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Generalized distances and their associate metrics. Impact on fixed point theory

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ABSTRACT. In the last years there is an abundance of fixed point theorems in literature, most of them established in various generalized metric spaces. Amongst the generalized spaces considered in those papers, we may find: cone metric spaces, quasimetric spaces (or b-metric spaces), partial metric spaces, G-metric spaces etc. In some recent papers [Haghi, R. H., Rezapour, Sh. and Shahzad, N., *Some fixed point generalizations are not real generalizations*, Nonlinear Anal., **74** (2011), 1799-1803], [Haghi, R. H., Rezapour, Sh. and Shahzad, N., *Be careful on partial metric fixed point results*, Topology Appl., **160** (2013), 450-454], [Samet, B., Vetro, C. and Vetro, F., *Remarks on G-Metric Spaces*, Int. J. Anal., Volume 2013, Article ID 917158, 6 pages http://dx.doi.org/10.1155/2013/917158], the authors pointed out that some of the fixed point theorems transposed from metric spaces to cone metric spaces, partial metric spaces or G-metric spaces, respectively, are sometimes not real generalizations. The main aim of the present note is to inspect what happens in this respect with b-metric spaces.

1. INTRODUCTION

Many of the most important nonlinear problems of applied mathematics reduce to solving a given equation which in turn may be reduced to finding the fixed points of a certain mapping or the common fixed points of two mappings.

A typical situation is illustrated by the Banach contraction mapping principle, which is a classical and powerful tool in nonlinear analysis, being used for solving several classes of nonlinear functional equations, and which can be briefly stated as follows.

Theorem 1.1. *Let (X, d) be a metric space and $T : X \to X$ a self mapping. If (X, d) is complete and T is a contraction, i.e., there exists a constant $a \in [0, 1)$ such that*

$$d(Tx, Ty) \le a\, d(x, y), \text{ for all } x, y \in X, \tag{1.1}$$

then T has a unique fixed point p and, for any $x_0 \in X$, the Picard iteration $\{T^n x_0\}$ converges to p.

The Banach contraction mapping principle has been generalized in several directions, see for example [4], [23] and [24], for recent surveys. This explains why the study of fixed and common fixed points of mapping satisfying certain contractive conditions attracted many researchers and stimulated an impressive research work in this field in the last four decades, see for example [23], [24].

One of the most productive directions of generalizing the contraction mapping principle was to extend the ambient space, i.e., to consider various generalized metric spaces instead of usual metric spaces. Amongst the generalized spaces considered by several authors we mention: cone metric spaces, quasimetric spaces (or b-metric spaces), partial

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metric spaces, G-metric spaces, w-metric spaces, τ-metric spaces etc. see [3]-[7], [17]-[93] and references therein.

But, as shown in some very recent papers [17], [18], [68], it is necessary to work cautiously in such spaces otherwise one could fall into a trap, because most of the fixed point theorems transposed from metric spaces to generalized metric spaces are not real generalizations. The papers [17], [18] point out such trivial generalizations in the case of cone metric spaces and partial metric spaces, respectively, while in [68] the authors study the same problem but for G-metric spaces.

Starting from these facts, the main aim of the present note is to inspect whether or not a similar situation may happen in the case of b-metric spaces.

2. Distance spaces

We use the terminology from [14]. We put $\mathbb{N} = \{1, 2, ...\}$ and $\omega = \{0, 1, 2, ...\}$. Let $cl_X A$ be the closure of the set A in the space X, and $|B|$ be the cardinality of the set B.

Definition 2.1. A distance space is a pair (X, ρ) consisting of a set X and a non-negative real-valued function ρ on the set $X \times X$ satisfying the following condition:

(D) $\rho(x, y) + \rho(y, x) = 0$ if and only if $x = y$.

A distance space (X, ρ) is said to be a symmetric space if it satisfies the following condition:

(S) $\rho(x, y) = \rho(y, x)$, for all $x, y \in X$.

Let (X, ρ) be a distance space. For any $x \in X$ and each $r > 0$ we put

$$B(x, r, \rho) = \{y \in X : \rho(x, y) < r\}.$$

The set $B(x, r, \rho)$ is the *ball with center x and radius r* or, simply, the r-ball about x. We say that a set $U \subseteq X$ is *ρ-open* if for any $x \in U$ we have $B(x, \epsilon, \rho) \subseteq U$ for some $\epsilon > 0$. The family $\mathcal{T}(\rho)$ of all ρ-open sets forms the topology of the distance space (X, ρ), i.e., the topology induced by the distance ρ.

Distinct concepts in the theory of distance spaces were discussed in [2, 20, 21, 22].

The distances ρ_1 and ρ_2 on a set X are called *equivalent distances* if for any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that for all $x, y \in X$:

- the inequality $\rho_1(x, y) \leq \delta$ implies $\rho_2(x, y) \leq \epsilon$;
- the inequality $\rho_2(x, y) \leq \delta$ implies $\rho_1(x, y) \leq \epsilon$.

Let (X, ρ) be a distance space. We say that $\{x_n \in X : n \in \mathbb{N}\}$ is a *Cauchy sequence* in (X, ρ) if for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $\rho(x_i, x_j) \leq \epsilon$ for all $i, j \geq n$. We say that a sequence $\{x_n \in X : n \in \mathbb{N}\}$ *converges* to a point $x \in X$ and we put $lim_{n\to\infty} x_n = x$ if $lim_{n\to\infty} \rho(x, x_n) = 0$. Let $Lim_{n\to\infty} x_n = \{x \in X : lim_{n\to\infty} x_n = x\}$.

(The sequence $\{x_n \in X : n \in \mathbb{N}\}$ is *trivial* or *almost-constant* if there exists m such that $x_n = x_m$ for all $n \geq m$.)

Example 2.1. Let $X = \{0\} \cup \{n^{-1} : n \in \mathbb{N}\}$, and $d : X \times X \to [0, +\infty)$ be given by $d(x, x) = 0$ for each $x \in X$, $d(0, n^{-1}) = d(n^{-1}, 0) = 2^{-n}$ and $d(n^{-1}, m^{-1}) = d(m^{-1}, n^{-1}) = 1$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Then (X, d) is a metrizable compact symmetric space.

The sequence $\{n^{-1} : n \in \mathbb{N}\}$ is a convergent but not Cauchy sequence. In such a symmetric space (X, d) any non-trivial sequence is not a Cauchy sequence.

A distance space (X, ρ) is *complete* if every Cauchy sequence in (X, ρ) is convergent to some point of X. If (X, ρ) is a complete distance space, then we say that the distance ρ is *complete*.

We mention the following general properties of distance spaces.

Proposition 2.1. *For any distance space (X, ρ) the following assertions are equivalent:*

1. The space $(X, \mathcal{T}_\rho)$ is a T_1-space;

2. $\rho(x, y) = 0$ if and only if $x = y$.

Proposition 2.2. *If the distances ρ_1 and ρ_2 on a set X are equivalent, then:*

1. $\mathcal{T}(\rho_1) = \mathcal{T}(\rho_2)$.

2. The sequence $\{x_n \in X : n \in \mathbb{N}\}$ is a Cauchy sequence in (X, ρ_1) if and only if the sequence $\{x_n \in X : n \in \mathbb{N}\}$ is a Cauchy sequence in (X, ρ_2).

3. The distance space (X, ρ_1) is complete if and only if the distance space (X, ρ_2) is complete.

Example 2.2. Let $X = \{a, b\} \cup \{c_n : n \in \mathbb{N}\}$, $d(x, x) = 0$ for each $x \in X$, $d : X \times X \to [0, +\infty)$ be given by $d(a, b) = d(b, a) = 1$, $d(a, c_n) = d(c_n, a) = d(b, c_n) = d(c_n, b) = n^{-1}$ and $d(c_n, c_m) = |n^{-1} - m^{-1}|$ for all $n, m \in \mathbb{N}$. Then (X, d) is a compact symmetric T_1-space.

The sequence $\{c_n : n \in \mathbb{N}\}$ is a convergent Cauchy sequence and $a, b \in Lim_{n\to\infty} c_n$.

The space $(X, \mathcal{T}(d))$ is not a Hausdorff space.

Example 2.3. (see [2], Example 2.2)

Let $Z = \{n + m^{-1} : n \in \omega, m \in \mathbb{N}\}$ be the subspace of the space of reals $\mathbb{R}$ with the Euclidean metric $d(x, y) = |x - y|$. For each natural number n we identify the points $\{n, n^{-1}\}$. Denote by X the quotient space and by $p : Z \longrightarrow X$ the natural projection. Let $A = \cup\{\{n, n^{-1}\} : n \in \mathbb{N}\}$ and $B = Z \setminus A$. If $x \in B$, then $p^{-1}(p(x)) = x$. If $x \in A$, then $p^{-1}(p(x)) = \{x, x^{-1}\}$. The function $\rho(u, v) = min\{d(x, y) : x \in p^{-1}(u), y \in p^{-1}(v)\}$ is a symmetric distance on the set X and the topology $\mathcal{T}(\rho)$ is the quotient topology on the set X. The space X is a completely regular Lindelöf space. At the point $O = p(0)$ the space X does not have the countable base and for each $r > 0$ does not exist an open set U of X such that $0 \in U \subseteq B(0, r, \rho)$.

3. Generalized metrics

Definition 3.2. A CF-distance space is a distance space (X, ρ) satisfying the following conditions:

$(CF1)$ $\rho(x, y) = 0$ if and only if $x = y$;

$(CF2)$ $\rho(x, y) = \rho(y, x)$, for all $x, y \in X$;

$(CF3)$ for any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that for all $x, y, z, \in X$ the inequalities $\rho(x, y) \leq \delta$ and $\rho(y, z) \leq \delta$ imply $\rho(x, z) \leq \epsilon$.

The notion of CF-metric (or of distance with the Fréchet-Chittenden condition) was introduced by M. Fréchet and studied by E. W. Chittenden (see [10]).

Definition 3.3. An F-distance space is a distance space (X, ρ) satisfying the following conditions:

$(F1)$ $\rho(x, y) = 0$ if and only if $x = y$;

$(F2)$ for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y, z, \in X$ the inequalities $\rho(x, y) \leq \delta$ and $\rho(y, z) \leq \delta$ imply $\rho(x, z) \leq \epsilon$ and $\rho(y, x) \leq \epsilon$.

Any convergent sequence in an F-distance space (X, ρ) is a Cauchy sequence.

We also mention the next three important facts.

Proposition 3.3. *Let (X, ρ) be a F-distance space. For all $x \in X$ and $r > 0$ there exists a ρ-open set $O(x, r, \rho)$ such that $x \in O(x, r, \rho) \subseteq B(x, r, \rho)$.*

Proposition 3.4. *Let (X, ρ) be an F-distance space. If $\rho_s(x, y) = 2^{-1}[\rho(x, y) + \rho(y, x)]$, then (X, ρ_s) is a CF-metric space, ρ and ρ_s are equivalent distances, and $\mathcal{T}(\rho_s) = \mathcal{T}(\rho)$.*

Proposition 3.5. *If ρ is a CF-metric on the set X, then $\rho_s = \rho$.*

Example 3.4. Let (X, ρ) be a (complete) metric space. Then (X, ρ) a (complete) CF-metric space.

Definition 3.4. A pair (X, ρ) is called a b-metric space (see [3, 5, 7, 13]) if there exists a number $s > 0$ such that for all points $x, y, z \in X$ the following conditions are satisfied:

$(B1)$. $\rho(x, y) = 0$ if and only if $x = y$;

$(B2)$. $\rho(x, y) = \rho(y, x)$;

$(B3)$. $\rho(x, z) \leq [\rho(x, y) + \rho(y, z)]$.

In this case the distance ρ is called a b-metric.

Example 3.5. Any B-metric is a CF-distance. Interesting examples of b-metric spaces are given in [5].

Definition 3.5. A pair (X, ρ) is called a b-quasimetric space or a B-distance space (see [3, 5, 7, 13]) if there exists a number $s > 0$ such that for all points $x, y, z \in X$ the following conditions are satisfied:

$(BQ1)$. $\rho(x, y) = 0$ if and only if $x = y$;

$(BQ2)$. $\rho(x, y) \leq s\rho(y, x)$;

$(BQ3)$. $\rho(x, z) \leq s[\rho(x, y) + \rho(y, z)]$.

In this case the distance ρ is called a b-quasimetric or a b-distance.

Example 3.6. Any B-metric space is a b-quasimetric space and an F-distance space.

Remark 3.1. For $s = 1$, the quasimetric is an usual metric. Any quasimetric ρ is a CF-metric.

Some interesting examples of quasimetric spaces are given in [3] and [5].

Example 3.7. Let (X, d) be a distance space, $p \geq q > 0$, $f : X \times X \to \mathbb{R}$ be a function and $f(x, y) \geq p$ for all $x, y \in X$. We put $d_f(x, y) = d(x, y)$, if $d(x, y) < q$ and $d_f(x, y) = f(x, y)$, if $d(x, y) \geq p$. Then d_f is a new distance on the set X and we say that d_f is the *f-modification* of the distance d. The following assertions are obvious:

- the distances d and d_f are equivalent;
- if the functions d and f are symmetric, then the function d_f is symmetric, too;
- the function d is an F-distance if and only if the function d_f is an F-distance.

4. The associate metric to an F-distance

Chittenden [10] has proven that any CF-metrizable space is metrizable. The Chittenden's method was improved by A. H. Frink [16]. E. W. Chittenden and A. H. Frink associated to any CF-metric ρ the function

$$\bar{\rho}(x, y) = inf\{\rho(x, z_1) + ... + \rho(z_i, z_{i+1}) + ... + \rho(z_n, y) : n \in \mathbb{N}, z_1, \ldots, z_n \in X\}, \quad (4.2)$$

and they had proven the following two facts.

Proposition 4.6. *Assume that (X, d) is a symmetric space and $d(x, z) \leq 2[d(x, y) + d(y, z)]$ for all $x, y, z \in X$. Then:*

1. d is a CF-distance.

2. $2^{-1}d(x, y) \leq \bar{d}(x, y) \leq d(x, y)$ for all $x, y \in X$.

3. $\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z)$ for all $x, y, z \in X$.

4. $\bar{d}$ is a metric on X equivalent to the distance d ($\bar{d}(x, y)$ will be called the associate metric to the distance $d(x, y)$).

Proposition 4.7. *Any CF-distance is equivalent to a metric.*

For any distance D on a set X the function $\bar{d}$ has the following properties:
- $\bar{d}(x,x) = 0$ and $\bar{d}(x,y) \le d(x,y)$ for all $x, y \in X$;
- $\bar{d}(x,z) \le \bar{d}(x,y) + \bar{d}(y,z)$ for all $x, y, z \in X$;
- if d is a symmetric function, then $\bar{d}$ is a symmetric function too.

Remark 4.2. The notion of CF-metric is important in the theory of uniform spaces (see [14], p. 527).

1. Let $\Delta(X) = \{(x,x) : x \in X\} \subseteq X \times X$ be the diagonal of a set X. Let $V \subseteq X \times X$. If $\Delta(X) \subseteq V$, then V is an entourage of the diagonal. We put $V^{-1} = \{(x,y) : (y,x) \in V\}$ and $nV = \{(x,y) \in X \times X : (x,z_1), (z_1,z_2), ..., (z_{n-1},y) \in V \textit{ for some } z_1, z_2, ..., z_n \in X\}$.

2. Let $\{V_n : n \in \mathbb{N}\}$ be a sequence of entourages of of the diagonal $\Delta(X) = \{(x,x) : x \in X\}$ in $X \times X$, $\Delta(X) = \cap\{V_n : n \in \mathbb{N}\}$, $V_1 = X \times X$ and $3V_{n+1} \subseteq V_n = V_n^{-1}$ for each $n \in \mathbb{N}$. Define a mapping $d : X \times X \longrightarrow \mathbb{R}$, where $d(x,y) = inf\{2^{-n} : (x,y) \in V_n\}$. Then d is a CF-metric on X, $d(x,z) \le 2[d(x,y) + d(y,z)]$ and $2^{-1}d(x,y) \le \bar{d}(x,y) \le d(x,y)$ for all $x, y, z \in X$ (see [14], Theorem 8.1.10).

3. Let ρ be CF-distance on a set X, $\epsilon_1 = 1$, $\epsilon_{n+1} = 2^{-n}\delta(\epsilon_n)$ and $V_n = \{(x,y) \in X \times X : \rho(x,y) < \epsilon_n\}$ for each $n \ge 1$. We put $d(x,y) = inf\{2^{-n} : (x,y) \in V_n\}$. Then:
- d is an CF-distance equivalent to the distance ρ;
- $d(x,z) \le 2[d(x,y) + d(y,z)]$ and $2^{-1}d(x,y) \le \bar{d}(x,y) \le d(x,y)$ for all $x, y, z \in X$.

5. On quasinorms

A pair (L, q) is called a quasinorm space, where L is a linear space and $q : L \longrightarrow \mathbb{R}$ is a function, if there exists a number $s > 0$ such that for all points $x, y \in L$ and any $\alpha \in \mathbb{R}$ the following conditions are satisfied:
$(QN1)$. $q(x) = 0$ if and only if $x = 0$;
$(QN2)$. $q(\alpha x) = |\alpha| q(x)$;
$(QN3)$. $q(x+y) \le s[q(x) + q(y)]$.
If $s \le 1$, then q is a norm.

Theorem 5.2. *Let q be a quasinorm on a linear space L and $q(x+y) \le 2[q(x) + q(y)]$ for all $x, y \in L$. Then on L there exists an equivalent norm n and a constant such that*

$$2^{-1}q(x) \le n(x) \le q(x),\ \forall x \in L.$$

Moreover, if $\rho(x,y) = q(x-y)$ is the quasimetric generated by the quasinorm q, then

$$\bar{\rho}(x,y) = n(x-y),\ \forall x, y \in L.$$

Proof. Follows from Proposition 4.6. □

Example 5.8. (see [5]) The space l_p $(0 < p < 1)$ of all real sequences $(x_n \in \mathbb{R} : n \in \mathbb{N})$, $t \in [0,1]$, with the quasinorm $\|x\|_p = (\Sigma\{|x_n|^p : n \in \mathbb{N}\})^{1/p}$ for each point $x = (x_n : n \in \mathbb{N}) \in l_p$, is a a quasinormed space. We have $\|x+y\|_p \le 2^{1/p}(\|x\|_p + \|\|_p\|)$ for all $x = (x_n : n \in \mathbb{N}) \in l_p$ and $y = (y_n : n \in \mathbb{N}) \in l_p$.

Example 5.9. (see [5]) The space L_p $(0 < p < 1)$ is another example of a quasinormed space. This example is similar to Example 5.8.

Remark 5.3. If X is the space from Example 5.8 or Example 5.9, and $\rho(x,y) = \|x-y\|$, for all $x, y \in X$, then $\bar{\rho}$ given by (4.2) is not a metric on X.

6. On fixed points

Remark 6.4. Let ρ be a symmetric distance on a set X, $f : X \to X$ be a mapping and $r > 0$. If

$$\rho(f(x), f(y)) \leq r\rho(x, y), \; \forall x, y \in X, \tag{6.3}$$

then the mapping f is continuous and

$$\bar{\rho}(f(x), f(y)) \leq r\bar{\rho}(x, y), \; \forall x, y \in X,$$

where $\overline{\rho}$ is the associated distance given by (4.2).

The assertion from Remark 6.4 remains true for set-value mappings $f : X \to \mathcal{P}(X)$ as well as for most of the single valued contractive mappings satisfying a linear contractive condition that extends or generalizes condition (6.3), see [4], [23] and [24].

Remark 6.5. Let ρ be a distance on a set X, $f : X \to X$ be a mapping such that

$$\rho(f(x), f(y)) < \rho(x, y),$$

provided $x, y \in X$ and $\rho(x, y) > 0$. Then the mapping f is continuous and has at most one fixed point.

In view of Remarks 5.3 and 6.4, we may conclude that working in b-metric spaces makes sense since if ρ is a quasimetric, then the associate metric $\overline{\rho}$ given by (4.2) is not always a metric.

Conjecture 6.1. *Let (X, ρ) be an F-distance space, $0 < r < 1$, $f : X \longrightarrow X$ be a mapping such that*

$$\rho(f(x), f(y)) \leq r\rho(x, y), \textit{ for all } x, y \in X.$$

1. If x_1 is a fixed point of the space X and $x_{n+1} = f(x_n)$, for each $n \geq 1$, then $\{x_n \in X : n \in \mathbb{N}\}$ is a Cauchy sequence in (X, ρ).
2. If the distance space (X, ρ) is complete, then there exists a unique point $x_0 \in X$ such that $f(x_0) = x_0$.

Related to this conjecture, we mention the following fact.

Theorem 6.3. *Let (X, ρ) be an F-distance space, $0 < r < 1$, $f : X \to X$ be a mapping and*

$$\rho(f(x), f(y)) \leq r\rho(x, y), \textit{ for all } x, y \in X.$$

The following assertions are equivalent:

1. There exists a point $a \in X$ such that $f(a) = a$.

2. If x_1 is a fixed point of the space X and $x_{n+1} = f(x_n)$, for each $n \geq 1$, then the sequence $\{x_n \in X : n \in \mathbb{N}\}$ is convergent.

3. If x_1 is a fixed point of the space X and $x_{n+1} = f(x_n)$, for each $n \geq 1$, then the sequence $\{x_n \in X : n \in \mathbb{N}\}$ contains a convergent subsequence.

4. For some fixed point $x_1 \in X$ the sequence $\{x_n \in X : n \in \mathbb{N}\}$, where $x_{n+1} = f(x_n)$, for each $n \geq 1$, contains a convergent subsequence.

Proof. Obviously, we can assume that d is an CF-distance. There exists a positive function $\mu(t) > 0$, $t > 0$, such that from $d(x, y) \leq \mu(t)$, $d(y, u) \leq \mu(t)$ and $d(u, z) \leq \mu(t)$ it follows $d(x, z) \leq t$.

Assume that $a \in X$ and $f(a) = a$. Fix $x_1 \in X$. Put $c = d(a, x_1)$ and $x_{n+1} = f(x_n)$, for each $n \geq 1$. Then $d(a, x_n) \leq r^{n-1}c$, for each $n \geq 1$. Hence $a = lim_{n\to\infty} x_n$. The implication $1 \to 2$ is proved.

The implications $2 \to 3 \to 4$ are obvious.

Assume now that $x_1 \in X$ and the sequence $\{x_n \in X : n \in \mathbb{N}\}$, where $x_{n+1} = f(x_n)$, for each $n \geq 1$, contains a convergent subsequence $\{x_{n_k} \in X : k \in \mathbb{N}\}$. Consider that $a = lim_{k\to\infty} x_{n_k}$. We affirm that $a = f(a)$. Assume that $a \neq f(a)$. Then $2p = d(a, f(a)) > 0$. By construction, $k \leq n_k$. We put $q = d(x_1, x_2)$. Then $d(x_n, x_{n+1}) \leq r^{n-1}q$ for each $n \geq 1$. There exists $m \in \mathbb{N}$ such that $r^{m-1}q \leq p$ and $d(a, x_{n_k}) \leq \mu(p)$ for each $k \geq m$. Then for $k \geq m$ we have $d(f(a), f(x_{n_k})) < d(a, x_{n_k}) \leq \mu(p)$ and $d(x_{n_r}, f(x_{n_k})) \leq \mu(p)$. Thus $d(a, f(a)) \leq p$, a contradiction. Hence $a = f(a)$. The implication $4 \to 1$ is proved. □

For other equivalent conditions similar to the ones in the previous theorem but in usual metric spaces, see [23], [24].

Corollary 6.1. *Let (X, ρ) be a compact F-distance space, $0 < r < 1$, $f : X \to X$ be a mapping such that*

$$\rho(f(x), f(y)) \leq r\rho(x, y), \text{ for all } x, y \in X.$$

Then there exists a unique point $a \in X$ such that $f(a) = a$.

By virtue of the above theorem, the following conjecture is also interesting.

Conjecture 6.2. *Let (X, ρ) be symmetric distance space, $f : X \longrightarrow X$ be a mapping and $(f(x), f(y)) < r\rho(x, y)$ for all distinct points $x, y \in X$. If the distance space $(X, \mathcal{T}(\rho))$ is a Hausdorff compact space, then there exists a unique point $x_0 \in X$ such that $f(x_0) = x_0$.*

Note that any Hausdorff compact space with a symmetric distance is metrizable (see [2], Theorem 2.5).

Conjectures 6.1 and 6.2 are particular cases of the following general problem.

General Problem 6.3. *Under which conditions on the distance d, the given assertion about fixed points in metric spaces (Y, ρ) remains true for the distance spaces (X, d)?*

Final note. References [27]-[93] are intended just to give an idea on the dynamics of research work related to fixed point theory in some generalized metric spaces: partial metric spaces, G-metric spaces and b-metric spaces, in the last few years.

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