is not surprising that f has two fixed points: $f(0,0)=0$ and $f(2,2)=2$.

One can also formulate other fixed point theorems, more general than Theorems 2.1-2.3, by adapting to the real line most of the results in [12], [18]-[22] and [26], obtained there in the general setting of a complete metric space.

It is also possible to get the conclusion (2.21) by adapting to the case $f:E^k\to E$, some of the fixed point principles established in [3] for functions $f:E\to E$, see [13].

3. Solving nonlinear cyclic systems

We illustrate our method for two representative examples of cyclic systems of the form

$$\begin{cases} x_1=f(x_2,x_3,\dots,x_{k+1})\\ x_2=f(x_3,x_4,\dots,x_{k+2})\\ \dots\\ x_{n-1}=f(x_n,x_1,\dots,x_{k-1})\\ x_n=f(x_1,x_2,\dots,x_k),\end{cases} \tag{3.22}$$

where $f:E^k\to\mathbb{R}$ is a real function of k real variables, $E\subset\mathbb{R}$ and k is a positive integer satisfying $1\leq k<n$.

Problem 3.1. *Solve in $\mathbb{R}^3$ the system*

$$\begin{cases} x=\sqrt{y+23}-\sqrt{y+7}\\ y=\sqrt{z+23}-\sqrt{z+7}\\ z=\sqrt{x+23}-\sqrt{x+7}.\end{cases} \tag{3.23}$$

Solution. Observe that the system (3.23) is of the form (3.22) with $n = 3$, $k = 1$ and $f(x) = \sqrt{x+23} - \sqrt{x+7}$, that is, it can be equivalently written in the form

$$x = f(y);\ y = f(z);\ z = f(x),$$

which shows that, if (a_1, a_2, a_3) is a solution of (3.23), then a_1, a_2, a_3 are periodic points of period 3 of f, that is, a_1, a_2, a_3 are solutions of the equation $a = f^3(a)$. As f satisfies (2.15) with $\alpha = \dfrac{1}{2\sqrt{23}} + \dfrac{1}{2\sqrt{7}} < 1$, by Theorem 2.2 it follows that the equation $x = f(x)$ has a unique solution $\overline{x}$ which is also the unique solution of $x = f^3(x)$.

By observing that $\overline{x} = 2$ is a solution of $x = \sqrt{x+23} - \sqrt{x+7}$, we deduce that $a_1 = a_2 = a_3 = 2$ and so $(2, 2, 2)$ is the unique solution of the system (3.23).

Problem 3.2. *Solve in* $\mathbb{R}^3$ *the system*

$$\left\{ \begin{array}{l} x = \sqrt{y+23} - \sqrt{z+7} \\ y = \sqrt{z+23} - \sqrt{x+7} \\ z = \sqrt{x+23} - \sqrt{y+7}. \end{array} \right. \tag{3.24}$$

Solution. Note that, despite the clear similarity between (3.23) and (3.24), the system (3.24) is however totally different from (3.23), as it is of the form (3.22), with $n = 3$, $k = 2$ and $f : B \to E$ is given by

$$f(x_1, x_2) = \sqrt{x_1+23} - \sqrt{x_2+7},\ (x_1, x_2) \in B,$$

where $E = [0, \infty)$ and $B = \left\{(x_1, x_2) \in E^2 : x_1 - x_2 + 16 \geq 0\right\}$. It is not difficult to show that f satisfies (2.6), that is, for all $(x_0, x_1), (x_1, x_2) \in B$, we have

$$|f(x_0, x_1) - f(x_1, x_2)| \leq \alpha_1 |x_0 - x_1| + \alpha_2 |x_1 - x_2|, \tag{3.25}$$

where $\alpha_1 = \dfrac{1}{2\sqrt{23}}$ and $\alpha_2 = \dfrac{1}{2\sqrt{7}}$, with $\alpha_1 + \alpha_2 < 1$.

Then, by Theorem 2.1 it follows that $f(x, x) = \sqrt{x+23} - \sqrt{x+7}$ has a unique fixed point $\overline{x}$ în $E = [0, \infty)$, which is also the unique fixed point of $f^n(x, x)$, for any $n > 1$.

By Problem 3.1 we know that $\overline{x} = 2$ and hence in view of Theorem 2.1, $\overline{x} = 2$ is the unique solution of the equation $x = f^3(x, x)$, that is, f has no other periodic points except for its unique fixed point.

Now, as an essential difference from the case of system (3.23), we have to show that, if (a_1, a_2, a_3) is a solution of the system (3.24), then a_1, a_2, a_3 are periodic points of period 3 of f, which is not at all immediate, as in the case of system (3.23). Using condition (3.25) and the equations of the system we find out that

$$|a_1 - a_2| = |f(a_2, a_3) - f(a_3, a_1)| \leq \alpha_1 |a_2 - a_3| + \alpha_2 |a_3 - a_1|,$$

$$|a_2 - a_3| = |f(a_3, a_1) - f(a_1, a_2)| \leq \alpha_1 |a_3 - a_1| + \alpha_2 |a_1 - a_2|,$$

$$|a_3 - a_1| = |f(a_1, a_2) - f(a_2, a_3)| \leq \alpha_1 |a_1 - a_2| + \alpha_2 |a_2 - a_3|,$$

from which, by summing up, one obtains:

$$S \leq (\alpha_1 + \alpha_2) \cdot S,$$

where we denoted $S = |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_1|$.

Since $\alpha_1 + \alpha_2 = \alpha < 1$, the inequality above necessarily implies that

$$|a_1 - a_2| + |a_2 - a_3| + |a_3 - a_1| = 0,$$

which means that $a_1 = a_2 = a_3 = a$, i.e., $a = f^3(a, a)$ and this shows that a_1, a_2, a_3 are indeed periodic points of f.

Since by Theorem 2.1, f has no other periodic points except for its fixed points, it follows that $(2, 2, 2)$ is the unique solution of the system (3.24).

For other examples with non unique fixed point and illustrative applications of more fixed point principles in solving cyclic systems, we refer to our recent paper [13].

Note that for the case $k = 1$, the method presented in this section has been introduced and illustrated by various examples in [3], [7] and [8].

4. Convergence of nonlinear recurrent sequences

In order to study the convergence of nonlinear recurrent sequences, we simply apply a fixed point principle, like Theorem 2.1, Theorem 2.2 or Theorem 2.3, by showing that this sequence is in fact the iterative method used to approximate the respective fixed points.

We present two simple sequences related to the cyclic systems treated in the previous section.

Problem 4.3. *Study the convergence of the sequence $\{x_n\}_{n\geq 0}$ defined by $x_0 \in [0, +\infty)$ and*

$$x_{n+1} = \sqrt{x_n + 45} - \sqrt{x_n + 5}, n \geq 0. \tag{4.26}$$

Solution. The sequence $\{x_n\}_{n\geq 0}$ in this problem is just the Picard iteration corresponding to the fixed point problem

$$x = f(x)$$

with $E = [0, +\infty)$ and $f(x) = \sqrt{x + 45} - \sqrt{x + 5}$. It is easy to show that f satisfies condition (2.15) with $\alpha = \dfrac{1}{6\sqrt{5}} + \dfrac{1}{2\sqrt{5}} < 1$ and hence by Theorem 2.2, it follows that $\{x_n\}_{n\geq 0}$ converges to the unique fixed point of f, which is $\overline{x} = 4$.

Problem 4.4. *Study the convergence of the sequence $\{x_n\}_{n\geq 0}$ defined by x_0, $x_1 \in [0, +\infty)$, $x_0 - x_1 \leq 40$ and*

$$x_{n+1} = \sqrt{x_n + 45} - \sqrt{x_{n-1} + 5}, n \geq 1. \tag{4.27}$$

Solution. Here we have $E = [0, +\infty)$ and $f : A \to E$ is given by

$$f(x_1, x_2) = \sqrt{x_1 + 45} - \sqrt{x_2 + 5},\ x_1, x_2 \in [0, +\infty),$$

where $A = \big\{(x_1, x_2) \in E^2 : x_1 - x_2 + 40 \geq 0\big\}$. It is easy to prove that f satisfies (2.18), that is, for all $(x_0, x_1), (x_1, x_2) \in A$,

$$|f(x_0, x_1) - f(x_1, x_2)| \leq \alpha_1 |x_0 - x_1| + \alpha_2 |x_1 - x_2|\,, \tag{4.28}$$

where $\alpha_1 = \dfrac{1}{6\sqrt{5}}$ and $\alpha_2 = \dfrac{1}{2\sqrt{5}}$ with $\alpha_1 + \alpha_2 = \dfrac{2}{3\sqrt{5}} < 1$.

Observe that the sequence $\{x_n\}_{n\geq 0}$ defined by (4.27) is actually the two-step iterative method

$$x_{n+1} = f(x_n, x_{n-1}),\ n \geq 1,$$

corresponding to the fixed point problem $x = f(x,x)$ and hence, by applying Theorem 2.1, we obtain that $\{x_n\}_{n\geq 0}$ converges to 4, the unique fixed point of f.

Now we invite the reader to apply the appropriate fixed point tool, see also the ones in [13], in order to study the convergence of the following nonlinear recurrent sequences:

$$3^{x_{n+1}} + 4^{x_n} = 5^{x_{n-1}},\ n \geq 1;\ x_0,\ x_1 \in \mathbb{R}.$$

$$x_{n+1} = 2x_{n-1}^3 - 7x_n^2 + 5x_{n-1} + 3x_n - 2,\ n \geq 1;\ x_0,\ x_1 \in \mathbb{R}.$$

and

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_{n-1}}\right),\ n \geq 1;\ x_0,\ x_1 \in (0, +\infty).$$

5. Conclusions and further directions of study

Let us note that, the argument used in solving the Problem 3.2, that is the proof of the fact that if (a_1, a_2, a_3) is a solution of the system (3.24), then we necessarily have $a_1 = a_2 = a_3$, is actually an independent method to solve that problem, without using any concept from fixed point theory.

The merit of our unified method is mainly the fact that it draws attention to the general principles that are behind this empirical method.

Secondly, we have to stress on the fact that not all cyclic systems of the form (3.22) do have the afore mentioned property. For example, the following cyclic system, see also [13] and [17],

$$x^2 + 2yz - 6x + 3 = 0; y^2 + 2zx - 6y + 3 = 0; z^2 + 2xy - 6z + 3 = 0, \qquad (5.29)$$

apart of the solution $(1, 1, 1)$ with all components equal, also admits the solutions $(-1, 5, -1)$, $(5, -1, -1)$, $(-1, -1, 5)$, which clearly do not satisfy the condition $a_1 = a_2 = a_3$ but, however, have two of the three components equal.

The open question is now the following: could we adapt the fixed point principles presented in this article (and also those in [13]) in such a way to obtain a more reliable method that could be used for solving cyclic systems of the form (5.29) or for studying recurrent sequences defined by similar equations?

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