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Triple fixed point theorems for mixed monotone Prešić-Kannan and Prešić-Chatterjea mappings in partially ordered metric spaces

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ABSTRACT. The aim of this paper is to extend the Kannan fixed point theorem from single-valued self mappings $T: X \to X$ to mappings $F: X^3 \to X$ satisfying a Prešić-Kannan type contractive condition:

$$d\left(F\left(x,y,z\right),F\left(y,z,u\right)\right)\leq\frac{k}{8}[d\left(x,F(x,y,z)\right)+d\left(y,F(y,x,y)\right)+$$

$$+d\left(z,F(z,y,x)\right)+d\left(y,F(y,z,u)\right)+d\left(z,F(z,y,z)\right)+d\left(u,F(u,z,y)\right)],$$

or a Prešić-Chatterjea type contractive condition:

$$d\left(F\left(x,y,z\right),F\left(y,z,u\right)\right)\leq\frac{k}{8}[d\left(x,F(y,z,u)\right)+d\left(y,F(z,y,z)\right)+$$

$$+d\left(z,F(u,z,y)\right)+d\left(y,F(x,y,z)\right)+d\left(z,F(y,x,y)\right)+d\left(u,F(z,y,x)\right)].$$

The obtained tripled fixed point theorems extend and unify several related results in literature.

1. INTRODUCTION

Our starting point for the present paper consists of the following fixed point results from literature.

1. **The Kannan fixed point theorem** in metric spaces.

Theorem 1.1 (R. Kannan [20], 1968). *Let (X,d) be a complete metric space and $T: X \to X$ be a mapping for which there exists $a \in \left[0, \frac{1}{2}\right)$ such that*

$$d(Tx,Ty)\leq a[\,d(x,Tx)+d(y,Ty)\,],\quad \text{for all } x,y\in X. \tag{1.1}$$

Then:

1) *T has a unique fixed point x^*, that is, there exists a unique $x^* \in X$ such that $T(x^*) = x^*$;*
2) *the sequence $\{x_n\}_{n\geq 0}$ defined by*

$$x_{n+1}=T(x_n)\,,\quad n=0,1,2,\ldots \tag{1.2}$$

converges to x^*, for any $x_0 \in X$.

2. **The Prešić fixed point theorem** in metric spaces.

Theorem 1.2 (S. Presić [32], 1965). *Let (X,d) be a complete metric space, k a positive integer, $\alpha_1,\alpha_2,\ldots,\alpha_k\in\mathbb{R}_+$, $\sum\limits_{i=1}^{k}\alpha_i=\alpha<1$ and $f: X^k \to X$ a mapping satisfying*

$$d(f(x_0,\ldots,x_{k-1}),f(x_1,\ldots,x_k))\leq\alpha_1 d(x_0,x_1)+\cdots+\alpha_k d(x_{k-1},x_k), \tag{1.3}$$

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for all $x_0, \dots, x_k \in X$.

Then:

1) *f has a unique fixed point x^*, that is, there exists a unique $x^* \in X$ such that $f(x^*, \dots, x^*) = x^*$;*
2) *the sequence $\{x_n\}_{n\geq 0}$ defined by*

$$x_{n+1} = f(x_{n-k+1}, \dots, x_n), \quad n = k-1, k, k+1, \dots \tag{1.4}$$

converges to x^*, for any $x_0, \dots, x_{k-1} \in X$.

3. **The Prešić-Kannan fixed point theorem** in metric spaces (see [23], [24]).

Theorem 1.3 (M. Păcurar [24], 2010). *Let (X, d) be a complete metric space, k a positive integer, $a \in \mathbb{R}$ a constant such that $0 \leq ak(k+1) < 1$ and $f : X^k \to X$ an operator satisfying the following condition:*

$$d(f(x_0, \dots, x_{k-1}), f(x_1, \dots, x_k)) \leq a \sum_{i=0}^{k} d(x_i, f(x_i, \dots, x_i)), \tag{1.5}$$

for any $x_0, x_1, \dots, x_k \in X$. Then

1) *f has a unique fixed point x^*, that is, there exists a unique $x^* \in X$ such that $f(x^*, \dots, x^*) = x^*$;*
2) *the sequence $\{y_n\}_{n\geq 0}$ defined by $y_{n+1} = f(y_n, y_n, \dots, y_n), n \geq 0$, converges to x^*;*
3) *the sequence $\{x_n\}_{n\geq 0}$ with $x_0, \dots, x_{k-1} \in X$ and $x_n = f(x_{n-k}, x_{n-k+1}, \dots, x_{n-1}), n \geq k$, also converges to x^*, with a rate estimated by:*

$$d(x_{n+1}, x^*) \leq M\theta^n, n \geq 0,$$

for a positive constant M and a certain $\theta \in (0, 1)$.

4. **The tripled fixed point theorem** in partially ordered metric spaces (see [5], [11]-[15]).

Definition 1.1. *(*[5]*)*An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of $F : X \times X \times X \to X$ if

$$F(x, y, z) = x, F(y, x, y) = y, \text{ and } F(z, y, x) = z.$$

Definition 1.2. *(*[5]*)*Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$. We say that F has the mixed monotone property if $F(x, y, z)$ is monotone nondecreasing in x and z, and is monotone non increasing in y, that is, for any $x, y, z \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z),$$

and

$$z_1, z_2 \in X, z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2).$$

Theorem 1.4 (V. Berinde and M. Borcut [5], 2011). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist the constants $j, k, l \in [0, 1)$ with $j + k + l < 1$ for which*

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w), \tag{1.6}$$

$\forall x \geq u, y \leq v, z \geq w$. If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), y_0 \geq F(y_0, x_0, y_0) \text{ and } z_0 \leq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ such that

$$x = F(x, y, z), y = F(y, x, y) \text{ and } z = F(z, y, x).$$

Based on the above results, our aim in this paper is to obtain tripled fixed point theorems for Prešić-Kannan and Prešić-Chatterjea mappings in partially ordered metric spaces, thus extending, generalising or unifying some of the previous mentioned results.

2. Main results

The following definitions and Propositions are taken from [15]. They will be useful in simplifying the arguments in the proofs of our main results.

Definition 2.3. Let X, Y, Z be nonempty sets and $F : X^3 \to Y$, $G : Y^3 \to Z$. We define the symmetric composition (or, the s-composition, for short) of F and G, by $G * F : X^3 \to Z$,

$$(G * F)(x, y, z) = G(F(x, y, z), F(y, x, y), F(z, y, x)) \, (x, y, z \in X).$$

For a nonempty set X, denote by P_X the projection mapping

$$P_X : X^3 \to X, P(x, y, z) = x \text{ for } x, y, z \in X.$$

The symmetric composition has the following properties.

Proposition 2.1. *(Associativity). If $F : X^3 \to Y$, $G : Y^3 \to Z$ and*

$$H : Z \times Z \times Z \to W, \text{ then } (H * G) * F = H * (G * F).$$

Proposition 2.2. *(Identity Element). If $F : X^3 \to Y$, then*

$$F * P_X = P_Y * F = F.$$

Proposition 2.3. *(Mixed Monotonicity). If $(X, \leq)$, $(Y, \leq)$, $(Z, \leq)$ are partially ordered sets and the mappings $F : X^3 \to Y, G : Y^3 \to Z$ are mixed monotone, then $G * F$ is mixed monotone, too.*

Proposition 2.4. *If $(X, \leq)$ is a partially ordered set and F is mixed monotone, then $F^n = F * F^{n-1} = F^{n-1} * F$ is mixed monotone for every n.*

The first main result of this paper is stated in the next theorem.

Theorem 2.5. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ such that*

$$d(F(x, y, z), F(y, z, u)) \leq \frac{k}{8}[d(x, F(x, y, z)) + d(y, F(y, x, y)) + d(z, F(z, y, x)) + \tag{2.7}$$
$$+ d(y, F(y, z, u)) + d(z, F(z, y, z)) + d(u, F(u, z, y))], \text{ for all } x \geq y, y \leq z, z \geq u.$$

Also suppose either

(a) F is continuous or

(b) X has the following property:

$$(i) \text{ if a nondecreasing sequence } \{x_n\} \to x, \text{ then } x_n \leq x \text{ for all } n,$$
$$(ii) \text{ if a nonincreasing sequence } \{y_n\} \to y, \text{ then } y_n \geq y \text{ for all } n.$$

If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), y_0 \geq F(y_0, x_0, y_0) \text{ and } z_0 \leq F(z_0, y_0, x_0), \tag{2.8}$$

then there exist $x, y, z \in X$ such that

$$x = F(x, y, z), y = F(y, x, y) \text{ and } z = F(z, y, x).$$

Proof. Consider the sequences $\{x_n\}, \{y_n\}, \{z_n\} \subset X$ defined by

$$x_{n+1} = F(x_n, y_n, z_n) = F^{n+1}(x_0, y_0, z_0), y_{n+1} = F(y_n, x_n, y_n) = F^{n+1}(y_0, x_0, y_0),$$
$$z_{n+1} = F(z_n, y_n, x_n) = F^{n+1}(z_0, y_0, x_0), (n = 0, 1, ...).$$

Since F^n is mixed monotone for every n, by Proposition 2.4, it follows by (2.8) that $\{x_n\}$ and $\{z_n\}$ are nondecreasing and $\{y_n\}$ is nonincreasing.

Indeed, due to the mixed monotone property of F, it is easy to show that

$$x_2 = F(x_1, y_1, z_1) \geq F(x_0, y_0, z_0) = x_1$$
$$y_2 = F(y_1, x_1, y_1) \leq F(y_0, x_0, y_0) = y_1$$
$$z_2 = F(z_1, y_1, x_1) \geq F(z_0, y_0, x_0) = z_1$$

and thus we obtain that the three sequences are satisfying the following conditions

$$x_0 \leq x_1 \leq ... \leq x_n \leq ...,$$
$$y_0 \geq y_1 \geq ... \geq y_n \geq ...,$$
$$z_0 \leq z_1 \leq ... \leq z_n \leq ....$$

Now, for $n \in \mathbb{N}$, denote

$$D_{x_{n+1}} = d(x_{n+1}, x_n), D_{y_{n+1}} = d(y_{n+1}, y_n), D_{z_{n+1}} = d(z_{n+1}, z_n)$$

and

$$D_{n+1} = D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}.$$

Using (2.7), we get

$$\begin{aligned} D_{x_{n+1}} &= d(x_{n+1}, x_n) = d(F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1})) \\ &\leq \frac{k}{8} [d(x_n, F_{x_n}) + d(y_n, F_{y_n}) + d(z_n, F_{z_n}) \\ &+ d(x_{n-1}, F_{x_{n-1}}) + d(y_{n-1}, F_{y_{n-1}}) + d(z_{n-1}, F_{z_{n-1}})] \\ &= \frac{k}{8} [D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}], \end{aligned}$$

so

$$D_{x_{n+1}} \leq \frac{k}{8} [D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}]. \tag{2.9}$$

Similarly, we obtain

$$D_{y_{n+1}} \leq \frac{k}{8} [D_{x_n} + 2D_{y_n} + D_{x_{n+1}} + 2D_{y_{n+1}}] \tag{2.10}$$

and

$$D_{z_{n+1}} \leq \frac{k}{8} [D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}]. \tag{2.11}$$

By (2.18), (2.19) and (2.20), we get

$$\begin{aligned} D_{n+1} &\leq \frac{k}{8} [3D_{x_n} + 4D_{y_n} + 2D_{z_n} + 3D_{x_{n+1}} + 4D_{y_{n+1}} + 2D_{z_{n+1}}] \\ &\leq \frac{k}{8} [4D_{x_n} + 4D_{y_n} + 4D_{z_n} + 4D_{x_{n+1}} + 4D_{y_{n+1}} + 4D_{z_{n+1}}] \\ &\leq \frac{k}{2} [D_n + D_{n+1}]. \end{aligned}$$

Therefore, for all $n \geq 1$, we have

$$D_{n+1} \leq \alpha \cdot D_n \leq ... \leq \alpha^n \cdot D_1, \text{ where } \alpha = \frac{k}{2-k} \in [0, 1), \text{ when } k \in [0, 1).$$

Because $D_{x_{n+1}} \leq D_{n+1}, D_{y_{n+1}} \leq D_{n+1}$ and $D_{z_{n+1}} \leq D_{n+1}$, we finally obtain

$$D_{x_{n+1}} \leq \alpha^n \cdot D_1, D_{y_{n+1}} \leq \alpha^n \cdot D_1 \text{ and } D_{z_{n+1}} \leq \alpha^n \cdot D_1, \tag{2.12}$$

which imply that $\{x_n\}, \{y_n\}, \{z_n\}$ are all Cauchy sequences in X. Indeed, let $m \geq n$, then

$$d(x_m, x_n) \leq D_{x_m} + D_{x_{m-1}} + ... + D_{x_{n+1}} \leq$$
$$\leq [\alpha^{m-1} + \alpha^{m-2} + ... + \alpha^n] \cdot D_1 = \frac{\alpha^n - \alpha^m}{1-\alpha} \cdot D_1 < \frac{\alpha^n}{1-\alpha} \cdot D_1.$$

Similarly, we can verify that $\{y_n\}$ and $\{z_n\}$ are also Cauchy sequences. Since X is a complete metric space, there exist $x, y, z \in X$ such that

$$\lim_{x\to\infty} x_n = x, \lim_{x\to\infty} y_n = y, \lim_{x\to\infty} z_n = z.$$

Finally, we claim that

$$x = F(x, y, z), y = F(y, x, y) \text{ and } z = F(z, y, x).$$

Suppose first that (a) holds and let $\epsilon > 0$. Since F is continuous at (x, y, z), for a given $\frac{\epsilon}{2} > 0$, there exists a $\delta > 0$ such that

$$d((x, y, z), (y, z, u)) = d(x, y) + d(y, z) + d(z, u) < \delta$$
$$\Rightarrow d(F(x, y, z), F(u, v, w)) < \frac{\epsilon}{2}.$$

Since

$$\lim_{x\to\infty} x_n = x, \lim_{x\to\infty} y_n = y, \lim_{x\to\infty} z_n = z,$$

for $\eta = \min\left(\frac{\epsilon}{2}, \frac{\delta}{2}\right)$, there exist n_0, m_0, p_0 such that, for $n \geq n_0, m \geq m_0, p \geq p_0$,

$$d(x_n, x) < \eta, d(y_n, y) < \eta, d(z_n, z) < \eta.$$

Now, for $n \in \mathbb{N}, n \geq max\{n_0, m_0, p_0\}$,

$$d(F(x, y, z), x) \leq d(F(x, y, z), x_{n+1}) + d(x_{n+1}, x)$$
$$= d(F(x, y, z), F(x_n, y_n, z_n)) + d(x_{n+1}, x) < \frac{\epsilon}{2} + \eta \leq \epsilon.$$

This implies that $x = F(x, y, z)$. Similarly, we can show that

$$y = F(y, x, y) \text{ and } z = F(z, y, x).$$

Suppose now that (b) holds. Since $\{x_n\}, \{z_n\}$ is non-decreasings and $x_n \to x$, $z_n \to z$, and as $\{y_n\}$ is non-increasing and $y_n \to y$, from (b) we have $x_n \leq x, y_n \geq y$ and $z_n \leq z$, for all n. Then by triangle inequality and (2.7), we get

$$\begin{aligned} d(x, F(x,y,z)) &\leq d(x, x_{n+1}) + d(x_{n+1}, F(x,y,z)) \\ &= d(x, x_{n+1})\, d(F(x_n, y_n, z_n), F(x,y,z)) \\ &\leq d(x, x_{n+1}) + \frac{k}{8}[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) \\ &\quad + d(x, F(x,y,z)) + d(y, F(y,x,y)) + d(z, F(z,y,x))], \end{aligned} \tag{2.13}$$

$$\begin{aligned} d(y, F(y,x,y)) &\leq d(y, y_{n+1}) + \frac{k}{8}[d(x_n, x_{n+1}) + 2d(y_n, y_{n+1}) \\ &\quad + d(x, F(x,y,z)) + 2d(y, F(y,x,y))], \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} d(z, F(z,y,x)) &\leq d(z, z_{n+1}) + \frac{k}{8}[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) \\ &\quad + d(x, F(x,y,z)) + d(y, F(y,x,y)) + d(z, F(z,y,x))]. \end{aligned} \tag{2.15}$$

By adding (2.22), (2.23), (2.24) we obtain

$$d(x, F(x,y,z)) + d(y, F(y,x,y)) + d(z, F(z,y,x))$$
$$\leq \frac{2}{2-k}[d(x, x_{n+1}) + d(y, y_{n+1}) + d(z, z_{n+1})]$$
$$+\frac{k}{4(2-k)}[3d(x_n, x_{n+1}) + 4d(y_n, y_{n+1}) + 2d(z_n, z_{n+1})].$$

So, by letting $n \to \infty$ in the previous inequality we get

$$d(x, F(x,y,z)) + d(y, F(y,x,y)) + d(z, F(z,y,x)) \leq 0,$$

which implies $d(x, F(x,y,z)) = 0, d(y, F(y,x,y)) = 0, d(z, F(z,y,x)) = 0$, that is, $x = F(x,y,z), y = F(y,x,y), z = F(z,y,x)$, as required.

□

Theorem 2.6. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0,1)$ such that*

$$d(F(x,y,z), F(y,z,u)) \leq \frac{k}{8}[d(x, F(y,z,u)) + d(y, F(z,y,z)) + \tag{2.16}$$
$$+d(z, F(u,z,y)) + d(y, F(x,y,z)) + d(z, F(y,x,y)) + d(u, F(z,y,x))].$$
$$\text{for all } x \geq y, y \leq z, z \geq u.$$

Also suppose either

(a) F is continuous or

(b) X has the following property:

$$(i) \text{ if a nondecreasing sequence } \{x_n\} \to x, \text{ then } x_n \leq x \text{ for all } n,$$
$$(ii) \text{ if a nonincreasing sequence } \{y_n\} \to y, \text{ then } y_n \geq y \text{ for all } n.$$

If there exist $x_0, y_0, z_0 \in X$ such that,

$$x_0 \leq F(x_0, y_0, z_0), y_0 \geq F(y_0, x_0, y_0) \text{ and } z_0 \leq F(z_0, y_0, x_0), \tag{2.17}$$

then there exist $x, y, z \in X$ such that,

$$x = F(x,y,z), y = F(y,x,y) \text{ and } z = F(z,y,x).$$

Proof. Let the sequences $\{x_n\}, \{y_n\}, \{z_n\} \subset X$ be defined by

$$x_{n+1} = F(x_n, y_n, z_n) = F^{n+1}(x_0, y_0, z_0), y_{n+1} = F(y_n, x_n, y_n) = F^{n+1}(y_0, x_0, y_0),$$
$$z_{n+1} = F(z_n, y_n, x_n) = F^{n+1}(z_0, y_0, x_0), (n = 0, 1, ...).$$

Since F^n is mixed monotone for every n [Proposition 2.4], it follows by (2.17) that $\{x_n\}$ and $\{z_n\}$ are nondecreasing and $\{y_n\}$ is nonincreasing. Indeed, due to the mixed monotone property of F, it is easy to show that

$$x_2 = F(x_1, y_1, z_1) \geq F(x_0, y_0, z_0) = x_1$$
$$y_2 = F(y_1, x_1, y_1) \leq F(y_0, x_0, y_0) = y_1$$
$$z_2 = F(z_1, y_1, x_1) \geq F(z_0, y_0, x_0) = z_1$$

and thus we obtain inductively that the three sequences satisfy the following conditions

$$x_0 \leq x_1 \leq ... \leq x_n \leq ...,$$
$$y_0 \geq y_1 \geq ... \geq y_n \geq ...,$$
$$z_0 \leq z_1 \leq ... \leq z_n \leq ....$$

Now, for $n \in N$, denote

$$D_{x_{n+1}} = d\left(x_{n+1}, x_n\right), D_{y_{n+1}} = d\left(y_{n+1}, y_n\right), D_{z_{n+1}} = d\left(z_{n+1}, z_n\right)$$

and

$$D_{n+1} = D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}.$$

Using (2.16), we get

$$\begin{aligned}
D_{x_{n+1}} &= d\left(x_{n+1}, x_n\right) = d(F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1})) \\
&\leq \frac{k}{8}\left[d\left(x_n, F_{x_{n-1}}\right) + d\left(y_n, F_{y_{n-1}}\right) + d\left(z_n, F_{z_{n-1}}\right)\right. \\
&\quad \left. + d\left(x_{n-1}, F_{x_n}\right) + d\left(y_{n-1}, F_{y_n}\right) + d\left(z_{n-1}, F_{z_n}\right)\right] \\
&= \frac{k}{8}\left[d\left(x_n, x_n\right) + d\left(y_n, y_n\right) + d\left(z_n, z_n\right)\right. \\
&\quad \left. + d\left(x_{n-1}, x_{n+1}\right) + d\left(y_{n-1}, y_{n+1}\right) + d\left(z_{n-1}, z_{n+1}\right)\right] \\
&= \frac{k}{8}\left[d\left(x_{n-1}, x_{n+1}\right) + d\left(y_{n-1}, y_{n+1}\right) + d\left(z_{n-1}, z_{n+1}\right)\right] \\
&\leq \frac{k}{8}\left[d\left(x_{n-1}, x_n\right) + d\left(y_{n-1}, y_n\right) + d\left(z_{n-1}, z_n\right)\right. \\
&\quad \left. + d\left(x_n, x_{n+1}\right) + d\left(y_n, y_{n+1}\right) + d\left(z_n, z_{n+1}\right)\right] \\
&= \frac{k}{8}\left[D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}\right],
\end{aligned}$$

therefore

$$D_{x_{n+1}} \leq \frac{k}{8}\left[D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}\right]. \tag{2.18}$$

Similarly, we obtain for the sequences $\left\{D_{y_{n+1}}\right\}, \left\{D_{z_{n+1}}\right\}$

$$\begin{aligned}
D_{y_{n+1}} &= d\left(y_{n+1}, y_n\right) = d(F(y_n, x_n, y_n), F(y_{n-1}, x_{n-1}, y_{n-1})) \\
&\leq \frac{k}{8}\left[d\left(y_n, F_{y_{n-1}}\right) + d\left(x_n, F_{x_{n-1}}\right) + d\left(y_n, F_{y_{n-1}}\right)\right. \\
&\quad \left. + d\left(y_{n-1}, F_{y_n}\right) + d\left(x_{n-1}, F_{x}n\right) + d\left(y_{n-1}, F_{y_n}\right)\right] \\
&= \frac{k}{8}\left[d\left(y_n, y_n\right) + d\left(x_n, x_n\right) + d\left(y_n, y_n\right)\right. \\
&\quad \left. + d\left(y_{n-1}, y_{n+1}\right) + d\left(x_{n-1}, x_{n+1}\right) + d\left(y_{n-1}, y_{n+1}\right)\right] \\
&= \frac{k}{8}\left[2d\left(y_{n-1}, y_{n+1}\right) + d\left(x_{n-1}, x_{n+1}\right)\right] \\
&\leq \frac{k}{8}\left[2d\left(y_{n-1}, y_n\right) + d\left(x_{n-1}, x_n\right) + d\left(x_n, x_{n+1}\right) + 2d\left(y_n, y_{n+1}\right)\right] \\
&= \frac{k}{8}\left[D_{x_n} + 2D_{y_n} + D_{x_{n+1}} + 2D_{y_{n+1}}\right],
\end{aligned}$$

so

$$D_{y_{n+1}} \leq \frac{k}{8}\left[D_{x_n} + 2D_{y_n} + D_{x_{n+1}} + 2D_{y_{n+1}}\right] \tag{2.19}$$

and

$$\begin{aligned}
D_{z_{n+1}} &= d\left(z_{n+1}, z_n\right) = d(F(z_n, y_n, x_n), F(z_{n-1}, y_{n-1}, x_{n-1})) \\
&\leq \frac{k}{8}\left[d\left(z_n, F_{z_{n-1}}\right) + d\left(y_n, F_{y_{n-1}}\right) + d\left(x_n, F_{x_{n-1}}\right)\right. \\
&\quad \left. + d\left(z_{n-1}, F_{z_n}\right) + d\left(y_{n-1}, F_{y_n}\right) + d\left(x_{n-1}, F_{x_n}\right)\right] \\
&= \frac{k}{8}\left[d\left(x_n, x_n\right) + d\left(y_n, y_n\right) + d\left(z_n, z_n\right)\right.
\end{aligned}$$

$$+d\left(x_{n-1},x_{n+1}\right)+d\left(y_{n-1},y_{n+1}\right)+d\left(z_{n-1},z_{n+1}\right)]$$
$$=\frac{k}{8}\left[d\left(x_{n-1},x_{n+1}\right)+d\left(y_{n-1},y_{n+1}\right)+d\left(z_{n-1},z_{n+1}\right)\right]$$
$$\leq\frac{k}{8}\left[d\left(x_{n-1},x_{n}\right)+d\left(y_{n-1},y_{n}\right)+d\left(z_{n-1},z_{n}\right)\right.$$
$$\left.+d\left(x_{n},x_{n+1}\right)+d\left(y_{n},y_{n+1}\right)+d\left(z_{n},z_{n+1}\right)\right]$$
$$=\frac{k}{8}\left[D_{x_n}+D_{y_n}+D_{z_n}+D_{x_{n+1}}+D_{y_{n+1}}+D_{z_{n+1}}\right],$$

therefore

$$D_{z_{n+1}}\leq\frac{k}{8}\left[D_{x_n}+D_{y_n}+D_{z_n}+D_{x_{n+1}}+D_{y_{n+1}}+D_{z_{n+1}}\right]. \tag{2.20}$$

By (2.18), (2.19) and (2.20), we get

$$D_{n+1}\leq\frac{k}{8}\left[3D_{x_n}+4D_{y_n}+2D_{z_n}+3D_{x_{n+1}}+4D_{y_{n+1}}+2D_{z_{n+1}}\right]$$
$$\leq\frac{k}{8}\left[4D_{x_n}+4D_{y_n}+4D_{z_n}+4D_{x_{n+1}}+4D_{y_{n+1}}+4D_{z_{n+1}}\right]$$
$$\leq\frac{k}{2}\left[D_n+D_{n+1}\right].$$

Therefore, for all $n\geq 1$, we have

$$D_{n+1}\leq\alpha\cdot D_n\leq ...\leq\alpha^n\cdot D_1,\ \text{where } \alpha=\frac{k}{2-k}\in[0,1),\ \text{when } k\in[0,1).$$

Because $D_{x_{n+1}}\leq D_{n+1}, D_{y_{n+1}}\leq D_{n+1}$ and $D_{z_{n+1}}\leq D_{n+1}$, we have

$$D_{x_{n+1}}\leq\alpha^n\cdot D_1, D_{y_{n+1}}\leq\alpha^n\cdot D_1 \text{ and } D_{z_{n+1}}\leq\alpha^n\cdot D_1, \tag{2.21}$$

which impliy that $\{x_n\},\{y_n\},\{z_n\}$ are Cauchy sequences in X. Indeed, let $m\geq n$, then

$$d\left(x_m,x_n\right)\leq D_{x_m}+D_{x_{m-1}}+...+D_{x_{n+1}}\leq$$
$$\leq\left[\alpha^{m-1}+\alpha^{m-2}+...+\alpha^{n}\right]\cdot D_1=\frac{\alpha^n-\alpha^m}{1-\alpha}\cdot D_1<\frac{\alpha^n}{1-\alpha}\cdot D_1.$$

Similarly, we can verify that $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences, too. Since X is a complete metric space, there exist $x,y,z\in X$ such that,

$$\lim_{x\to\infty}x_n=x,\ \lim_{x\to\infty}y_n=y,\ \lim_{x\to\infty}z_n=z.$$

Finally, we claim that

$$x=F\left(x,y,z\right),y=F\left(y,x,y\right)\text{and } z=F\left(z,y,x\right).$$

Suppose first that (b) holds and let $\epsilon>0$. Since F is continuous at (x,y,z), for a given $\frac{\epsilon}{2}>0$, there exists a $\delta>0$ such that,

$$d\left((x,y,z),(y,z,u)\right)=d\left(x,y\right)+d\left(y,z\right)+d\left(z,u\right)<\delta$$
$$\Rightarrow d\left(F\left(x,y,z\right),F\left(y,z,u\right)\right)<\frac{\epsilon}{2}.$$

Since

$$\lim_{x\to\infty}x_n=x,\ \lim_{x\to\infty}y_n=y,\ \lim_{x\to\infty}z_n=z,$$

for $\eta=\min\left(\frac{\epsilon}{2},\frac{\delta}{2}\right)$, there exist n_0,m_0,p_0 such that, for $n\geq n_0,m\geq m_0,p\geq p_0$,

$$d\left(x_n,x\right)<\eta,d\left(y_n,y\right)<\eta,d\left(z_n,z\right)<\eta.$$

Now, for $n \in \mathbb{N}, n \geq max\{n_0, m_0, p_0\}$,

$$d(F(x,y,z), x) \leq d(F(x,y,z), x_{n+1}) + d(x_{n+1}, x)$$
$$= d(F(x,y,z), F(x_n, y_n, z_n)) + d(x_{n+1}, x) < \frac{\epsilon}{2} + \eta \leq \epsilon.$$

This implies that $x = F(x,y,z)$. Similarly, we can show that

$$y = F(y,x,y) \text{ and } z = F(z,y,x).$$

Suppose now that (b) holds. Since $\{x_n\}, \{z_n\}$ is non-decreasings and $x_n \to x$, $z_n \to z$, and as $\{y_n\}$ is non-increasing and $y_n \to y$, from (b) we have $x_n \leq x, y_n \geq y$ and $z_n \leq z$, for all n. Then by triangle inequality and (2.16), we get

$$\begin{aligned} d(x, F(x,y,z)) &\leq d(x, x_{n+1}) + d(x_{n+1}, F(x,y,z)) \\ &= d(x, x_{n+1}) + d(F(x_n, y_n, z_n), F(x,y,z)) \\ &\leq d(x, x_{n+1}) + \frac{k}{8}[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) \\ &\quad + d(x, F(x,y,z)) + d(y, F(y,x,y)) + d(z, F(z,y,x))], \end{aligned} \tag{2.22}$$

$$\begin{aligned} d(y, F(y,x,y)) \leq d(y, y_{n+1}) + \frac{k}{8}[d(x_n, x_{n+1}) + 2d(y_n, y_{n+1}) \\ + d(x, F(x,y,z)) + 2d(y, F(y,x,y))], \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} d(z, F(z,y,x)) \leq d(z, z_{n+1}) + \frac{k}{8}[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) \\ + d(x, F(x,y,z)) + d(y, F(y,x,y)) + d(z, F(z,y,x))]. \end{aligned} \tag{2.24}$$

Adding (2.22), (2.23), (2.24) we obtain

$$\begin{aligned} d(x, F(x,y,z)) + d(y, F(y,x,y)) + d(z, F(z,y,x)) \\ \leq \frac{2}{2-k}[d(x, x_{n+1}) + d(y, y_{n+1}) + d(z, z_{n+1})] \\ + \frac{k}{4(2-k)}[3d(x_n, x_{n+1}) + 4d(y_n, y_{n+1}) + 2d(z_n, z_{n+1})] \end{aligned}$$

So, letting $n \to \infty$ one get

$$d(x, F(x,y,z)) + d(y, F(y,x,y)) + d(z, F(z,y,x)) \leq 0.$$

Hence $x = F(x,y,z), y = F(y,x,y), z = F(z,y,x)$. □

We end this section with an example that illustrates Theorem 2.5

Example 2.1. *(see Example in* [15]*) Let* $X = [0,1]$ *be endowed with the usual metric* $d(x,y) = |x-y|$ *and let* $F : X^3 \to X$ *be given by* $F(x,y,z) = \frac{11}{80}$*, for* $(x,y,z) \in \left[0, \frac{4}{5}\right] \times [0,1]^2$ *and* $F(x,y,z) = \frac{1}{20}$*, for* $(x,y,z) \in \left[\frac{4}{5}, 1\right] \times [0,1]^2$.

Then F *satisfies the Kannan-Prešić contractive condition* (2.7) *with* $k = \frac{14}{15} < 1$ *but does not satisfy the contractive condition* (1.6) *in Theorem 1.4 (a detailed proof follows the steps in* [15]*).*

F is not continuous but X satisfies assumption (b) in Theorem 2.5. Moreover, by taking $x_0 = 0$, $y_0 = \frac{1}{5}$ and $z_0 = \frac{1}{8}$, one can easily check that (2.8) *is fulfilled.*

Thus, the assumptions in Theorem 2.5 are satisfied and hence F has a unique tripled fixed point, $\left(\frac{11}{80}, \frac{11}{80}, \frac{11}{80}\right)$.

3. Conclusions

Our aim in this paper was to extend the Kannan fixed point theorem and Chatterjea fixed point theorem from single-valued self mappings $T : X \to X$ in metric spaces to mappings $F : X^3 \to X$ satisfying a Prešić-Kannan type contractive condition and a Prešić-Chatterjea type contractive condition, respectively, in partially ordered metric spaces.

As the theory of tripled fixed point is a natural continuation of coupled fixed point theory, similar results to the ones established here could be obtained for mappings $F : X^2 \to X$, see for example [2], [3], [4], [8], [9], and also for the case of multidimensional fixed point, see [21] for the case of quadruple fixed points.

By combining Banach contraction mapping principle (see for example [1]), Kannan fixed point theorem [20] and Chatterjea fixed point theorem [17], Zamfirescu [34] obtained a very interesting fixed point theorem in metric spaces that encompasses all these three classical fundamental fixed point theorems.

A similar result in the case of mixed monotone mappings $F : X^3 \to X$ in partially ordered metric spaces will unify Theorem 7 in [5] and Theorems 2.5 and 2.7 in the present paper.

In the case the mappings F in this paper are of a single variable, i.e., $F : X \to X$, then by Theorems 2.5 and 2.7 in the present paper we obtain a Kannan fixed point theorem and a Chatterjea fixed point theorem, respectively, in partially ordered metric spaces, see [10] and [18], where related results are obtained by other means.

Amongst the most important particular cases of Theorems 2.5 and 2.7 in this paper, we mention the ones in [20], [17], [24] and [23]. For other related results see also [1], [6], [7], [10], [16], [32], [22]-[28], [33].

Note that the first contractive condition considered in the present paper, that is,

$$d\left(F\left(x,y,z\right),F\left(y,z,u\right)\right) \le \frac{k}{8}[d\left(x,F(x,y,z)\right)+d\left(y,F(y,x,y)\right)+d\left(z,F(z,y,x)\right)+ \quad (3.25)$$
$$+d\left(y,F(y,z,u)\right)+d\left(z,F(z,y,z)\right)+d\left(u,F(u,z,y)\right)], \text{ for all } x \ge y, y \le z, z \ge u,$$

is similar but essentially different of the corresponding contractive condition used in [15],

$$d\left(F\left(x,y,z\right),F\left(u,v,w\right)\right) \le \frac{k}{8}[d\left(x,F(x,y,z)\right)+d\left(y,F(y,x,y)\right)+ \quad (3.26)$$
$$+d\left(z,F(z,y,x)\right)+d\left(u,F(u,v,w)\right)+d\left(v,F(v,u,v)\right)+d\left(w,F(w,v,u)\right)],$$
$$\text{for all } x \ge u, y \le v, z \ge w.$$

Indeed, while in (3.26) the arguments of the two $F's$ appearing in the lefthand side are independent, that is, $F(x,y,z)$ and $F(u,v,w)$, in the case of condition (3.25), two of the three independent variables coincide. Therefore, condition (3.25) is theoretically more general than (3.26) and thus our Theorem 2.5 in the present paper also generalises the main result result in [15].

Note also that the denominator of the coefficient appearing in the two contracting conditions (3.25) and (3.26), i.e., the number 8, appears to be tributary to the technique of proof we have used. It is our feeling that, the coefficient should be more naturally $\frac{k}{6}$ instead of $\frac{k}{8}$, because we have 6 terms in the right hand side of these inequalities and this would be in accordance with the original form of Kannan [20] and Chatterjea [17] contractive conditions:

there exists $a \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq a[\,d(x, Tx) + d(y, Ty)\,], \quad \textit{for all } x, y \in X. \tag{3.27}$$

and, respectively, there exists $b \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq b[\,d(x, Ty) + d(y, Tx)\,], \quad \textit{for all } x, y \in X, \tag{3.28}$$

where, each time, the sum of all coefficients in the right hand side of the inequality is less than 1. (Note that in the case of conditions (3.25) and (3.26) this sum is less than $\frac{6}{8}$).

Last, but not least, let us note that the tripled fixed point (x, y, z) in Theorem 2.5 and Theorem 2.6 can be obtained as the limit of the triple sequence (x_n, y_n, z_n) defined by

$$x_{n+1} = F(x_n, y_n, z_n) = F^{n+1}(x_0, y_0, z_0), y_{n+1} = F(y_n, x_n, y_n) = F^{n+1}(y_0, x_0, y_0),$$
$$z_{n+1} = F(z_n, y_n, x_n) = F^{n+1}(z_0, y_0, x_0), (n = 0, 1, ...),$$

where (x_0, y_0, z_0) is the triple satisfying condition (2.8) or (2.17), respectively.

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