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The role of the Pompeiu-Hausdorff metric in fixed point theory

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ABSTRACT. The main aim of this note is to highlight the role of the Pompeiu-Hausdorff metric in fixed point theory and, subsidiarily, to touch some issues related to the history of this fundamental concept in modern mathematics. This will allow us to conclude that what is nowadays almost generally called *Hausdorff metric (distance)* and very seldom *Hausdorff-Pompeiu metric (distance)* or *Pompeiu-Hausdorff metric (distance)*, should be fairly and correctly named *Pompeiu-Hausdorff metric (distance)*.

1. INTRODUCTION

The distance between two (closed) sets is nowadays a fundamental tool in mathematics, computer science and many other autonomous and interdisciplinary research fields. It is difficult to imagine how researchers could work without it in hand but not too many past and current researchers that used or are using this concept in their work, in a way or another, are really aware on how this concept appeared and even how to name it correctly and completely. Therefore, there is no general awareness about the fact that it was introduced more than one hundred years ago, in 1905, by D. Pompeiu (1873-1954), and thereafter established in the general setting of a metric space and largely disseminated, by F. Hausdorff (1868-1942), since 1914.

To motivate our approach, we would like to start by giving some concrete arguments to our introductory sentences. Let us search in some important electronic databases in order to build a closer image on the relevance and use of the Pompeiu-Hausdorff distance.

A search in Web of Science database for the syntagm "Hausdorff metric" appearing in either Article title, Abstract or Keywords produces 572 results, articles in various fields of research that use that concept. We enumerate the first 100 Web of Science Categories (by record count) displayed:

mathematics applied (276 articles); mathematics (268); statistics probability (73); computer science artificial intelligence (69); computer science theory methods (64); physics mathematical (27); mathematics interdisciplinary applications (25); engineering electrical electronic (24); operations research management science (24); physics multidisciplinary (16); computer science software engineering (15); automation control systems (14); computer science information systems (14); computer science interdisciplinary applications (13); mechanics (10); computer science cybernetics (8); multidisciplinary sciences (8); chemistry analytical (1); economics (5); engineering multidisciplinary (5); imaging science photographic technology (4); optics (4); acoustics (3); computer science hardware architecture

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(3); management (3); neurosciences (3); social sciences mathematical methods (3); engineering mechanical (2); geosciences multidisciplinary (2); remote sensing (2); telecommunications (2); electrochemistry (1); engineering aerospace (1); engineering manufacturing (1); environmental sciences (1); geography (1); geography physical (1); information science library science (1); instruments instrumentation (1); logic (1); microscopy (1); physics fluids plasmas (1); psychology mathematical (1); radiology nuclear medicine medical imaging (1); biochemistry molecular biology (1); robotics (1); biophysics (1); transportation science technology (1).

A similar search in Web of Science database but this time for the syntagm "Hausdorff distance" appearing in either Article title, Abstract or Keywords produces 1386 results. It is also very interesting to see how are these results (more than two times compared to the previous search) distributed among the Web of Science Categories. One gets the following data:

computer science artificial intelligence (407); engineering electrical electronic (300); mathematics applied (254); mathematics (232); computer science theory methods (218); computer science software engineering (162); imaging science photographic technology (151); computer science information systems (131); optics (90); radiology nuclear medicine medical imaging (88); computer science interdisciplinary applications (82); automation control systems (73); engineering biomedical (67); telecommunications (57); statistics probability (40); remote sensing (38); robotics (31); operations research management science (29); computer science cybernetics (25); mathematical computational biology (23); physics mathematical (23); mathematics interdisciplinary applications (21); instruments instrumentation (17); computer science hardware architecture (16); physics applied (13); engineering mechanical (11); medical informatics (11); multidisciplinary sciences (11); neurosciences (10); engineering industrial (9); mechanics (9); transportation science technology (9); engineering civil (8); engineering multidisciplinary (8); geosciences multidisciplinary (8); surgery (8); acoustics (7); engineering manufacturing (6); physics multidisciplinary (6); dentistry oral surgery medicine (5); engineering aerospace (5); geochemistry geophysics (5); geography physical (5); physics fluids plasmas (5); biochemical research methods (4); information science library science (4); management (4); neuroimaging (4); oncology (4); biotechnology applied microbiology (3); cardiac cardiovascular systems (3); economics (3); education scientific disciplines (3); environmental sciences (3); geography (3); nanoscience nanotechnology (3); water resources (3); materials science multidisciplinary (2); peripheral vascular disease (2); physics particles fields (2); social sciences mathematical methods (2); astronomy astrophysics (2); biology (2); clinical neurology (2); construction building technology (2); education educational research (2); energy fuels (2); biochemistry molecular biology (1); biophysics (1); business (1); chemistry analytical (1); chemistry multidisciplinary (1); chemistry physical (1); crystallography (1); dermatology (1); electrochemistry (1); endocrinology metabolism (1); engineering marine (1); environmental studies (1); genetics heredity (1); health care sciences services (1); limnology (1); logic (1); materials science biomaterials (1); medicine general internal (1); medicine research experimental (1); meteorology atmospheric sciences (1); oceanography (1); psychology (1); psychology mathematical (1); reproductive biology (1); spectroscopy (1); transportation (1); zoology (1).

As we can see, the list of research fields that use Pompeiu-Hausdorff distance (metric) is quite impressive !

It is important to stress on the fact that these lists do not include all research or survey works indexed in WoS that actually use the Pompeiu-Hausdorff metric (under the form

Hausdorff metric or Hausdorff distance) but just those papers which mention explicitely this concept in the title of the work, in its Abstract or in Keywords. Many other papers could also use the Pompeiu-Hausdorff metric without mentioning it in these contexts, and so we could not detect them, by the mentioned filters from the WoS database or other mathematics oriented databases like Zentralblatt MATH or /and MathSciNet.

An example in this respect is the paper [2], appeared 1975, which establishes a very interesting evaluation of the Pompeiu-Hausdorff distance between the spectra of two linear operators T and T' (the name is here correctly used: the Pompeiu-Hausdorff distance). There are many other papers in the same situation.

It is quite clear from the above lists that the great majority of the research papers indexed in WoS that use Pompeiu-Hausdorff metric under the name *Hausdorff metric* are from Mathematics applied (276 articles) and Mathematics (268), while, under the name *Hausdorff distance*, the greatest number of papers are from Computer Science artificial inteligence (407) and Engineering electrical electronic (300).

But Web of Science is a very selective database (there exist many good, very good and excellent serials which are not indexed there) and, additionally, it covers a shorter period than Zentralblatt MATH and MathScinet. SCOPUS is, in some sense, complementing Web of Science, by including some of the good, very good and excellent serials not covered by Web of Science, but covers even a shorter period of time.

So, the best mirrors of mathematics oriented publications are, from this point of view, Zentralblatt MATH and MathScinet, that will be used later in Section 3.

Note that by searching after the same syntagms as above, in the SCOPUS database gives very close figures to those from Web of Science: 735 results for ”Hausdorff metric” and 1823 results for ”Hausdorff distance”.

In view of the previous discussion, it is the main aim of this note to highlight the role of Pompeiu-Hausdorff metric in a very specific mathematical field, i.e., in fixed point theory.

Subsidiarily, we intend to touch some historical issues related to the origins of this fundamental concept in modern mathematics. This will allow us to conclude that, what is nowadays almost generally called *Hausdorff metric (distance)* and very seldom *Hausdorff-Pompeiu metric (distance)* or *Pompeiu-Hausdorff metric (distance)*, should be always correctly named *Pompeiu-Hausdorff metric (distance)*.

2. Pompeiu's definition and Hausdorff's definition

Pompeiu defined the concept of distance between two closed sets, in the context of complex analysis, in his PhD thesis defended at University of Paris (the Sorbonne), in 1905, under the direction of H. Poincaré, and published it in the same year in *Annales de la Faculté de Sciences de Toulouse* (*Annals of Faculty of Sciences in Toulouse*) [28].

Pompeiu actually needed this distance in order to rigorously define the distance between two curves in the complex plane and also to introduce by means of this distance the concept of limit of a sequence of sets.

If we use the current terminology and notations, Pompeiu proceeded as follows, see his Complete Works [29], page 12, where the whole content of [28] is reprinted.

Let A, B be two closed and bounded sets. If $a \in A$, then the distance between the point a and the set B is by definition

$$d(a,B) = \min\{d(a,b) : b \in B\},$$

where $d(a,b)$ is the (Euclidean) distance between the points a and b (do not forget that Pompeiu was working in the complex plane !).

Further, Pompeiu defined the *asymmetric distance* (**écart**, in French) between the sets A and B as

$$D(A, B) = \max\{d(a, B) : a \in A\}.$$

He noted immediately that $D(A, B)$ is not symmetric and that $D(A, B) = 0$ if and only if "all points of A belong to B", that is, $A \subset B$. Therefore, he also considered the *asymmetric distance* (écart) between the sets B and A,

$$D(B, A) = \max\{d(b, A) : b \in B\},$$

and noted that $D(B, A) = 0$ if and only if $B \subset A$.

In order to endow the distance between two sets with its most natural property, i.e., with the *symmetry*, Pompeiu considered a very natural way to symmetrize his concept, by defining the *distance between the sets A and B* (**écart mutuel**, in French), denoted here by $P(A, B)$, by

$$P(A, B) = D(A, B) + D(B, A), \tag{2.1}$$

and concluded that $P(A, B) = 0$ if and only if $D(A, B) = 0$ and $D(B, A) = 0$, that is, if and only if $A = B$.

What did Hausdorff ?

In the first edition of his famous book published in 1914 [14], Hausdorff considered all the basic concepts introduced by Pompeiu, but in the general setting of a metric space, and adopted an alternative way to symmetrize the asymmetric distances $D(A, B)$ and $D(B, A)$, by defining what is currently denoted by $H(A, B)$ and commonly named *Hausdorff metric*:

$$H(A, B) = \max\{D(A, B), D(B, A)\}. \tag{2.2}$$

The two definitions are clearly equivalent, by virtue of the double inequality

$$\frac{1}{2} \cdot (u + v) \leq max\{u, v\} \leq 1 \cdot (u + v),$$

which yields

$$\frac{1}{2} \cdot P(A, B) \leq H(A, B) \leq 1 \cdot P(A, B).$$

Hausdorff cited correctly Pompeiu's contribution and in this way he explicitly acknowledged Pompeiu's priority: in the first edition of his book [14] (at page 463), in its shorter second edition [15] (at page 280), as well as in the third edition [16] and its two translations (Russian translation [17], at page 293 and English translation [18], at page 343).

Which is then the reason why, even if Hausdorff explicitly mentioned Pompeiu's priority, a fact confirmed in the monograph of Kuratowski from 1933 [21], and also cited correctly Pompeiu's work in [14], [15] and [16] (and its translations [17] and [18]), the posterity however credited only Hausdorff as creator of this fundamental concept ?

This issue is discussed in more details in our paper [3], and hence in the following we are focusing only on the role of the Pompeiu-Hausdorff distance in metrical fixed point theory.

3. Pompeiu-Hausdorff metric in fixed point theory

The first fixed point theorem explicitly formulated in literature is a *topological* fixed point theorem, i.e., the Brower's fixed point theorem, published in 1912 [6], see also the theorem established by Poincaré in 1883 [26] and shown much later to be equivalent to Brower's fixed point theorem. This fixed point theorem is concerned with single valued functions defined on a compact and convex set in $\mathbb{R}^n$.

In fact, the first formulation of a functional equation as an equivalent fixed point problem appears also to have been done by Poincaré. In his famous memoir of 1890 [27] on the three-body problem, crowned with King Oscar Prize, Poincaré reduced the study of the T-periodic solutions of a differential system in $\mathbb{R}^n$:

$$x' = f(t, x) \tag{3.3}$$

to the study of the fixed points of the operator P_T:

$$x = P_T(x)$$

where P_T is defined on R^n as the solution of (3.3) verifying the initial condition

$$x(s) = y.$$

As it can be inferred, the operator involved in this fixed point approach is single valued, while the ambient space is that of continuous functions on $\mathbb{R}^n$.

Following Poincaré's fixed point approach, Picard applied systematically the method of successive approximations, extracted directly from the fixed point formulation of a functional equation, to various differential equations problems [25].

By dressing Picard's technique of proof with more abstract clothes, i.e., by stating it in what we are nowadays calling a Banach space, this led Banach [1] in 1922 to state the well known fixed point theorem for contractions. Banach's fixed point theorem is the first metrical fixed point theorem in literature, being essentially based on the contraction condition, which in turn is expressed by means of the *distance / metric* of a metric space.

In the same year, 1922, the study of boundary value problems for nonlinear ordinary differential equations was motivating Birkhoff and Kellogg's extension of Brouwer's fixed point theorem to some abstract spaces (function spaces) [4]. Further developments in topological fixed point theory are due to Schauder [30], for the case of topological vector spaces, and to other mathematicians.

Consequently, the first fixed point theorems for set valued mappings were not metrical but *topological* type fixed point theorems. This list was opened in 1941 by Kakutani's fixed point theorem [20], for the case of mappings defined on convex compact subsets of a Euclidean space, and continued with Ky Fan-Glicksberg fixed point theorem (see [11] and [13]), in the case of infinite dimensional locally convex topological vector spaces, and with many other important contributions, see for example [34].

On the other hand, the first *metrical* fixed point theorems for set valued mappings appeared slightly later. The list opened in 1968 with the papers of Markin [23], Nadler [24] and with the paper from 1969 by Covits and Nadler [8]. These fixed point theorems are essentially based on the use of Pompeiu-Hausdorff metric, in order to express the generalized contraction condition, like in the next example taken from [24].

Example 3.1. Let (X, d) be a metric space and $T : X \to \mathcal{CB}(X)$ a set-valued mapping. T is called a α-contraction, if there exists a constant $\alpha \in (0, 1)$ such that

$$H(Tx, Ty) \leq \alpha\, d(x, y), \text{ for all } x, y \in X.$$

So, 1968 marks the entrance of Pompeiu-Hausdorff metric in the scene of *metrical* fixed point theory, but under the name "Hausdorff metric" or "Hausdorff distance", which is still used in the great majority of cases.

This, however, happened after the publication of some articles that explicitly pointed out Pompeiu's priority in defining the set distance, see for example the paper from 1954 (the year of Pompeiu's death) written by T. Ganea [12] and also the one authored by Calude in 1973 (the year that marked Pompeiu' s centenary) [7].

One possible explanation of the fact that Pompeiu' s priority has been disregarded for a long period of time and still continues to be disregarded is the fact that the above mentioned papers (and some others) were written in Romanian, French or Russian and have been published in journals with limited spreading, even though they were reviewed by both Zentralblatt MATH and Mathematical Reviews. Other arguments, collected from [3], are presented in the following.

The fact that Pompeiu-Hausdorff metric has been named mainly after Hausdorff is certainly due to the history of his famous book, *Grundzüge der Mengenlehre*, see the extended review [9]. Its first edition has been published in 1914 (in German) [14]. A second edition of the book has been printed in 1927 by W. de Gruyter [15], but limited by the publisher to 320 pages only (the first edition comprised 408 pages). A slightly extended third edition has been published in 1935 [16]. The latter one has been immediately translated into Russian (in 1937), editors being Alexandrov and Kolmogorov themselves, while the English version was published 20 years later [18] and has been reprinted many times since then. The English version [18] is mainly responsible for crediting Hausdorff as the unique author of the fundamental concept of distance between two sets, especially because in this version of *Mengenlehre* the mention of Pompeiu's definition and priority is done in a rather hidden and unclear way, see [3] for more historical details.

Coming back to Fixed Point Theory, we stress on the fact that there exists a vast literature on the fixed point theory of multi-valued mappings, which is essentially based on the use of Pompeiu-Hausdorff metric, but which cannot be presented here completely, see [34]-[38] for most of the references.

If we search in MathSciNet for reviewed papers under the MSC code 47H04 (Set valued mappings) and which are also mentioning the word "Hausdorff" anywhere, we find 211 results. If we do the same search but with the word "Pompeiu" anywhere, we find only 11 results. Therefore, only 5% of the papers in this category are mentioning Pompeiu's name, for the rest of 95% being credited Hausdorff only.

If we search now in MathSciNet for reviewed papers under the MSC code 47H10 (Fixed-point theorems) and which are mentioning the word "Hausdorff" anywhere, we find 574 results, of which only 20 papers are containing the word "Pompeiu" anywhere. This means that no more than 3% of papers in this category are mentioning Pompeiu's name, for the rest of 97% being credited Hausdorff only.

What is important at this stage is to persuade people working in fixed point theory and using Pompeiu-Hausdorff metric about the fact that they should name correctly this fundamental concept in their work. Most of them are already doing so, but the great majority of them are not.

The main merits in promoting *Pompeiu-Hausdorff metric* in fixed point theory, under this name or under the alternative form *Hausdorff-Pompeiu metric*, belongs to Professor I. A. Rus and his fixed point research group in Cluj-Napoca, see [34]-[38] and the references therein.

In contrast to the general situation in (non Romanian) fixed point theory works and in almost all other important fields of research where the Pompeiu-Hausdorff metric is exploited, there are however some monographs that are correctly naming it, see for example the recent books [32] (published in 1998), [10] (2009) and [22] (2010).

4. Conclusions

Although Pompeiu has been scientifically active more than 45 years (he died in 1954), after the moment he introduced the distance between two sets (in 1905), he did not more work on any topic directly related to or using this distance. He is known mainly for the Pompeiu problem and for the Cauchy-Pompeiu formula in complex analysis, which are still actual and important topics of research, but also for the *areolar derivative* in analysis and the Pompeiu's theorem in elementary geometry, see [5], [7], [12], [19], [29].

Most probably, he simply ignored the developments of this fundamental concept and especially its launch and use in topology starting with the early forties. Otherwise, we would expect him to have pretended some paternity together with Hausdorff.

At least the following three reasons are plausible for his lack of interest in this respect:

1. He worked on very deep problems in complex analysis, that are currently still important research topics. He published 161 papers in the period 1902-1951, almost all in complex analysis, according to Zentralblatt MATH;

2. He did not write any monograph (on topics related to or using the distance between two sets);

3. Hausdorff's book was a fundamental reference book in the field of set theory and topology, with many seminal concepts and results, which established the awareness that the concept of distance between two sets was due to Hausdorff only.

Despite the fact that some important recent monographs [10], [22], [32] are correctly naming Pompeiu-Hausdorff metric, there is no general awareness of Pompeiu's merits amongst the non Romanian scientists using this important tool in almost all research fields where it is used and, in particular, in fixed point theory.

Hence, we believe that it is now the moment to re-consider the terminology regarding the Pompeiu-Hausdorff metric, not only in fixed point theory, but in all fields of mathematics and interdisciplinary domains that are using it.

So, what is nowadays almost generally called *Hausdorff metric (distance)* and very seldom *Hausdorff-Pompeiu metric (distance)* or *Pompeiu-Hausdorff metric (distance)*, should be correctly named *Pompeiu-Hausdorff metric (distance)*. This would thus do a very delayed but well deserved justice to Pompeiu's seminal contribution.

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REFERENCES

[1] Banach, S., *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133–181

[2] Bercovici, H., Gologan, R. and Timotin, D., *Generalizations of some functions defined on* $L(H)$ (in Romanian), Stud. Cerc. Mat., **27** (1975), No. 2, 131–156

[3] Berinde, V. and Păcurar, M., *Hausdorff metric should be correctly named Pompeiu-Hausdorff metric* (in preparation)

[4] Birkhoff, G. D. and Kellogg, O. D., *Invariant points in function space*, Trans. Amer. Math. Soc., **23** (1922), 96–115

[5] Birsan, T. and Tiba, D., One hundred years since the introduction of the set distance by Dimitrie Pompeiu, in *System modeling and optimization*, 35–39, IFIP Int. Fed. Inf. Process., 199, Springer, New York, 2006

[6] Brouwer, L. E. J., *Ueber Abbildungen von Mannigfaltigkeiten*, Math. Ann., **71** (1912), 97–115

[7] Calude, C., *The Pompeiu distance between closed sets* (Romanian), "Gheorghe Ţiţeica and Dimitrie Pompeiu" Symposium on Geometry and Global Analysis (Bucharest, 1973) (Romanian), pp. 281–287. Editura Acad. R.S.R., Bucharest, 1976

[8] Covitz, H. and Nadler, S. B., Jr., *Multi-valued contraction mappings in generalized metric spaces*, Israel J. Math., **8** (1970), 5-11

[9] Deiser, O., Review ZBL 1175.01034, republished in Eur. Math. Soc. Newsletter, 88 (2013), 65–68

[10] Dontchev, A. L. and Rockafellar, R. T., *Implicit Functions and Solution Mappings: A view from Variational Analysis*, Springer, 2009

[11] Fan, Ky., *Fixed-point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. U. S. A., **38** (1952), 121–126

[12] Ganea, T., *The distance between closed subsets defined by D. Pompeiu* (in Romanian), Acad. Republ. Popul. Romîne, Studii Cerc. Mat., **5** (1954), 25–28

[13] Glicksberg, I. L., *A further generalization of the Kakutani fixed theorem, with application to Nash equilibrium points*, Proc. Amer. Math. Soc., **3** (1952), 170-174

[14] Hausdorff, F., *Grundzüge der Mengenlehre*, Viet. Leipzig, 1914

[15] Hausdorff, F., *Mengenlehre*, Zweite neubearbeitete Auflage, Walter de Gruyter, Berlin, 1927

[16] Hausdorff, F., *Mengenlehere*, 3 Auflage, Walter de Gruyter, Berlin, 1935

[17] Hausdorff, F., *Teoria Mojestv*, Ob. Nau.-Teh. Izdatelstvo NKTP SSSR, Moskva-Leningrad, 1937

[18] Hausdorff, F., *Set Theory*, Chelsea Publishing Company, New York, 1957

[19] Istrăţescu, V. I., *Fixed Point Theory. An Introduction*, Kluwer Academic Publishers, 2001

[20] Kakutani, S., *A generalization of Brouwer's fixed point theorem*, Duke Math. J., **8** (1941), 457–459

[21] Kuratowski, C., *Topologie: Espaces mtrisables, espaces complets*, Warschau, Monografie matematyczne, 1933

[22] Lorenz, T., *Mutational Analysis: A Joint Framework for Cauchy Problems in and Beyond Vector Spaces*, Springer, Lecture Notes in Mathematics, Issue 1996, Berlin Heidelberg, 2010

[23] Markin, J. T., *A fixed point theorem for set valued mappings*, Bull. Amer. Math. Soc., **74** (1968), 639–640

[24] Nadler, Sam B., Jr., *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 475–488

[25] Picard, E., *Memoire sur la théorie des équations aux déerivées partielles et la méthode des approximations successives*, J. Math. Pures Appl., 6 (1890), 145-210

[26] Poincaré, H., *Sur certaines solutions particulières du problème des trois corps*, C.R. Acad. Sci. Paris Sér. I, **97** (1883), 251–252

[27] Poincaré, H., *Sur le problème des trois corps et les équations de la dynamique*, Acta Mathematica, **13** (1890), 1–270

[28] Pompeiu, D., *Sur la continuité des fonctions de variables complexes* (These), Gauthier-Villars, Paris, 1905; Ann.Fac.Sci.de Toulouse, **7** (1905), 264–315

[29] Pompeiu, D., *Oeuvre mathématique. [Opera matematică.]*, Editura Academiei Republicii Populare Romîne, Bucureşti, 1959

[30] Schauder, J., *Der Fixpunktsatz in Funktionalrumen*, Studia Math., **2** (1930), 171–180

[31] Toader, Gh., *Remarques sur la distance entre les ensembles*, Bul. Inst. Politeh. Iaşi, Secţ. I, **17(21)** (1971), No.3-4, 103–105

[32] Rockafellar, R. T., Wets, Roger J.-B., *Variational analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 317, Springer-Verlag, Berlin, 1998

[33] Rus, I. A., *Fixed point theorems for multivalued mappings in complete metric spaces*, Math. Japon., **20** (1975), 21–24

[34] Rus, I. A., *Principles and Applications of the Fixed Point Theory* (in Romanian), Editura Dacia, Cluj-Napoca, 1979

[35] Rus, I. A., *Basic problems of the metric fixed point theory revisited (II)*, Stud. Univ. Babeş-Bolyai, **36** (1991), 81–99

[36] Rus, I. A., *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001

[37] Rus, I. A., Petruşel, A. and Petruşel, G., *Fixed point theory: 1950-2000. Romanian contributions*. House of the Book of Science, Cluj-Napoca, 2002

[38] Rus, I. A., Petruşel, A. and Petruşel, G., *Fixed Point Theory*. Cluj University Press, Cluj-Napoca, 2008

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