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Controlling autonomous scalar discrete dynamical systems generated by non self Lipschitzian functions

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ABSTRACT. We apply fixed point techniques of Krasnoselskij type for stabilizing autonomous scalar discrete dynamical systems in the case of Lipschitzian functions.

1. INTRODUCTION

Discrete dynamical systems, even in one dimension, constitute suitable modeling tools for a large variety of real world phenomena. They also occur as components in hybrid dynamical systems which combine continuous and discrete behaviours. Time discretizations of continuous dynamical systems as well as the first return Poincaré maps associated to periodic orbits in continuous dynamical systems also lead to discrete dynamical systems.

Our aim in this paper is to give stability results for autonomous discrete dynamical systems generated by Lipschitz functions.

In this section we recall some related concepts and the fixed point theorems of the Krasnoselskij type iterations we use to establish the results of the paper.

The discrete dynamical system generated by the function $f : [a, b] \to [a, b]$, $a, b \in \mathbb{R}$, $a < b$, will be denoted by $[[a, b], f]$. In $[[a, b], f]$ the trajectory of an $x_0 \in [a, b]$ is the sequence $(x_n)_{n\in\mathbb{N}}$ given by the Picard iteration $x_{n+1} = f(x_n), n \in \mathbb{N}$. Motivated even by practical interpretations there is a main interest in analyzing the dependences on the starting points of the trajectories.

Denote F_f the set of fixed points of f. In case that the function $f : [a, b] \to [a, b]$ is continuous, the intermediate value theorem applied to the continuous function $f(x) - x$ and the inequalities $f(a) \geq a$, $f(b) \leq b$ assure that $F_f \neq \varnothing$; moreover, the set F_f is compact, since it is a bounded, closed subset of $\mathbb{R}$; thus F_f, as a nonempty compact subset of $\mathbb{R}$, has a least element and a greatest element.

In the discrete dynamical system $[[a, b], f]$ a fixed point x^* of f is considered, see [5, 7], as

-*attracting* or *stable* if there exists an open interval I which contains x^* such that $f(x) \in I$ for all $x \in I$ and $\lim\limits_{n\to\infty} f^n(x) = x^*$ for all $x \in I$;

-*repelling* or *instable* if there exists an open interval I which contains x^* such that for every $x \in I \setminus \{x^*\}$ there exists $n \in \mathbb{N}^*$ with $f^n(x) \notin I$.

If $[[a, b], f]$ is generated by a contraction (i. e. f satisfies a Lipschitz condition, see below, with constant $L < 1$) then all trajectories converge to the unique fixed point of f according to the contraction principle.

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If in $[[a,b], f]$ the function f is monotone increasing, then all trajectories converge monotonously, and if in addition f is continuous, they converge to some fixed points of f. If the function f is decreasing, then each trajectory splits in monotone convergent subsequences $(x_{2k})_{k\in\mathbb{N}}$, $(x_{2k+1})_{k\in\mathbb{N}}$, and if in addition f is continuous, then the subsequences converge to some fixed points of f^2. For results on Picard iterations with monotone and continuous functions see [9].

Regarding sequences of real numbers we consider the following special properties of monotony as in [3]. We call the sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ *s-increasing* if either $x_n < x_{n+1}$ for all $n \in \mathbb{N}$, or there is a $k \in \mathbb{N}$ such that $x_0 < x_1 < \ldots < x_{k-1} < x_k = x_{k+1} = x_{k+2} = \ldots$. We call the sequence $(x_n)_{n\in\mathbb{N}}$ *s-decreasing* if either $x_n > x_{n+1}$ for all $n \in \mathbb{N}$, or there is a $k \in \mathbb{N}$ such that $x_0 > x_1 > \ldots > x_{k-1} > x_k = x_{k+1} = x_{k+2} = \ldots$.

Slightly differently from [1] where the strict monotony is required, through this paper, as in [3], for fixed points in $[[a,b], f]$ we consider the following special types of stability. We call a fixed point x^* of f within $[[a,b], f]$ as

-*monotonously attracting from below* if there exists $\epsilon > 0$ such that all trajectories starting with $x_0 \in (x^* - \epsilon, x^*)$ are s-increasing and converge to x^*;

-*monotonously attracting from above* if there exists $\epsilon > 0$ such that all trajectories starting with $x_0 \in (x^*, x^* + \epsilon)$ are s-decreasing and converge to x^*;

-*monotonously stable* if it is monotonously attracting both from below and from above.

We deal with two families of functions related to f, as in [3], with $\gamma \in \mathbb{R}^*$

$$\overline{f}_\gamma : [a,b] \to \mathbb{R}, \overline{f}_\gamma(x) = x + \gamma(f(x) - x),$$

$$\widetilde{f}_\gamma : [a,b] \to \mathbb{R}, \widetilde{f}_\gamma(x) = x(1 + \gamma(f(x) - x)).$$

Comparing the fixed point sets

$$F_f = F_{\overline{f}_\gamma}, \gamma \in \mathbb{R}^*.$$

If $0 \notin [a,b]$ or $0 \in F_f$ then also

$$F_f = F_{\widetilde{f}_\gamma}, \gamma \in \mathbb{R}^*.$$

We will associate to $[[a,b], f]$ certain discrete dynamical systems $\left[J, \overline{f}_\gamma\right]$ respectively $\left[J, \widetilde{f}_\gamma\right]$ with the intervals $J \subset [a,b]$ and the values $\gamma \in \mathbb{R}^*$ suitably chosen.

We refer to $\left[J, \overline{f}_\gamma\right]$ as a *variation controlled* discrete dynamical system with control parameter γ.

In $\left[J, \overline{f}_\gamma\right]$ the trajectory of a $y_0 \in J$ is $(y_n)_{n\in\mathbb{N}}$ defined by $y_{n+1} = \overline{f}_\gamma(y_n)$, $n \in \mathbb{N}$, i. e.

$$y_{n+1} = y_n + \gamma(f(y_n) - y_n), n \in \mathbb{N}.$$

When $J = [a,b]$, in the system $\left[[a,b], \overline{f}_\gamma\right]$ with $\gamma \in (0,1)$ given, this is exactly a Krasnoselskij iteration applied to f. For results on Krasnoselskij iterations within more general settings see [2].

In $\left[J, \widetilde{f}_\gamma\right]$ the trajectory of a $z_0 \in J$ is $(z_n)_{n\in\mathbb{N}}$ given by $z_{n+1} = \widetilde{f}(z_n), n \in \mathbb{N}$, i. e.

$$z_{n+1} = z_n(1 + \gamma(f(z_n) - z_n)), n \in \mathbb{N}.$$

The relation

$$\frac{z_{n+1} - z_n}{z_n} = \gamma(f(z_n) - z_n)$$

constitutes the basis for referring to the system $\left[J, \widetilde{f}_\gamma\right]$ as a *growth-rate controlled* discrete dynamical system with control parameter γ, [8].

The fixed point theorems we recall below focus on functions f satisfying a Lipschitz condition

$$|f(u_1) - f(u_2)| \leq L\,|u_1 - u_2|\,, u_1, u_2 \in [a, b]\,,$$

where $L > 0$ is a constant, and propose monotonous Krasnoselskij type iterations of the function f to approximate fixed points of f. The iterations are of the form $x_{n+1} = \overline{f}_\gamma(x_n)$ respectively $x_{n+1} = \widetilde{f}_\gamma(x_n)$, without the functions involved, $\overline{f}_\gamma$ and $\widetilde{f}_\gamma$, being devised to be monotone or contractive on vicinities of the limit points.

Theorem 1.1. *([3], Theorem 2.1) Let $a, b \in \mathbb{R}$, $a < b$, $f : [a, b] \to [a, b]$ satisfy the Lipschitz condition with $L > 0$, and let $x_0 \in [a, b]$.*

i) If $f(x_0) > x_0$, letting $\gamma \in \left(0, \frac{1}{L+1}\right]$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = x_n + \gamma\,(f(x_n) - x_n)$, is s-increasing and convergent to $\min\,(F_f \cap [x_0, b])$.

ii) If $f(x_0) > x_0$ and $F_f \cap [a, x_0] \neq \varnothing$, letting $\gamma \in \left[-\frac{1}{L+1}, 0\right)$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = x_n + \gamma\,(f(x_n) - x_n)$, is s-decreasing and convergent to $\max\,(F_f \cap [a, x_0])$.

iii) If $f(x_0) < x_0$, letting $\gamma \in \left(0, \frac{1}{L+1}\right]$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = x_n + \gamma\,(f(x_n) - x_n)$, is s-decreasing and convergent to $\max\,(F_f \cap [a, x_0])$.

iv) If $f(x_0) < x_0$ and $F_f \cap [x_0, b] \neq \varnothing$, letting $\gamma \in \left[-\frac{1}{L+1}, 0\right)$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = x_n + \gamma\,(f(x_n) - x_n)$, is s-increasing and convergent to $\min\,(F_f \cap [x_0, b])$.

Theorem 2.2 from [3], and its proof, can be easily extended for nonself functions as follows.

Theorem 1.2. *Let $a, b \in \mathbb{R}$, $0 \leq a < b$, $f : [a, b] \to \mathbb{R}$ satisfy the Lipschitz condition with $L > 0$, and let $x_0 > 0$, $x_0 \in [a, b]$.*

i) If $f(x_0) > x_0$ and $F_f \cap [x_0, b] \neq \varnothing$, consider $p = \min\,(F_f \cap [x_0, b])$. Letting $\gamma \in \left(0, \frac{1}{p\,(L+1)}\right]$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = x_n\,(1 + \gamma\,(f(x_n) - x_n))$, is s-increasing and convergent to p.

ii) If $f(x_0) > x_0$ and $F_f \cap [a, x_0] \neq \varnothing$, consider $q = \max\,(F_f \cap [a, x_0])$. Letting $\gamma \in \left[-\frac{1}{x_0\,(L+1)}, 0\right)$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = x_n\,(1 + \gamma\,(f(x_n) - x_n))$, is s-decreasing and convergent to q.

iii) If $f(x_0) < x_0$ and $F_f \cap [a, x_0] \neq \varnothing$, consider $q = \max\,(F_f \cap [a, x_0])$. Letting $\gamma \in \left(0, \frac{1}{x_0\,(L+1)}\right]$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = x_n\,(1 + \gamma\,(f(x_n) - x_n))$, is s-decreasing and convergent to q.

iv) If $f(x_0) < x_0$ and $F_f \cap [x_0, b] \neq \varnothing$, consider $p = \min\,(F_f \cap [x_0, b])$. Letting $\gamma \in \left[-\frac{1}{p\,(L+1)}, 0\right)$, the sequence $(x_n)_{n\in\mathbb{N}}$, $x_{n+1} = x_n\,(1 + \gamma\,(f(x_n) - x_n))$, is s-increasing and convergent to p.

The particular case of $\gamma = \frac{1}{L+1}$ from i) and iii) in Theorem 1.1 recovers Hillam's result [6] which for $L = 1$ is a real line version of the Krasnoselskij's fixed point theorem.

The iterations considered in Theorem 1.2 are related to the growth-rate controlling mechanism studied under some assumptions on f' at the fixed point of f by Huang, W. [8].

Our main results in this paper are stated in Section 2 as Theorems 2.1-2.3.

2. Main results

The theorems we develop here have potential applicability in stabilizing autonomous scalar discrete dynamical systems generated by Lipschitz functions.

2.1. **Stability by variation control.** The following result is a consequence of Theorem 1.1.

Theorem 2.3. *Let $a, b \in \mathbb{R}$, $a < b$ and $f : [a, b] \to [a, b]$ satisfy the Lipschitz condition with $L > 0$ on the interval $[a, b]$. For $\gamma \in \mathbb{R} \setminus \{0\}$ define $\overline{f}_\gamma : [a, b] \to \mathbb{R}$, $\overline{f}_\gamma (x) = x + \gamma (f (x) - x)$. Let $c \in [a, b]$. The following holds.*

i) If $f(c) > c$, consider $p = \min (F_f \cap [c, b])$. Let $\gamma \in \left(0, \dfrac{1}{L+1}\right]$. In the dynamical system $\left[[c, p], \overline{f}_\gamma\right]$ the fixed point p is monotonously attracting from below.

ii) If $f(c) > c$ and $F_f \cap [a, c] \neq \varnothing$, consider $q = \max (F_f \cap [a, c])$. Let $\gamma \in \left[-\dfrac{1}{L+1}, 0\right)$. In $\left[[q, c], \overline{f}_\gamma\right]$ the fixed point q is monotonously attracting from above.

iii) If $f(c) < c$, consider $q = \max (F_f \cap [a, c])$. Let $\gamma \in \left(0, \dfrac{1}{L+1}\right]$. In $\left[[q, c], \overline{f}_\gamma\right]$ the fixed point q is monotonously attracting from above.

iv) If $f(c) < c$ and $F_f \cap [c, b] \neq \varnothing$, consider $p = \min (F_f \cap [c, b])$. Let $\gamma \in \left[-\dfrac{1}{L+1}, 0\right)$. In $\left[[c, p], \overline{f}_\gamma\right]$ the fixed point p is monotonously attracting from below.

Proof. The Lipschitz function f is continuous, so $f : [a, b] \to [a, b]$ possesses at least one fixed point. As we discussed earlier $F_f = F_{\overline{f}_\gamma}$.

i) Remark that $f(c) > c$ and $f(b) \leq b$ assure $F_f \cap [c, b] \neq \varnothing$. We show that $c \leq \overline{f}_\gamma(x) \leq p$ for all $x \in [c, p]$, i. e. the discrete dynamical system $\left[[c, p], \overline{f}_\gamma\right]$ is correctly defined. By the definition of p is clear that f has no fixed points in the interval $[c, p)$, so the continuous function $f(x) - x$ preserves its sign on the interval $[c, p)$; since $f(c) - c > 0$, it follows that $f(x) - x > 0$ for all $x \in [c, p)$. With $\gamma > 0$ it results that $\overline{f}_\gamma(x) - x = \gamma(f(x) - x) > 0$ and $\overline{f}_\gamma(x) > x \geq c$ for all $x \in [c, p)$. To prove $\overline{f}_\gamma(x) \leq p$ for all $x \in [c, p)$ suppose there is a $t \in [c, p)$ with $\overline{f}_\gamma(t) > p$. From $t < p < \overline{f}_\gamma(t)$ it follows successively

$$|p - t| < \left|\overline{f}_\gamma(t) - t\right| = |\gamma(f(t) - t)| = \gamma\, |f(t) - t| =$$
$$\gamma\, |f(t) - f(p) + p - t| \leq \gamma\, (|f(t) - f(p)| + |p - t|) \leq$$
$$\gamma\, (L\, |t - p| + |p - t|) = \gamma\, (L + 1)\, |p - t| \leq |p - t|,$$

which is a contradiction.

For all $x_0 \in [c, p)$, as the inequality $f(x_0) > x_0$ holds, i) from Theorem 1.1 applies, thus the sequence $x_{n+1} = \overline{f}_\gamma(x_n)$, that is the trajectory of x_0 in $\left[[c, p], \overline{f}_\gamma\right]$, is s-increasing and convergent to p.

ii) We show that $q \le \overline{f}_\gamma(x) \le c$ for all $x \in [q, c]$, i. e. $\left[[q, c], \overline{f}_\gamma\right]$ is correctly defined. By the definition of q is clear that f has no fixed points in $(q, c]$, so the continuous function $f(x) - x$ preserves its sign on the interval $(q, c]$; since $f(c) - c > 0$, it follows that $f(x) - x > 0$ for all $x \in (q, c]$. With $\gamma < 0$ it results that $\overline{f}_\gamma(x) - x = \gamma(f(x) - x) < 0$ and $\overline{f}_\gamma(x) < x \le c$ for all $x \in (q, c]$. To prove $\overline{f}_\gamma(x) \ge q$ for all $x \in (q, c]$ suppose there is a $t \in (q, c]$ with $\overline{f}_\gamma(t) < q$. From $\overline{f}_\gamma(t) < q < t$ it follows successively

$$|q - t| < \left|\overline{f}_\gamma(t) - t\right| = |\gamma(f(t) - t)| = |\gamma| \, |f(t) - t| =$$

$$|\gamma| \, |f(t) - f(q) + q - t| \le |\gamma| \left(|f(t) - f(q)| + |q - t|\right) \le$$

$$|\gamma| \left(L \, |t - q| + |q - t|\right) = |\gamma| \, (L + 1) \, |q - t| \le |q - t| \, ,$$

which is a contradiction.

For all $x_0 \in (q, c]$, as the inequality $f(x_0) > x_0$ holds, ii) from Theorem 1.1 applies, thus the sequence $x_{n+1} = \overline{f}_\gamma(x_n)$ i. e. the trajectory of x_0 in $\left[[q, c], \overline{f}_\gamma\right]$ is s-decreasing and convergent to q.

iii) Remark that $f(c) < c$ and $f(a) \ge a$ assure $F_f \cap [a, c] \ne \varnothing$. We show that $q \le \overline{f}_\gamma(x) \le c$ for all $x \in [q, c]$, i. e. $\left[[q, c], \overline{f}_\gamma\right]$ is correctly defined. By the definition of q is clear that f has no fixed points in $(q, c]$, so the continuous function $f(x) - x$ preserves its sign on the interval $(q, c]$; since $f(c) - c < 0$, it follows that $f(x) - x < 0$ for all $x \in (q, c]$. With $\gamma > 0$ it results that $\overline{f}_\gamma(x) - x = \gamma(f(x) - x) < 0$ and $\overline{f}_\gamma(x) < x \le c$ for all $x \in (q, c]$. To prove $\overline{f}_\gamma(x) \ge q$ for all $x \in (q, c]$ suppose there is a $t \in (q, c]$ with $\overline{f}_\gamma(t) < q$. From $\overline{f}_\gamma(t) < q < t$ it follows successively

$$|q - t| < \left|\overline{f}_\gamma(t) - t\right| = |\gamma(f(t) - t)| = \gamma \, |f(t) - t| =$$

$$\gamma \, |f(t) - f(q) + q - t| \le \gamma \left(|f(t) - f(q)| + |q - t|\right) \le$$

$$\gamma \left(L \, |t - q| + |q - t|\right) = \gamma \, (L + 1) \, |q - t| \le |q - t| \, ,$$

which is a contradiction.

For all $x_0 \in (q, c]$, as the inequality $f(x_0) < x_0$ holds, iii) from Theorem 1.1 applies, thus the sequence $x_{n+1} = \overline{f}_\gamma(x_n)$ i. e. the trajectory of x_0 in $\left[[q, c], \overline{f}_\gamma\right]$ is s-decreasing and convergent to q.

iv) We show that $c \le \overline{f}_\gamma(x) \le p$ for all $x \in [c, p]$, i. e. $\left[[c, p], \overline{f}_\gamma\right]$ is correctly defined. By the definition of p is clear that f has no fixed points in $[c, p)$, so the continuous function $f(x) - x$ preserves its sign on the interval $[c, p)$; since $f(c) - c < 0$, it follows that $f(x) - x < 0$ for all $x \in [c, p)$. With $\gamma < 0$ it results that $\overline{f}_\gamma(x) - x = \gamma(f(x) - x) > 0$ and $\overline{f}_\gamma(x) > x \ge c$ for all $x \in [c, p)$.To prove $\overline{f}_\gamma(x) \le p$ for all $x \in [c, p)$ suppose there is a $t \in [c, p)$ with $\overline{f}_\gamma(t) > p$. From $t < p < \overline{f}_\gamma(t)$ it follows successively

$$|p - t| < \left|\overline{f}_\gamma(t) - t\right| = |\gamma(f(t) - t)| = |\gamma| \, |f(t) - t| =$$

$$|\gamma| \, |f(t) - f(p) + p - t| \le |\gamma| \left(|f(t) - f(p)| + |p - t|\right) \le$$

$$|\gamma| \left(L \, |t - p| + |p - t|\right) = |\gamma| \, (L + 1) \, |p - t| \le |p - t| \, ,$$

which is a contradiction.

For all $x_0 \in [c, p)$, as the the inequality $f(x_0) < x_0$ holds, iv) from Theorem 1.1 applies, thus the sequence $x_{n+1} = \overline{f}_\gamma(x_n)$ i. e. the trajectory of x_0 in $\left[[c, p], \overline{f}_\gamma\right]$ is s-increasing and convergent to p. □

2.2. **Stability by growth-rate control.** The subsequent theorems are inspired by the growth-rate controlling mechanism studied under some assumptions on f' at the fixed point of f by Huang, W. [8].

The following theorem is a consequence of Theorem 1.2.

Theorem 2.4. *Let $a, b \in [0, +\infty)$, $a < b$ and $f : [a, b] \to \mathbb{R}$ satisfy the Lipschitz condition with $L > 0$ on the interval $[a, b]$. For $\gamma \in \mathbb{R} \setminus \{0\}$ define $\widetilde{f}_\gamma : [a, b] \to \mathbb{R}$, $\widetilde{f}_\gamma(x) = x(1 + \gamma(f(x) - x))$. Let $c \in [a, b]$. The following holds.*

i) If $f(c) > c$ and $F_f \cap [c, b] \neq \varnothing$, consider $p = \min(F_f \cap [c, b])$. Let $\gamma \in \left(0, \dfrac{1}{p(L+1)}\right]$. In the dynamical system $\left[[c, p], \widetilde{f}_\gamma\right]$ the fixed point p is monotonously attracting from below.

ii) If $f(c) > c$ and $F_f \cap [a, c] \neq \varnothing$, consider $q = \max(F_f \cap [a, c])$. Let $\gamma \in \left[-\dfrac{1}{c(L+1)}, 0\right)$. In $\left[[q, c], \widetilde{f}_\gamma\right]$ the fixed point q is monotonously attracting from above.

iii) If $f(c) < c$ and $F_f \cap [a, c] \neq \varnothing$, consider $q = \max(F_f \cap [a, c])$. Let $\gamma \in \left(0, \dfrac{1}{c(L+1)}\right]$. In $\left[[q, c], \widetilde{f}_\gamma\right]$ the fixed point q is monotonously attracting from above.

iv) If $f(c) < c$ and $F_f \cap [c, b] \neq \varnothing$, consider $p = \min(F_f \cap [c, b])$. Let $\gamma \in \left[-\dfrac{1}{p(L+1)}, 0\right)$. In $\left[[c, p], \widetilde{f}_\gamma\right]$ the fixed point p is monotonously attracting from below.

Proof. The function f is continuous as it satisfies a Lipschitz condition.

i) We show that $c \leq \widetilde{f}_\gamma(x) \leq p$ for all $x \in [c, p]$, i. e. the discrete dynamical system $\left[[c, p], \widetilde{f}_\gamma\right]$ is correctly defined. By the definition of p is clear that f has no fixed points in the interval $[c, p)$, so the continuous function $f(x) - x$ preserves its sign on the interval $[c, p)$; since $f(c) - c > 0$, it follows that $f(x) - x > 0$ for all $x \in [c, p)$. With $\gamma > 0$ and $x \geq 0$ it results that $\widetilde{f}_\gamma(x) - x = \gamma x(f(x) - x) \geq 0$ and $\widetilde{f}_\gamma(x) \geq x \geq c$ for all $x \in [c, p)$. To prove $\widetilde{f}_\gamma(x) \leq p$ for all $x \in [c, p)$ suppose there is a $t \in [c, p)$ with $\widetilde{f}_\gamma(t) > p$. From $t < p < \widetilde{f}_\gamma(t)$ it follows successively

$$|p - t| < \left|\widetilde{f}_\gamma(t) - t\right| = |\gamma t(f(t) - t)| = \gamma t\,|f(t) - t| =$$

$$\gamma t\,|f(t) - f(p) + p - t| \leq \gamma t\,(|f(t) - f(p)| + |p - t|) \leq$$

$$\gamma t\,(L\,|t - p| + |p - t|) = \gamma t\,(L+1)\,|p - t| < \gamma p\,(L+1)\,|p - t| \leq |p - t|,$$

which is a contradiction.

For all $x_0 \in (c, p)$, as the inequalities $x_0 > 0$ and $f(x_0) > x_0$ hold, i) from Theorem 1.2 applies, thus the sequence $x_{n+1} = \widetilde{f}_\gamma(x_n)$, that is the trajectory of x_0 in $\left[[c, p], \widetilde{f}_\gamma\right]$, is s-increasing and convergent to p.

ii) We show that $q \leq \widetilde{f}_\gamma(x) \leq c$ for all $x \in [q, c]$, i. e. $\left[[q, c], \widetilde{f}_\gamma\right]$ is correctly defined. By the definition of q is clear that f has no fixed points in $(q, c]$, so the continuous function $f(x) - x$ preserves its sign on the interval $(q, c]$; since $f(c) - c > 0$, it follows that $f(x) - x > 0$ for all $x \in (q, c]$. With $\gamma < 0$ and $x > 0$ it results that $\widetilde{f}_\gamma(x) - x = \gamma x(f(x) - x) < 0$ and $\widetilde{f}_\gamma(x) < x \leq c$ for all $x \in (q, c]$. To prove $\widetilde{f}_\gamma(x) \geq q$ for all $x \in (q, c]$, suppose there is a $t \in$

$(q\,,c]$ with $\widetilde{f}_\gamma(t) < q$. From $\widetilde{f}_\gamma(t) < q < t$ it follows successively

$$|q-t| < \left|\widetilde{f}_\gamma(t) - t\right| = |\gamma t(f(t)-t)| = |\gamma|\cdot t\cdot |f(t)-t| =$$

$$|\gamma|\cdot t\cdot |f(t)-f(q)+q-t| \le |\gamma|\, t\,(|f(t)-f(q)|+|q-t|) \le$$

$$|\gamma|\, t\,(L\,|t-q|+|q-t|) = |\gamma|\, t\,(L+1)\,|q-t| \le |\gamma|\, c\,(L+1)\,|q-t| \le |q-t|\,,$$

which is a contradiction.

For all $x_0 \in (q,c]$, as the inequalities $x_0 > 0$, $f(x_0) > x_0$ hold and $\gamma \in \left[-\dfrac{1}{x_0\,(L+1)}, 0\right)$ is guaranteed by $\gamma \in \left[-\dfrac{1}{c\,(L+1)}, 0\right)$, ii) from Theorem 1.2 applies, thus the sequence $x_{n+1} = \widetilde{f}_\gamma(x_n)$ i. e. the trajectory of x_0 in $\left[[q,c]\,,\widetilde{f}_\gamma\right]$ is s-decreasing and convergent to q.

iii) We show that $q \le \widetilde{f}_\gamma\,(x) \le c$ for all $x \in [q,c]$, i. e. $\left[[q,c]\,,\widetilde{f}_\gamma\right]$ is correctly defined. By the definition of q is clear that f has no fixed points in $(q,c]$, so the continuous function $f(x)-x$ preserves its sign on the interval $(q,c]$; since $f(c)-c<0$, it follows that $f(x)-x<0$ for all $x \in (q,c]$. With $\gamma > 0$ and $x > 0$ it results that $\widetilde{f}_\gamma(x) - x = \gamma x(f\,(x)-x) < 0$ and $\widetilde{f}_\gamma(x) < x \le c$ for all $x \in (q,c]$. To prove $\widetilde{f}_\gamma(x) \ge q$ for all $x \in (q\,,c]$ suppose there is a $t \in (q\,,c]$ with $\widetilde{f}_\gamma(t) < q$. From $\widetilde{f}_\gamma(t) < q < t$ it follows successively

$$|q-t| < \left|\widetilde{f}_\gamma(t) - t\right| = |\gamma t(f(t)-t)| = \gamma t\,|f(t)-t| =$$

$$\gamma t\,|f(t)-f(q)+q-t| \le \gamma t\,(|f(t)-f(q)|+|q-t|) \le$$

$$\gamma t\,(L\,|t-q|+|q-t|) = \gamma t\,(L+1)\,|q-t| \le \gamma c\,(L+1)\,|q-t| \le |q-t|\,,$$

which is a contradiction.

For all $x_0 \in (q,c]$, as the inequalities $x_0 > 0$, $f(x_0) < x_0$ hold and $\gamma \in \left(0, \dfrac{1}{x_0\,(L+1)}\right]$ is guaranteed by $\gamma \in \left(0, \dfrac{1}{c\,(L+1)}\right]$, iii) from Theorem 1.2 applies, thus the sequence $x_{n+1} = \widetilde{f}_\gamma(x_n)$ i. e. the trajectory of x_0 in $\left[[q,c]\,,\widetilde{f}_\gamma\right]$ is s-decreasing and convergent to q.

iv) We show that $c \le \widetilde{f}_\gamma\,(x) \le p$ for all $x \in [c,p]$, i. e. $\left[[c,p]\,,\widetilde{f}_\gamma\right]$ is correctly defined. By the definition of p is clear that f has no fixed points in $[c\,,p)$, so the continuous function $f(x)-x$ preserves its sign on the interval $[c\,,p)$; since $f(c)-c<0$, it follows that $f(x)-x<0$ for all $x \in [c\,,p)$. With $\gamma < 0$ and $x \ge 0$ it results that $\widetilde{f}_\gamma(x) - x = \gamma x(f\,(x)-x) \ge 0$ and $\widetilde{f}_\gamma(x) \ge x \ge c$ for all $x \in [c\,,p)$. To prove $\widetilde{f}_\gamma(x) \le p$ for all $x \in [c\,,p)$ suppose there is a $t \in [c,p)$ with $\widetilde{f}_\gamma(t) > p$. From $t < p < \widetilde{f}_\gamma(t)$ it follows successively

$$|p-t| < \left|\widetilde{f}_\gamma(t) - t\right| = |\gamma t(f(t)-t)| = |\gamma|\cdot t\cdot |f(t)-t| =$$

$$|\gamma|\cdot t\cdot |f(t)-f(p)+p-t| \le |\gamma|\, t\,(|f(t)-f(p)|+|p-t|) \le$$

$$|\gamma|\, t\,(L\,|t-p|+|p-t|) = |\gamma|\, t\,(L+1)\,|p-t| < |\gamma|\, p\,(L+1)\,|p-t| \le |p-t|\,,$$

which is a contradiction.

For all $x_0 \in (c,p)$, as the inequalities $x_0 > 0$ and $f(x_0) < x_0$ hold, iv) from Theorem 1.2 applies, thus the sequence $x_{n+1} = \widetilde{f}_\gamma(x_n)$ i. e. the trajectory of x_0 in $\left[[c,p]\,,\widetilde{f}_\gamma\right]$ is s-increasing and convergent to p. □

Remark 2.1. There is a range of γ, namely with values sufficiently close to 0, that independently on c or p satisfies the conditions form i) and that from iii) in Theorem 2.4: $\gamma \in \left(0, \dfrac{1}{b(L+1)}\right]$; respectively the conditions from ii) and that from iv) in Theorem 2.4: $\gamma \in \left[-\dfrac{1}{b(L+1)}, 0\right)$.

The following theorem can be proved similarly to the previous one.

Theorem 2.5. *Let $a, b \in (-\infty, 0]$, $a < b$ and $f : [a,b] \to \mathbb{R}$ satisfy the Lipschitz condition with $L > 0$ on the interval $[a,b]$. For $\gamma \in \mathbb{R} \setminus \{0\}$ define $\widetilde{f}_\gamma : [a,b] \to \mathbb{R}$, $\widetilde{f}_\gamma(x) = x(1 + \gamma(f(x) - x))$. Let $c \in [a,b]$. The following holds.*

i) If $f(c) > c$ and $F_f \cap [c,b] \neq \varnothing$, consider $p = \min(F_f \cap [c,b])$. Let $\gamma \in \left[\dfrac{1}{c(L+1)}, 0\right)$. In the dynamical system $\left[[c,p], \widetilde{f}_\gamma\right]$ the fixed point p is monotonously attracting from below.

ii) If $f(c) > c$ and $F_f \cap [a,c] \neq \varnothing$, consider $q = \max(F_f \cap [a,c])$. Let $\gamma \in \left(0, \dfrac{1}{-q(L+1)}\right]$. In $\left[[q,c], \widetilde{f}_\gamma\right]$ the fixed point q is monotonously attracting from above.

iii) If $f(c) < c$ and $F_f \cap [a,c] \neq \varnothing$, consider $q = \max(F_f \cap [a,c])$. Let $\gamma \in \left[\dfrac{1}{q(L+1)}, 0\right)$. In $\left[[q,c], \widetilde{f}_\gamma\right]$ the fixed point q is monotonously attracting from above.

iv) If $f(c) < c$ and $F_f \cap [c,b] \neq \varnothing$, consider $p = \min(F_f \cap [c,b])$. Let $\gamma \in \left(0, \dfrac{1}{-c(L+1)}\right]$. In $\left[[c,p], \widetilde{f}_\gamma\right]$ the fixed point p is monotonously attracting from below.

Remark 2.2. There is a range of γ, namely with values sufficiently close to 0, that independently on c or q satisfies the conditions form i) and that from iii) in Theorem 2.5: $\gamma \in \left[\dfrac{1}{a(L+1)}, 0\right)$; respectively the conditions from ii) and that from iv) in Theorem 2.5: $\gamma \in \left(0, \dfrac{1}{-a(L+1)}\right]$.

By choosing a convenient value for the control parameter γ when applying Theorem 2.3, 2.4 or 2.5 one may assure an acceptable rate of convergence for the controlled trajectories, and at the same time, for the function $\overline{f}_\gamma$, or $\widetilde{f}_\gamma$ respectively, to be not very different, on the considered interval, from the given function f.

Periodic points should be studied in a future work.

3. Numerical examples

Consider the discrete dynamical system $[[-3/2, 3/2], f]$, $f(x) = 2\left|x^2 - 1\right| - 1$. This function $f : [-3/2, 3/2] \to [-3/2, 3/2]$ satisfies a Lipschitz condition with $L = 6$, it is not differentiable at $x \in \{-1, 1\}$, and its fixed points set is

$$F_f = \left\{-1, \frac{1}{2}, \frac{3}{2}\right\}.$$

Remark that there is a fixed point of f, i. e. -1, where f is not differentiable. Figure 1a plots the graph of f.

In $[[-3/2,3/2], f]$ the trajectories of the starting points 0.7 and -0.6, as initiated in Figures 1b and 2a respectively, seem to be chaotic. On the horizontal axis is graphed n, on the vertical axis $f^n(0.7)$, respectively $f^n(-0.6)$. We will use the theorems of Section 2 to obtain stabilized systems.

Stabilizing by variation control. In $\left[[1/2,0.7], \overline{f}_{0.125}\right]$ the fixed point $1/2$ is monotonously attracting from above by Theorem 2.3 iii). In $\left[[0.7,3/2], \overline{f}_{-0.125}\right]$ the fixed point $3/2$ is monotonously attracting from below by Theorem 2.3 iv). In $\left[[-0.6,1/2], \overline{f}_{0.125}\right]$ the fixed point $1/2$ is monotonously attracting from below by Theorem 2.3 i). In $\left[[-1,-0.6], \overline{f}_{-0.125}\right]$ the fixed point -1 is monotonously attracting from above by Theorem 2.3 ii). Figure 2b displays the stabilized trajectories of 0.7 in the controlled systems $\left[[1/2,0.7], \overline{f}_{0.125}\right]$, by Diamond symbols, and $\left[[0.7,3/2], \overline{f}_{-0.125}\right]$, by Box symbols, as well as the stabilized trajectories of -0.6 in the controlled systems $\left[[-0.6,1/2], \overline{f}_{0.125}\right]$, by Circle symbols, and $\left[[-1,-0.6], \overline{f}_{-0.125}\right]$, by Asterisk symbols.

Stabilizing by growth-rate control. In $\left[[1/2,0.7], \widetilde{f}_{0.2}\right]$ the fixed point $1/2$ is monotonously attracting from above by Theorem 2.4 iii). In $\left[[0.7,3/2], \widetilde{f}_{-0.09}\right]$ the fixed point $3/2$ is monotonously attracting from below by Theorem 2.4 iv). In $\left[[-1,-0.6], \widetilde{f}_{0.09}\right]$ the fixed point -1 is monotonously attracting from above by Theorem 2.5 ii). Figure 3a displays the stabilized trajectories of 0.7 in the controlled systems $\left[[1/2,0.7], \widetilde{f}_{0.2}\right]$, by Diamond symbols, and $\left[[0.7,3/2], \widetilde{f}_{-0.09}\right]$, by Box symbols, as well as the stabilized trajectory of -0.6 in the controlled system $\left[[-1,-0.6], \widetilde{f}_{0.09}\right]$ by Circle symbols.

Figure 3b plots the graph of $\overline{f}_{0.125}$ (Dash) and the graph of $\overline{f}_{-0.125}$ (Dash Dot). Figure 4a plots the graphs of $\widetilde{f}_{0.2}$ (Dash), $\widetilde{f}_{0.09}$ (Dash Dot) and $\widetilde{f}_{-0.09}$ (Dot).

Figures 4b and 5a are stability diagrams of $\left\{ \overline{f}_\gamma \middle| \gamma \in [-1/7,1/7] \setminus \{0\} \right\}$, for the starting points 0.7 and -0.6 respectively. A stability diagram here suggests, through the levels attained on the vertical axis, the fixed points monotonously attracting form below (above) in the controlled systems depending on the values of the control parameter γ. Figures 4b and 5a are realized by depicting the 20 points $\left(\gamma, \overline{f}^n_\gamma(0.7)\right)$, $n \in \{101, 102, \ldots, 120\}$, and respectively $\left(\gamma, \overline{f}^n_\gamma(-0.6)\right)$, $n \in \{101, 102, \ldots, 120\}$, for 201 equidistant values of $\gamma \in [-1/7,1/7] \setminus \{0\}$.

Figure 5b shows a stability diagram of $\left\{ \widetilde{f}_\gamma \middle| \gamma \in [0.01,0.2] \right\}$ for the starting point 0.7 by depicting the points $\left(\gamma, \widetilde{f}^n_\gamma(0.7)\right)$, $n \in \{101, 102, \ldots, 120\}$ with 101 equidistant values of $\gamma \in [0.01, 0.2]$. Figure 6 shows a stability diagram of $\left\{ \widetilde{f}_\gamma \middle| \gamma \in [-0.1,0.1] \setminus \{0\} \right\}$ for the starting point -0.6 by depicting the points $\left(\gamma, \widetilde{f}^n_\gamma(-0.6)\right)$, $n \in \{101, 102, \ldots, 120\}$ with 101 equidistant values of $\gamma \in [-0.1, 0.1] \setminus \{0\}$.

Remark 3.3. The functions $\overline{f}_\gamma$ and $\widetilde{f}_\gamma$ in Theorems 2.3-2.5 are not required to be monotone or contractive on the interval between c and the fixed point p respectively q. In $\left[[0.7,3/2], \widetilde{f}_{-0.09}\right]$ the fixed point $3/2$ is monotonously attracting from below; $\widetilde{f}_{-0.09}$ is not a contraction on the interval $[0.7, 3/2]$. In $\left[[1/2,1], \widetilde{f}_{0.2}\right]$ the fixed point $1/2$ is monotonously attracting from above according to Theorem 2.4. iii), while the function $\widetilde{f}_{0.2}$ is not monotone on $[1/2, 1]$, see its graph in Figure 4a.

Remark 3.4. Figure 6 also shows that the sequence $\left(\widetilde{f}_{\gamma}^{n}(-0.6)\right)_{n\in\mathbb{N}}$ with $\gamma \in [-0.1, 0)$ converges to the fixed point 0 of $\widetilde{f}_{\gamma}$ which is not a fixed point of f - actually f has no fixed points between -0.6 and 0.

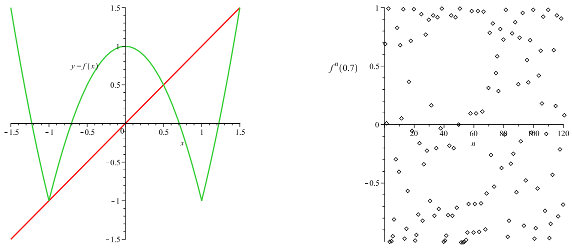

FIGURE 1. a) left and b) right

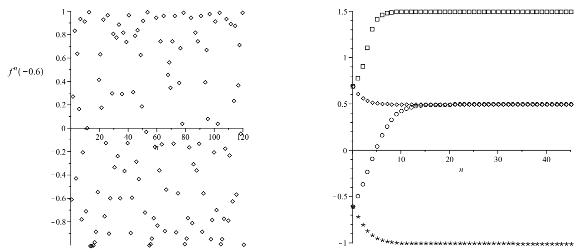

FIGURE 2. a) left and b) right

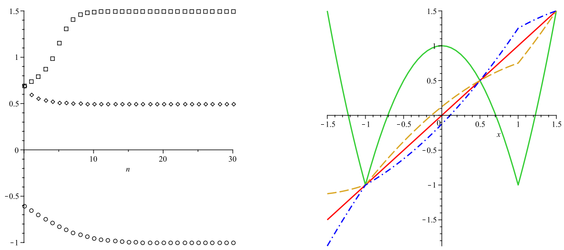

FIGURE 3. a) left and b) right

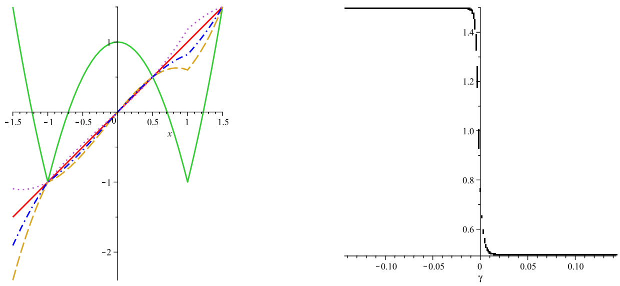

FIGURE 4. a) left and b) right

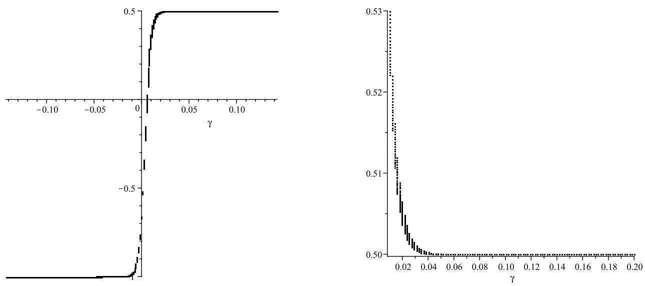

FIGURE 5. a) left and b) right

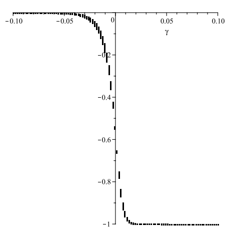

FIGURE 6.

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