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A generalization of Mortici lemma

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ABSTRACT. The aim of this note is to obtain a generalization of a very simple, elegant but powerful convergence lemma introduced by Mortici [Mortici, C., *Best estimates of the generalized Stirling formula*, Appl. Math. Comp., **215** (2010), no. 11, 4044–4048; Mortici, C., *Product approximations via asymptotic integration*, Amer. Math. Monthly, **117** (2010), no. 5, 434–441; Mortici, C., *An ultimate extremely accurate formula for approximation of the factorial function*, Arch. Math. (Basel), 93 (2009), no. 1, 37–45; Mortici, C., *Complete monotonic functions associated with gamma function and applications*, Carpathian J. Math., **25** (2009), no. 2, 186–191] and exploited by him and other authors in an impressive number of recent and very recent papers devoted to constructing asymptotic expansions, accelerating famous sequences in mathematics, developing approximation formulas for factorials that improve various classical results etc.

We illustrate the new result by some important particular cases and also indicate a way for using it in similar contexts.

1. INTRODUCTION

C. Mortici introduced the following simple, elegant but powerful lemma which was first used in [11], [12], [7], [8] to accelerate some sequences and to construct asymptotic expansions, and afterwards was intensively used in [7]-[51] and many other related papers. We state here this lemma as given and proved by its author in [51].

Lemma 1.1. (Mortici, [51])
If $\{\omega_n\}_{n\geq 1}$ is convergent to 0 and there exists the limit

$$\lim_{n\to\infty} n^k(\omega_n - \omega_{n+1}) = l \in \mathbb{R}, \tag{1.1}$$

with $k > 1$, then there exists the limit

$$\lim_{n\to\infty} n^{k-1}\omega_n = \frac{l}{k-1}. \tag{1.2}$$

Remark 1.1. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers converging to a and b, respectively, for which there exists the limit

$$\lim_{n\to\infty} \frac{a_n - a}{b_n - b} = L \in \mathbb{R}.$$

If $L \neq 0$, then we say (see for example [1] and [2]) that $\{a_n\}$ and $\{b_n\}$ *have the same rate (order) of convergence*. If $L = 0$, then we say that $\{a_n\}$ *converges faster* to a than $\{b_n\}$ converges to b.

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In view of the above concept, the interpretation of Lemma 1.1 is the following one: if the sequences $\{\omega_n - \omega_{n+1}\}_{n\geq 1}$ and $\left\{\dfrac{1}{n^k}\right\}$, with $k > 1$, have the same order of convergence, then the same happens with the sequences $\{\omega_n\}_{n\geq 1}$ and $\left\{\dfrac{1}{n^{k-1}}\right\}$.

Lemma 1.1 is actually a consequence of the classical Cesaro-Stolz lemma for the case $(0/0)$ (see [5]), an auxiliary result that has been used by Mortici in [29] directly and not in the form of the derived Lemma 1.1:

Lemma 1.2. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers with the following properties*
(i) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_n = 0$;
(ii) $\{b_n\}$ *is strictly increasing;*
(iii) there exists the limit

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = l \in \overline{\mathbb{R}}.$$

Then there exists the limit of the sequence $\dfrac{a_n}{b_n}$ *and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

In applications, however, Lemma 1.1 is more powerful than Lemma 1.2, as shown by the following selected examples which are taken from papers that use Lemma 1.1.

- In the paper [10] Mortici considers the sequence $\{\omega_n\}_{n\geq 1}$ given by

$$\omega_n = \ln\Gamma(n) - \ln\sqrt{2\pi} + b - \ln(n+b) + n + \frac{1}{2}\Psi(n+c),$$

  where Γ is the Euler gamma function, $\Psi = \Gamma'/\Gamma$ is the digamma function and a, b, c are some parameters.
- In [25] Mortici considers the sequence $\{\omega_n\}_{n\geq 1}$ given by

$$\omega_n = \ln\Gamma(n+1) - \ln\sqrt{2\pi} + p - n\ln(n+p) + n - \frac{1}{6}\ln(n^3 + an^2 + bn + c),\ p \in [0,1],$$

  where a, b, c are parameters depending on p;
- In [14] Mortici considers the sequence $\{\omega_n\}_{n\geq 1}$ defined either by the equality

$$n! = \sqrt{\frac{2\pi}{e}}\left(\frac{n+2}{e}\right)^{n+\frac{3}{2}} \cdot \frac{1}{n+a} \cdot e^{\omega_n},$$

  or by

$$n! = \sqrt{\frac{2\pi}{e}}\left(\frac{n+3}{e}\right)^{n+\frac{5}{2}} \cdot \frac{1}{n^2+bn+c} \cdot e^{\omega_n},$$

  where a, b, c are some parameters;
- In [11] Mortici considers the sequence $\{\omega_n\}_{n\geq 1}$ defined by the equality

$$n! = \sqrt{2\pi}e^{b-a}\left(\frac{n+a}{e}\right)^{n+b} \cdot (n+c)^{\frac{1}{2}-b} \cdot e^{\omega_n},$$

  with a, b parameters;

- In [32] Mortici considers the sequence $\{\omega_n\}_{n\geq 1}$ given by

$$\omega_n(a,b,c)=\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-2}+\frac{b}{a+n-1}-\ln\left(\frac{a+n-1}{a}+c\right),$$

with $a,b,c\in\mathbb{R}$ such that $c>-\dfrac{a+1}{a}$.

- In [30] Mortici considers the sequence $\{\omega_n\}_{n\geq 1}$ given by

$$\omega_n=1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln\frac{n^2+an+b}{n+c},$$

with $a,b,c\in\mathbb{R}$ parameters.

- In [30] Mortici considers the sequence $\{\omega_n\}_{n\geq 1}$ given by

$$\omega_n=1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln\frac{n^2+an+b}{n+c},$$

with $a,b,c\in\mathbb{R}$ parameters.

- In [3] Chen considers the sequence $\{\omega_n\}_{n\geq 1}$ given by

$$\omega_n=\Psi(n+1)-\ln\left(n+\frac{1}{2}\right)+q\,\Psi''(n+p),$$

where Γ is the Euler gamma function, $\Psi=\Gamma'/\Gamma$ is the digamma function and p,q are parameters.

- In [4] Chen considers the sequence $\{\omega_n\}_{n\geq 1}$ given by

$$\omega_n=\sum_{k=0}^{n}\frac{1}{16^k}\binom{2k}{k}^2-c_0+\frac{b}{n+c},$$

where c_0 is given and b,c are parameters.

Many other papers in the list [7]-[51] are using directly Lemma 1.1, while Lemma 1.2 was used in significantly less cases, amongst which we mention the paper [29].

2. Main result

Lemma 2.3. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive real numbers such that the series $\sum\limits_{n=1}^{\infty}a_n$ converges to S with the same rate as $\{b_n\}$ converges to 0.*

If $\{\omega_n\}_{n\geq 1}$ is a sequence of real numbers converging to ω such that $\{\omega_n-\omega_{n+1}\}_{n\geq 1}$ and $\{a_n\}$ have the same rate of convergence, then $\{\omega_n-\omega\}_{n\geq 1}$ and $\{b_n\}$ have the same rate of convergence, too.

Proof. Although the proof easily follows by Cesaro-Stolz lemma in the case $(0/0)$, we give here an independent proof, as the proof of Lemma 1.2 cannot be found easily in textbooks, under this form, except for the book [5] which is very scarce, even in Russian. Let, by hypothesis, $l,L\in\mathbb{R}$ be such that

$$\lim_{n\to\infty}\frac{\omega_n-\omega_{n+1}}{a_n}=l\in\mathbb{R}\tag{2.3}$$

and, respectively,

$$\lim_{n\to\infty}\frac{S-\sum\limits_{i=1}^{n-1}a_i}{b_n}=L\in\mathbb{R}.\tag{2.4}$$

Let $\epsilon > 0$. Then there exists $n_0 = n_0(\epsilon)$ such that

$$l - \epsilon \leq \frac{\omega_n - \omega_{n+1}}{a_n} \leq l + \epsilon,\ n \geq n_0. \tag{2.5}$$

If we write (2.5) in the equivalent form (note that $a_n > 0$)

$$(l - \epsilon)a_n \leq \omega_n - \omega_{n+1} \leq (l + \epsilon)a_n,\ n \geq n_0,$$

and sum up the inequalities obtained for $n, n+1, \ldots, n+p-1$ $(p \geq 2)$, one obtains

$$(l - \epsilon) \sum_{i=n}^{n+p-1} a_i \leq \omega_n - \omega_{n+p} \leq (l + \epsilon) \sum_{i=n}^{n+p-1} a_i,\ n \geq n_0. \tag{2.6}$$

By letting $p \to \infty$ in (2.6), we obtain

$$(l - \epsilon)\left(S - \sum_{i=1}^{n-1} a_i\right) \leq \omega_n - \omega \leq (l + \epsilon)\left(S - \sum_{i=1}^{n-1} a_i\right),\ n \geq n_0. \tag{2.7}$$

Now, divide (2.7) by $b_n > 0$ and use (2.4) to get, by letting $n \to \infty$ in the resulted inequality,

$$\lim_{n\to\infty} \frac{\omega_n - \omega}{b_n} = l \cdot L, \tag{2.8}$$

as required. □

Remark 2.2. In the particular case $a_n = \dfrac{1}{n^k}$ $(k > 1)$, $b_n = na_n$ and $\omega = 0$, by Lemma 2.3 we get Lemma 1.1, i.e., the Lemma 2.1 in [32]. Essential here is the fact that the series $\sum\limits_{n=1}^{\infty} a_n$ involved in Lemma 2.3 will be in this particular case the generalized harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n^k}$$

which is known to converge to Riemann zeta function $\zeta(k)$, for $k > 1$, see [6], with the convergence order given by the relation

$$\lim_{n\to\infty} n^{k-1}\left(\zeta(k) - \sum_{i=1}^{n-1} \frac{1}{i^k}\right) = \frac{1}{k-1}.$$

A particular case. As shown in the first section, in the paper [10] the sequence $\{\omega_n\}_{n\geq 1}$ is given by

$$\omega_n = \ln\Gamma(n) - \ln\sqrt{2\pi} + b - \ln(n+b) + n + \frac{1}{2}\Psi(n+c),$$

where Γ is the Euler gamma function, $\Psi = \Gamma'/\Gamma$ is the digamma function and a, b, c are some parameters, while the sequences $\{a_n\}$ and $\{b_n\}$ are

$$a_n = \frac{1}{n^k} \text{ and } b_n = \frac{1}{n^{k-1}},\ \text{for } k = 2, 3, 4,$$

depending on the values of parameters a, b, c (there exist three distinct cases).

3. Conclusions

Lemma 1.1 is based essentially on the assumption $k > 1$. If $0 < k \leq 1$, then it cannot be applied, since in this case the generalized harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n^k}$$

is divergent. In such cases, we could be interested to find instead of generalized harmonic series another "comparison" series which is still convergent, even though its order of convergence is very slow.

On the contrary, we could be interested to use "comparison" sequences which converge faster than the sequence $\left\{\frac{1}{n^k}\right\}$, used overall in the papers that use Lemma 1.1. For example, as it is known, see [1], the recursive sequence that calculates $\sqrt{a}$ for $a > 0$, that is,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right),\ n \geq 0,\ x_0 > 0,$$

converges to $\sqrt{a}$ faster than any of the sequences $\left\{\frac{1}{n^k}\right\}, k = 1, 2, \ldots.$

These contexts would require the use of our Lemma 2.3 rather than of Lemma 1.1.

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