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Applications of the PL homotopy algorithm for the computation of fixed points to unconstrained optimization problems

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ABSTRACT. This paper describes the main aspects of the "piecewise-linear homotopy method" for fixed point approximation proposed by Eaves and Saigal [Eaves, C. B. and Saigal, R., *Homotopies for computation of fixed points on unbounded regions*, Mathematical Programming, **3** (1972), No. 1, 225–237]. The implementation of the method is developed using the modern programming language C# and then is used for solving some unconstrained optimization problems. The PL homotopy algorithm appears to be more reliable than the classical Newton method in the case of the problem of finding a local minima for Schwefel's function and other optimization problems.

1. INTRODUCTION

There exists a vast literature on the iterative approximation of fixed points, see for example the recent monographs [2], [4] and [5] and references therein. The fundamental problem of this field of research could be briefly stated as follows.

We have to solve a certain nonlinear fixed point equation

$$x = Tx, \tag{1.1}$$

where T is a given self operator of a space X. Suppose X and T are such that the equation (1.1) has at least one solution (usually called a *fixed point* of T). A typical situation of this kind is illustrated by the well known Brouwer's fixed point theorem, see [10].

Theorem 1.1. *Every continuous function f from a convex compact subset K of a Euclidean space to K itself has a fixed point.*

Under the assumptions of Theorem 1.1, the Picard iteration associated to (1.1), defined by $x_0 \in X$ and

$$x_{n+1} = Tx_n,\ n = 0, 1, 2, \dots, \tag{1.2}$$

which is successfully used in many cases to solve nonlinear fixed point equations, does not converge, in general.

This is the reason for which several authors tried to find appropriate algorithms to compute fixed points of continuous mappings like the ones in Theorem 1.1, see [4].

In 1967 Herbert Scarf proposed a method for approximating fixed points of continuous mappings [15]. The algorithm proposed by Scarf, which is also a numerically implementable constructive proof of the Brouwer fixed point theorem, has its origins in

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the Lemke-Howson complementary pivoting algorithm for solving linear complementarity problems [12]. Beside the generalization and applications in fixed point theory, the Lemke-Howson algorithm is also famous for its applications in finding Nash equilibrium points for bimatrix games. Several improvements to the algorithm developed by Scarf were made by Terje Hansen in 1967, see [16] and by Harold W. Kuhn in 1968 [11]. But the decisive advancements came in 1972, when Eaves [7] and then Eaves and Saigal [8] described a piecewise-linear (PL) homotopy deformation algorithm as an improvement for the algorithm proposed by Scarf. Another PL algorithm, related to the one proposed by Eaves and Saigal, was presented by Orin H. Merrill in 1972 [13]. The main practical advantage of the PL homotopy methods is that they don't require smoothness of the underlying map, and in fact they can be used to calculate fixed points of set-valued maps. Although PL methods can be viewed in the more general context of complementary pivoting algorithms usually are considered in the special class of homotopy or continuation methods [1].

Starting from this background, we will describe in this paper the main aspects of the piecewise-linear homotopy method for fixed point approximation proposed by Eaves and Saigal. A detailed description of the implementation of the algorithm using the modern programming language C# is given in [3]. This implementation of the algorithm is used for solving some unconstrained optimization problems and we compare the results with results obtained using the classical Newton's method.

2. Piecewise-linear homotopy algorithms

The homotopy methods are useful alternatives and aides for the Newton methods in solving systems of n nonlinear equations in n variables:

$$F(x) = 0, \quad F : \mathbb{R}^n \to \mathbb{R}^n. \tag{2.3}$$

mainly when very little a priori knowledge regarding the zero points of F is available and so, a poor starting value could cause a divergent Newton iteration sequence. The idea of the homotopy method is to consider a new function $G : \mathbb{R}^n \to \mathbb{R}^n$, related to F, with a known solution, and then to gradually deform this new function into the original function F. Typically one can define the convex homotopy:

$$H(x,t) = t \cdot G(x) + (1-t) \cdot F(x) \tag{2.4}$$

and can try to trace the implicitly defined curve

$$H^{-1}(0) = \{x \in \mathbb{R}^n \mid \exists\, t \in [0,1] \text{ such that } H(x,t) = 0\} \tag{2.5}$$

from a starting point $(x_0, 1)$ to a solution point $(x^*, 0)$. The implicit function theorem ensures that the set $H^{-1}(0)$ is at least locally a curve under the assumption that $(x_0, 1)$ is a regular value of H, i.e. the Jacobian $H'(x_0, 1)$ has full rank n. However, because there is no smoothness condition on F, a more complex approach involving piecewise-linear approximations is needed.

We define the "refining" triangulation J_3 of $\mathbb{R}^n \times (0,1]$ such that the vertices of this triangulation are given by the set of points:

$$J_3^0 = \{(v_1, \ldots, v_{n+1}) \mid v_{n+1} = 2^{-k}, k \in \mathbf{N} \text{ and } \frac{v_i}{v_{n+1}} \in \mathbb{Z}\}.$$

So, every $(n+1)$-simplex of this triangulation is contained in some slab $\mathbb{R}^n \times [2^{-k}, 2^{-k-1}]$, $k \in \mathbf{N}$.

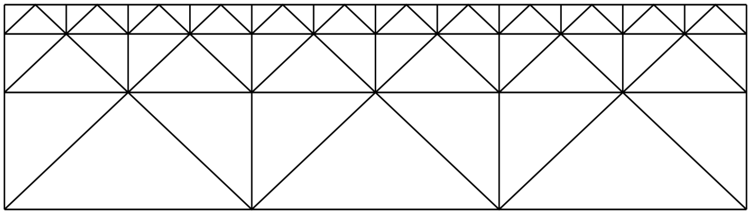

FIGURE 1. J_3 triangulation of $\mathbb{R} \times \mathbb{R}$

Let $\sigma = [v_1, v_2, \ldots, v_{n+1}, v_{n+2}] \in J_3$ be an $(n+1)$-simplex and let $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be the following canonical projection: $\pi(x,t) = t$. We define the level of σ as $\max_{i=\overline{1,n+2}} \pi(v_i)$, which is the maximum of the last co-ordinates of all vertices of σ. We call J_3 a refining triangulation of $\mathbb{R}^n \times \mathbb{R}$ because the diameter of σ tends to zero as the level of σ tends to zero.

We define the piecewise linear homotopy map H_{J_3} which interpolates H on the vertices of the given refining triangulation J_3:

- $H_{J_3}(x,1) = G(x)$
- $H_{J_3}(x,0) = F(x)$
- $H_{J_3}(x,t) = \sum_{i=1}^{n+2} \lambda_i H(v_i,t)$, where:
  (v_i,t) are vertices of a simplex $\sigma \in J_3$
  $(x,t) = \sum_{i=1}^{n+2} \lambda_i (v_i,t)$, $\quad \sum_{i=1}^{n+2} \lambda_i = 1$, $\lambda_i \geq 0$.

The algorithm will trace the unique component of the polygonal path $H_{J_3}^{-1}(0)$ which contains $(x_0,1)$, with nodes on the n-faces of the triangulation J_3.

The algorithm starts with the unique simplex σ_0 which contains the initial point $(x_0,1)$. Then, for each $i = 0, 1, \ldots$ it will perform the following steps in a loop:

- It will trace the restriction of $H_{J_3}^{-1}(0)$ to the current simplex σ_i, from the point (x_i,t_i) and finds the intersection point (x_{i+1},t_{i+1}) with some other facet of σ_i. This step is called "door-in-door-out step", see [1]. Sometimes this step is also called linear programming step because it involves the solving of linear equations in a manner typical for linear programming methods.
- It performs a pivoting step, which means to find the new simplex σ_{i+1} which is adjacent to the current simplex and which contains the point (x_{i+1},t_{i+1}). This step is usually performed using only a few operations which define the pivoting rules of the triangulation.

The generated sequence $(x_0,1), (x_1,t_1), \ldots$ will converge to a solution $(x^*,0)$ of the homotopy map H.

3. APPLICATIONS IN UNCONSTRAINED OPTIMIZATION

The implementation of the PL homotopy algorithm is used in this section to solve some unconstrained optimization problems which are well known as test problems for numerical algorithms. The results are then compared with the results obtained for the same problems by using the classical Newton's method.

3.1. **Example 1.** The PL homotopy algorithm is used to find the minimum of the following function

$$F(x) = x \cdot \arctan(x^3 - 2x - 7). \quad (3.6)$$

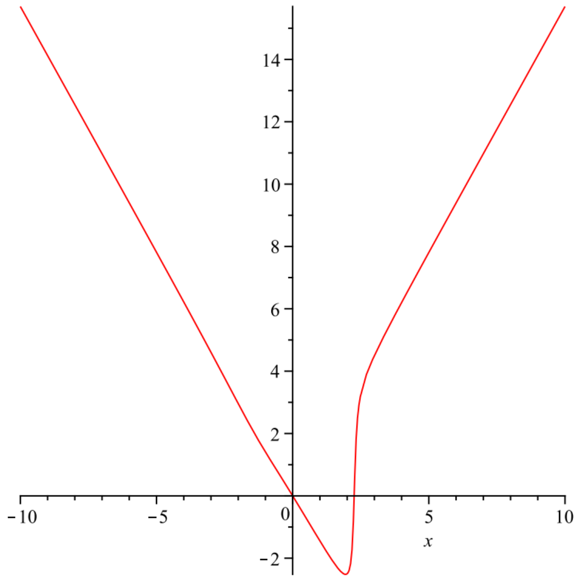

FIGURE 2. Visualization of $x \cdot \arctan(x^3 - 2x - 7)$

In order to solve the equation the following homotopy $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined and used in the algorithm

$$H(x,t) = \begin{cases} x - x_0 & \text{for } t \leq 0, \\ F'(x) & \text{for } t > 0. \end{cases} \quad (3.7)$$

The following table presents the number of iterations required by Newton's method and PL homotopy method to obtain an the minimum of the function $F(x)$ using different initial approximations x_0.

x_0	Newton	PL Homotopy
0	-	7
1	-	6
2	4	3
3	-	5
4	-	7
100	-	14
1000	-	17

3.2. **Example 2.** The PL homotopy algorithm is used to find a local minima of Schwefel's function [14]

$$F(x,y) = -x \cdot \sin(\sqrt{|x|}) - y \cdot \sin(\sqrt{|y|}) \quad (3.8)$$

In order to solve the problem the following homotopy map is defined and used in the algorithm

$$H : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2, \; H(x,t) = \begin{cases} x - x_0 & \text{for } t \leq 0, \\ \nabla F(x) & \text{for } t > 0. \end{cases} \quad (3.9)$$

Schwefel's function is is a well known benchmark optimization problem and presents many local minima and maxima, the closest local minima to the origin being $(\frac{5\pi}{3}, \frac{5\pi}{3})$. It is important to notice that the point $(0, 0)$ can not be used as initial approximation for Newton's method. This is not the case with the PL homotopy method, which also works with this initial value. Another interesting fact, outlined in the the following table, is that the Newton's method converges very slowly when the initial approximation is near $(0, 0)$, but the PL homotopy method is efficient also in this cases.

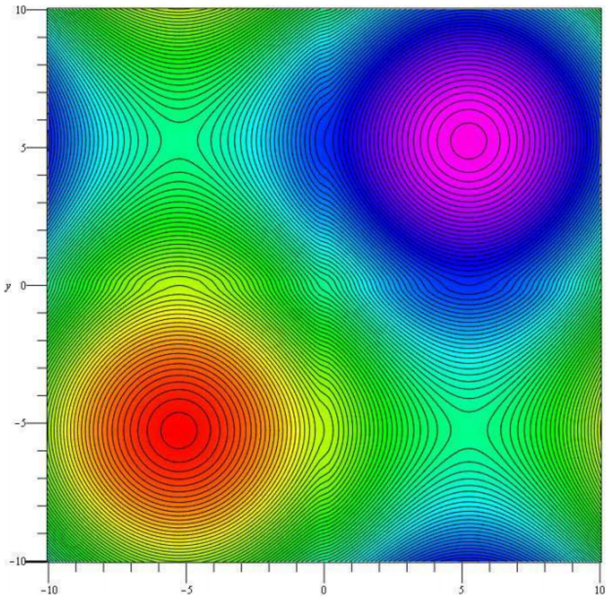

FIGURE 3. Visualization of Schwefel's function

The following table presents the number of iterations required by Newton's method and PL homotopy method to obtain an approximate local minima of Schwefel's function using different initial approximations (x_0, y_0).

x_0, y_0	Newton	PL Homotopy
5, 5	3	9
4, 4	4	17
3, 3	5	24
2, 2	6	31
1, 1	5	39
0.1, 0.1	15	44
0.01, 0.01	97	46
0.001, 0.001	908	46
0.0001, 0.0001	9009	46
0, 0	-	46

4. CONCLUSIONS AND FUTURE WORK

The main practical advantages of the PL homotopy methods is that they don't require smoothness of the underlying map. Also another important feature of these methods is that they can be applied when no a priori knowledge regarding the solutions of the system to be solved is available.

The implementation of this algorithm in a modern programming language will make possible the study of the newer developments related to the piecewise-linear homotopy methods, see for example [9]. Also as an important research direction is to study the feasibility of parallelizing some parts of this algorithm.

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