Chapter 4
The Retraction-Displacement Condition in the Theory of Fixed Point Equation with a Convergent Iterative Algorithm

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Abstract Let (X, d) be a complete metric space and $f : X \to X$ be an operator with a nonempty fixed point set, i.e., $F_f := \{x \in X : x = f(x)\} \neq \emptyset$. We consider an iterative algorithm with the following properties:

(1) for each $x \in X$ there exists a convergent sequence $(x_n(x))$ such that $x_n(x) \to x^*(x) \in F_f$ as $n \to \infty$;
(2) if $x \in F_f$, then $x_n(x) = x$, for all $n \in \mathbb{N}$.

In this way, we get a retraction mapping $r : X \to F_f$, given by $r(x) = x^*(x)$. Notice that, in the case of Picard iteration, this retraction is the operator f^∞, see I.A. Rus (Picard operators and applications, Sci. Math. Jpn. 58(1):191–219, 2003). By definition, the operator f satisfies the retraction-displacement condition if there is an increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous at 0 and satisfies $\psi(0) = 0$, such that

$$d(x, r(x)) \leq \psi(d(x, f(x))), \text{ for all } x \in X.$$

In this paper, we study the fixed point equation $x = f(x)$ in terms of a retraction-displacement condition. Some examples, corresponding to Picard, Krasnoselskii, Mann and Halpern iterative algorithms, are given. Some new research directions and open questions are also presented.

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4.1 Introduction

In this paper we will consider the following two conditions involving a ~~singlevalued~~ single-valued operator f from a metric space X to itself.

Definition 1. Let (X, d) be a metric space and $f : X \to X$ be an operator so that its fixed point set F_f is nonempty. Let $r : X \to F_f$ be a set retraction. Then, by definition, f satisfies the (ψ, r) retraction-displacement condition (ψ-condition in [11], (ψ, r)-operator in [41], ψ-weakly Picard operator in the case of Picard iterations in [37], the collage condition in [2]) if:

(i) $\psi : R_+ \to R_+$ is increasing, continuous at 0 and $\psi(0) = 0$;
(ii) $d(x, r(x)) \leq \psi(d(x, f(x)))$, for every $x \in X$.

Remark 1. If $F_f = \{x^*\}$, then the (ψ, r) retraction-displacement condition takes the following form:

(i) $\psi : R_+ \to R_+$ is increasing, continuous at 0 and $\psi(0) = 0$;
(ii) $d(x, x^*) \leq \psi(d(x, f(x)))$, for every $x \in X$.

We will call it the (x^*, ψ) retraction-displacement condition.

Definition 2. Let (X, d) be a metric space and $f : X \to X$ be an operator so that its fixed point set F_f is nonempty. Let

$$\begin{cases} x_{n+1} = f_n(x_n), n \in N \\ x_0 \in X \end{cases} \tag{4.1}$$

(where $f_n : X \to X$ is a sequence of ~~singlevalued~~ single-valued operators, for $n \in N$) be an iterative algorithm such that ~~:~~

(i) $F_{f_n} = F_f$;
(ii) the sequence $(x_n)_{n \in N}$ converges to an element $x^*(x_0) \in F_f$ as $n \to +\infty$.

If we denote by $r : X \to F_f$ given by $r(x) := x^*(x)$ the retraction defined above, then, by definition, the above algorithm satisfies a retraction-displacement condition if the operator f satisfies a (ψ, r) retraction condition.

Notice that r is the limit operator of the iterative algorithm.

In this paper we study the fixed point equation $x = f(x)$ in terms of a retraction-displacement condition. Some examples, corresponding to Picard, Krasnoselskii, Mann and Halpern iterative algorithms, are given. Some new research directions are also presented.

Through the paper we will denote by R the set of real numbers, by N the set of natural numbers, by R_+ the set ~~f~~ of positive numbers, by R_+^* the set of strict positive numbers and by $N^* := N \setminus \{0\}$. We also denote by R_+^m the space of all m-dimensional vectors with positive components.

4.2 Generalized Contractions: Examples

We will start this section by presenting some notions and results concerning generalized contractions, which will be used in our main section.

A first generalization of the Banach contraction principle involves the concept of comparison function.

Definition 3 ([36,39]). A function $\varphi : R_+ \to R_+$ is called a comparison function if it satisfies:

$(i)_\varphi$ φ is increasing;
$(ii)_\varphi$ the sequence $(\varphi^n(t))_{n\in N}$ converges to 0 as $n \to \infty$, for all $t \in R_+$.

If the condition $(ii)_\varphi$ is replaced by the condition:

$(iii)_\varphi$ $\sum_{k=0}^{\infty} \varphi^k(t) < \infty$, for any $t > 0$,

then φ is called a strong comparison function.

Moreover, if the condition $(iii)_\varphi$ is replaced by the condition:

$(iv)_\varphi$ $t - \varphi(t) \to +\infty$, as $t \to +\infty$,

then φ is said to be a strict comparison function.

As a consequence of the above definition, we have the following lemmas.

Lemma 1 ([36,39]). *If $\varphi : R_+ \to R_+$ is a comparison function, then the following hold:*

(i) $\varphi(t) < t$, *for any* $t > 0$*;*
(ii) $\varphi(0) = 0$*;*
(iii) φ *is continuous at* 0.

Lemma 2 ([3,24,39]). *If $\varphi : R_+ \to R_+$ is a strong comparison function, then the following hold:*

(i) φ *is a comparison function;*
(ii) *the function* $s : R_+ \to R_+$*, defined by*

$$s(t) = \sum_{k=0}^{\infty} \varphi^k(t), \ t \in R_+, \tag{4.2}$$

is increasing and continuous at 0*;*
(iii) *there exist* $k_0 \in N$, $a \in (0,1)$ *and a convergent series of nonnegative terms* $\sum_{k=1}^{\infty} v_k$ *such that*

$$\varphi^{k+1}(t) \le a\varphi^k(t) + v_k, \text{ for } k \ge k_0 \text{ and any } t \in R_+.$$

Remark 2. Some authors use the notion of (c)-comparison function defined by the statements (i) and (iii) in Lemma 2. Actually, the concept of (c)-comparison function coincides with that of strong comparison function.

Example 1. (1) $\varphi : R_+ \to R_+, \varphi(t) = at$, where $a \in [0, 1[$, is a strong comparison function and a strict comparison function. In this case, f is called a contraction with constant $a \in [0, 1[$.

(2) $\varphi : R_+ \to R_+, \varphi(t) = \dfrac{t}{1+t}$ is a strict comparison function, but is not a strong comparison function.

(3) $\varphi : R_+ \to R_+$ defined by

$$\varphi(t) := \begin{cases} \frac{1}{2}t, & t \in [0, 1] \\ t - \frac{1}{2}, & t > 1 \end{cases}$$

is a strong comparison function.

(4) $\varphi : R_+ \to R_+, \varphi(t) = at + \dfrac{1}{2}[t]$, where $a \in \left]0, \dfrac{1}{2}\right[$ is a strong comparison function.

For other considerations on comparison functions, see [3,17,18,36,39], and the references therein.

4.3 The Retraction-Displacement Condition in the Theory of Weakly Picard Operators

The first part of the following result is known as Matkowski's Theorem (see [20]), while the second part belongs to Rus [36].

Theorem 1. *Let (X, d) be a complete metric space and $f : X \to X$ be a φ-contraction, i.e., $\varphi : R_+ \to R_+$ is a comparison function and*

$$d(f(x), f(y)) \leq \varphi(d(x, y)) \text{ for all } x, y \in X.$$

Then f is a Picard operator, i.e., f has a unique fixed point $x^* \in X$ and $\lim_{n\to+\infty} f^n(x) = x^*$, for all $x \in X$. Moreover, if $\varphi : R_+ \to R_+$ is a strict comparison function, then f is a ψ-Picard operator, i.e., f is a Picard operator and

$$d(x, x^*) \leq \psi_\varphi(d(x, f(x))), \text{ for all } x \in X,$$

where $\psi_\varphi : R_+ \to R_+$ is given by $\psi_\varphi(t) := \sup\{s | s - \varphi(s) \leq t\}$.

The second conclusion of the above theorem gives an answer to the following very general problem.

Problem 1. Let (X, d) be a metric space and $f : X \to X$. Which generalized contractions f are ψ-Picard operators? Which generalized contractions f satisfy a (ψ, r) retraction-displacement condition?

A general result concerning the above problem is the following.

Theorem 2. *Let (X, d) be a metric space, $f : X \to X$ be an operator, $\varphi : R_+ \to R_+$ a strict comparison function and $\theta : R_+ \to R_+$ be an increasing function, continuous at 0 with $\theta(0) = 0$. We suppose that:*

(i) $F_f = \{x^*\}$;
(ii) $d(f(x), x^*) \le \varphi(d(x, x^*)) + \theta(d(x, f(x)))$, *for all* $x \in X$.

Then:

$$d(x, x^*) \le \psi_\varphi(d(x, f(x)) + \theta(d(x, f(x)))), \text{ for all } x \in X, \tag{4.3}$$

i.e., f is a ψ-Picard operator with $\psi = \psi_\varphi \circ \left(1_{R_+} + \theta\right)$

Proof. We have

$$d(x, x^*) \le d(x, f(x)) + d\left(f(x), x^*\right) \le$$
$$\le d(x, f(x)) + \varphi\left(d(x, x^*)\right) + \theta(d(x, f(x))).$$

From the definition of ψ_φ we get the conclusion. We remark that the function $\psi = \psi_\varphi \circ \left(1_{R_+} + \theta\right)$ is increasing, continuous at 0 and $\psi(0) = 0$.

A class of ψ-Picard operators (with a particular $\psi(t) := \alpha t$, for some $\alpha \in [0, 1[$ and with a ~~particulate~~ particular $\theta(t) = Lt$, for some $L \ge 0$) is given by the following consequence.

Corollary 1. *Let (X, d) be a metric space and $f : X \to X$ be an operator. We suppose:*

(a) $F_f = \{x^*\}$;
(b) (see [32]) there exists $\alpha \in [0, 1[$ *and* $L \ge 0$ *such that*

$$d(f(x), x^*) \le \alpha d(x, x^*) + Ld(x, f(x)), \text{ for all } x \in X.$$

Then:

$$d(x, x^*) \le \frac{1 + L}{1 - \alpha} d(x, f(x))), \text{ for all } x \in X.$$

We will present now some examples of generalized contractions which satisfy the assumptions (a) and (b) in the above theorems. For example, using the Hardy and Rogers type condition, we can prove the following result.

Theorem 3. *Let (X, d) be a complete metric space and $f : X \to X$ be an operator. We suppose there exist $a, b, c \in R_+$ with $a + 2b + 2c < 1$ such that, for all $x, y \in X$, we have*

$$d(f(x), f(y)) \leq ad(x, y) + b(d(x, f(x)) + d(y, f(y))) + c(d(x, f(y)) + d(y, f(x))).$$

Then:

(i) $F_f = \{x^*\}$;
(ii) $d(x, x^*) \leq \frac{1+b-c}{1-a-2c} d(x, f(x)))$, *for all* $x \in X$.

Proof. Let $x_0 \in X$ and $x_n := f^n(x_0), n \in N^*$. Then, by ~~Hardy-Rogers~~Hardy–Rogers' fixed point theorem we get $F_f = \{x^*\}$. Now, we also have ~~:~~

$$d(f(x), x^*) = d(f(x), f(x^*)) \leq$$
$$ad(x, x^*) + b(d(x, f(x) + d(x^*, f(x^*)) + c(d(x, f(x^*)) + d(x^*, f(x))).$$

Thus

$$d(f(x), x^*) \leq \frac{a + c}{1 - c} d(x, x^*) + \frac{b}{1 - c} d(x, f(x), \text{ for all } x \in X.$$

The conclusion follows now from Theorem 2.

We will discuss now the case of so-called Suzuki type contractions.

Theorem 4. *Let (X, d) be a metric space, $\theta : R_+ \to R_+$ such that $\theta(0) = 0$, $\varphi : R_+ \to R_+$ be a strict comparison function and $f : X \to X$ be an operator. We suppose that* ~~:~~

(i) $F_f = \{x^*\}$;
(ii) $x, y \in X, \theta(d(f(x), x)) \leq d(x, y) \Longrightarrow d(f(x), f(y)) \leq \varphi(d(x, y))$.

Then:

$$d(x, x^*) \leq \psi_\varphi(d(x, f(x))), \text{ for all } x \in X. \tag{4.4}$$

Proof. We take in (ii) $x = x^*$ and we apply Theorem 2.

Remark 3. It is worth to notice that there exists operators which satisfy all the assumptions in Theorem 2, but which are not Picard operators. For example, $f : R \to R, f(x) := 2x$. In this case, $F_f := \{0\}$ and

$$|f(x)| \leq \frac{1}{2}|x| + 2|x - f(x)|, \text{ for all } x \in X,$$

but f is not a Picard operator.

In the next part of this section, the case of the cyclic φ-contractions is discussed. One of the most important ~~generalization~~ generalizations of the Banach Contraction Principle was given in 2003 by Kirk, Srinivasan and Veeramani, using the concept of cyclic operator. More precisely, they proved in [19] the following result.

Theorem 5 ([19, Theorem 2.4]). *Let* $\{A_i\}_{i=1}^m$ *be nonempty subsets of a complete metric space and suppose* $f : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ *satisfies the following conditions:*

(1) $f(A_i) \subseteq A_{i+1}$ *for* $1 \le i \le m$, *where* $A_{m+1} = A_1$;
(2) $d(f(x), f(y)) \le \varphi(d(x, y))$, *for all* $x \in A_i$, *and all* $y \in A_{i+1}$, *for* $1 \le i \le m$, *where the mapping* $\varphi : R_+ \to R_+$ *is upper semi-continuous from the right and satisfies the condition* $0 \le \varphi(t) < t$ *for* $t > 0$.

Then f *has a unique fixed point.*

An extension of this result was given by Păcurar and Rus in [24].

Theorem 6 ([24, Theorem 2.1]). *Let* $\{A_i\}_{i=1}^m$ *be nonempty subsets of a complete metric space and suppose* $f : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ *satisfies the following conditions:*

(1) $f(A_i) \subseteq A_{i+1}$ *for* $1 \le i \le m$, *where* $A_{m+1} = A_1$;
(2) there exists a comparison function $\varphi : R_+ \to R_+$ *such that*

$$d(f(x), f(y)) \le \varphi(d(x, y)), \text{ for all } x \in A_i \text{ and all } y \in A_{i+1},\ 1 \le i \le m.$$

Then:

(a) f *has a unique fixed point* $x^* \in \bigcap_{i=1}^m A_i$ *and, for each* $x \in A := \bigcup_{i=1}^m A_i$ *the sequence* $x_n := (f^n(x))_{n \in N}$ *converges to* x^* *as* $n \to +\infty$;
(b) the following estimates take place:

$$d(x_n, x^*) \le s(\varphi^n(d(x_0, f(x_0))) \text{ and } d(x_n, x^*) \le s(\varphi(d(x_n, f(x_n))), \text{ for all } n \in N^*;$$

(c) the following relation holds:

$$d(x, x^*) \le s(d(x, f(x))), \text{ for all } x \in A,$$

where $s : R_+ \to R_+$ *is defined by* $s(t) := \sum_{k=0}^{\infty} \varphi^k(t)$.

Our next result is an extension of the previous result.

Theorem 7. *Let* $\{A_i\}_{i=1}^m$ *be nonempty subsets of a complete metric space and suppose* $f : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ *satisfies the following conditions:*

(1) $f(A_i) \subseteq A_{i+1}$ *for* $1 \leq i \leq m$, *where* $A_{m+1} = A_1$;
(2) there exists a strict comparison function $\varphi : R_+ \to R_+$ *such that*

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \text{ for all } x \in A_i \text{ and all } y \in A_{i+1},\ 1 \leq i \leq m.$$

Then:

(a) f *is a Picard operator, i.e.,* f *has a unique fixed point and, for each* $x \in A := \bigcup_{i=1}^m A_i$ *the sequence* $x_n := (f^n(x))_{n \in N}$ *converges to* x^* *as* $n \to +\infty$;
(b) $x^* \in \bigcap_{i=1}^m A_i$;
(c) the following relation holds:

$$d(x, x^*) \leq \psi_\varphi(d(x, f(x))), \text{ for all } x \in A,$$

where $\psi_\varphi : R_+ \to R_+$ *is given by* $\psi_\varphi(t) := \sup\{s | s - \varphi(s) \leq t\}$.

Proof. The conclusions (a) and (b) follow by Theorem 6. The conclusion (c) follows (since $x^* \in \bigcap_{i=1}^m A_i$) by the following relations:

$$d(x, x^*) \leq d(x, f(x)) + d(f(x), f(x^*)) \leq d(x, f(x)) + \varphi(d(x, x^*)).$$

Another general open problem is the following one.

Problem 2. Let (X, d) be a metric space and $f : X \to X$ be such that there exists $n_0 \in N^*$ such that f^{n_0} is a generalized contraction. Under which conditions f is a ψ-Picard operator (or a ψ-weakly Picard operator)? Under which conditions f satisfies the ψ-condition with respect to a set retraction r?

For the above problem, we have the following result (see [22,35,46,52], etc.).

Theorem 8. *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be such that there exists* $n_0 \in N^*$ *such that* f^{n_0} *is a contraction with constant* $a \in [0, 1[$. *We also suppose that* f *is L-Lipschitz with constant* $L \geq 1$. *Then, the following conclusions hold:*

(i) f *is a Picard operator and* $F_f = \{x^*\}$;
(ii) (a) if $L > 1$, *then*

$$d(x, x^*) \leq \frac{L^{n_0} - 1}{(1 - a)(L - 1)} d(x, f(x)), \text{ for all } x \in X.$$

(b) if $L = 1$*, then*

$$d(x, x^*) \leq \frac{n_0}{1-a} d(x, f(x)), \text{ for all } x \in X.$$

Proof. We only need to prove (ii). Since f^{n_0} is a contraction with constant $a \in [0, 1[$, we get that f is a $\frac{1}{1-a}$-Picard operator. Thus, for all $x \in X$, we have:

$$d(x, x^*) \leq \frac{1}{1-a} d(x, f^{n_0}(x)) \leq$$

$$\frac{1}{1-a}(d(x, f(x)) + d(f(x), f^2(x)) + d(f^{n_0-1}(x), f^{n_0}(x)))$$

$$\leq \frac{L^{n_0}-1}{(1-a)(L-1)} d(x, f(x)).$$

In a similar way one obtain (b).

A third general problem is the following.

Problem 3. Let (X, d) be a metric space and $f : X \to X$ be such that there exists $n_0 \in N^*$ such that $f|_{f^{n_0}(X)} : f^{n_0}(X) \to f^{n_0}(X)$ is a generalized contraction. Under which conditions f is a ψ-Picard operator (or a ψ-weakly Picard operator)? Under which conditions f satisfies the ψ-condition with respect to a set retraction r?

A first answer to the above problem is the following theorem.

Theorem 9. *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be such that there exists* $n_0 \in N^*$ *such that* $f|_{f^{n_0}(X)} : f^{n_0}(X) \to f^{n_0}(X)$ *is a contraction with constant* $a \in [0, 1[$*. Suppose that* f *is Lipschitz with constant* $L \geq 1$*. Then, the following conclusions hold:*

(i) f *is a Picard operator and* $F_f = \{x^*\}$*;*
(ii) (a) if $L > 1$*, then*

$$d(x, x^*) \leq \left(\frac{L^{n_0}-1}{L-1} + \frac{L^{n_0}}{1-a}\right) \cdot d(x, f(x)), \text{ for all } x \in X.$$

(b) if $L = 1$*, then*

$$d(x, x^*) \leq \left(n_0 + \frac{1}{1-a}\right) \cdot d(x, f(x)), \text{ for all } x \in X.$$

Proof. (a) Since $f|_{f^{n_0}(X)}$ is a contraction with constant a, we obtain that $f|_{\overline{f^{n_0}(X)}}$ is a contraction with constant a, too. Thus, $F_f := \{x^*\}$. Moreover, since f^{n_0} is a $\frac{1}{1-a}$-Picard operator, for all $x \in X$, we have:

$$d(f^{n_0}(x), x^*) \leq \frac{1}{1-a} d(f^{n_0}(x), f(f^{n_0}(x))) \leq \frac{L^{n_0}}{1-a} d(x, f(x)).$$

On the other hand, for all $x \in X$, we can write:

$$d(x,x^*) \leq d(x,f^{n_0}(x)) + d(f^{n_0}(x),x^*) \leq$$
$$(d(x,f(x)) + d(f(x),f^2(x)) + d(f^{n_0-1}(x),f^{n_0}(x))) + \frac{L^{n_0}}{1-a}d(x,f(x)) \leq \cdots \leq$$
$$\frac{L^{n_0}-1}{L-1}d(x,f(x)) + \frac{L^{n_0}}{1-a}d(x,f(x)).$$

Notice now that (b) follows by a similar approach.

Notice now that in Theorem 2 and Corollary 1 the uniqueness of the fixed point can be deduced by the imposed condition (b). In the absence of the uniqueness assumption for the fixed point, we can prove the following extension of Corollary 1.

Corollary 2. *Let* (X, d) *be a metric space and* $f : X \to X$ *be an operator. We suppose:*

(a) for each $x \in X$ *there exists a sequence* $(x_n(x))_{n\in N}$ *and there exists* $x^*(x) \in F_f$ *with* $\lim_{n\to+\infty} x_n(x) = x^*(x)$*. In particular, if* $x \in F_f$*, then* $x_n(x) = x$*, for all* $n \in N$*. Thus,* $r : X \to F_f \;\; x \mapsto x^*(x)$ *is a set retraction.*

(b) there exists $\alpha \in [0, 1[$ *and* $L \geq 0$ *such that*

$$d(f(x), r(x)) \leq \alpha d(x, r(x)) + Ld(x,f(x)), \text{ for all } x \in X.$$

Then

$$d(x, r(x)) \leq \frac{1+L}{1-\alpha}d(x,f(x)), \text{ for all } x \in X.$$

Proof. Notice that, for every $x \in X$, we have

$$d(x, r(x)) \leq d\,(x,f(x)) + d\,(f(x), r(x)) \leq$$
$$\leq d\,(x,f(x)) + \alpha d(x, r(x)) + Ld(x,f(x)).$$

As an illustrative example, we have the following result for graphic contractions.

Theorem 10. *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be an operator. We suppose:*

(a) there exists $a \in [0, 1[$ *such that*

$$d(f(x),f^2(x)) \leq ad(x,f(x)), \text{ for all } x \in X.$$

(b) f *has closed graph.*

Then:

(i) for each $x \in X$, the sequence $x_n := f^n(x)$, $n \in N^*$ converges to an element $x^*(x) = f^\infty(x) \in F_f$;

(ii)

$$d(x, f^\infty(x)) \leq \frac{1}{1-a} d(x, f(x)) \text{ for all } x \in X.$$

Proof. (i) is the ~~well known~~ well-known Graphic Contraction Principle. For the sake of completeness, we recall here the proof. Let $x_0 \in X$ and $x_n := f^n(x_0)$, $n \in N$. Then, by (a), the sequence (x_n) is Cauchy and, by the completeness of the space (X, d), there exists $x^*(x_0) \in X$ such that $\lim_{n\to+\infty} x_n = x^*(x_0)$. By (b), we get that $x^*(x_0) \in F_f$. Now, for each $x \in X$, we also have ~~:~~

$$d(f(x), f^\infty(x)) \leq d(f(x), f^2(x)) + d(f^2(x), f^3(x)) + \cdots + d(f^n(x), f^\infty(x)) \leq$$

$$ad(x, f(x)) + a^2 d(x, f(x)) + \cdots + a^{n-1} d(x, f(x)) + d(f^n(x), f^\infty(x)) =$$

$$= \frac{a(1-a^{n-1})}{1-a} d(x, f(x)) + d(f^n(x), x^*(x)).$$

Letting $n \to +\infty$, we get that

$$d(f(x), f^\infty(x)) \leq \frac{a}{1-a} d(x, f(x)), \text{ for all } x \in X.$$

Now we can conclude

$$d(x, f^\infty(x)) \leq d(x, f(x)) + d(f(x), f^\infty(x)) \leq$$

$$\leq d(x, f(x)) + \frac{a}{1-a} d(x, f(x)) = \frac{1}{1-a} d(x, f(x)).$$

In the case of Caristi–Browder operators (see [18, 39]) we have a similar result.

Theorem 11. *Let (X, d) be a complete metric space, $f : X \to X$ be an operator and $\varphi : X \to R_+$ be a given function. We suppose:*

(a) $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$, for all $x \in X$;
(b) f has closed graph.

Then:

(i) $F_f \neq \emptyset$;
(ii) for each $x \in X$, the sequence $x_n := f^n(x)$, $n \in N^*$ converges to an element $f^\infty(x) \in F_f$;
(ii) if, additionally, there is $\alpha \in R_+^*$ such that $\varphi(x) \leq \alpha d(x, f(x))$, then

$$d(x, f^\infty(x)) \leq \alpha d(x, f(x)), \text{ for all } x \in X.$$

Proof. For (i) and (ii), let us consider $x \in X$. From (a) it follows

$$\sum_{k=0}^{n} d(f^k(x), f^{k+1}(x)) \leq \varphi(x) - \varphi(f^{n+1}(x)) \leq \varphi(x).$$

This implies that $(f^n(x))_{n\in N}$ is a convergent sequence. Let us denote by $f^{\infty}(x) \in X$ its limit. From (b) we have that $f^{\infty}(x) \in F_f$.

For (iii), notice that for each $x \in X$, we have

$$d(x, f^{n+1}(x)) \leq \sum_{k=0}^{n} d(f^k(x), f^{k+1}(x)) \leq \varphi(x) \leq \alpha d(x, f(x)).$$

Thus, $d(x, f^{\infty}(x)) \leq \alpha d(x, f(x))$, for all $x \in X$.

Remark 4. For other considerations on weakly Picard operator theory, see [6, 13, 27, 37, 40, 45], etc.

Remark 5. For generalized contractions conditions and related results, see [3, 17, 18, 23, 28, 30, 31, 34, 36, 39, 46, 52], etc.

4.4 The Retraction-Displacement Condition in the Case of Other Iterative Algorithms

Let $(X, +, R, \|\cdot\|)$ be a Banach space, $Y \subset X$ a nonempty convex subset, $f : Y \to Y$ an operator, $0 < \lambda < 1$ and $\Lambda := (\lambda_n)_{n\in N}$ with $0 < \lambda_n < 1$, $n \in N$.

4.4.1 Krasnoselskii Algorithm

By the Krasnoselskii perturbation of f we understand the operator $f_\lambda : Y \to Y$ defined by

$$f_\lambda(x) = (1-\lambda)x + \lambda f(x),\ x \in Y.$$

For this perturbation of f we have (see [3, 10, 42, 50, 51], etc.):

Theorem 12. *Let f_λ be defined as above. Then:*

(i) $F_{f_\lambda} = F_f$. *In general* $F_{f_\lambda^n} \neq F_{f^n}$, $n \geq 2$.
(ii) *If f is l-Lipschitz, then f_λ is l-Lipschitz.*
(iii) *If f is a φ-contraction, then f_λ is a φ_λ-contraction.*
(iv) *If in addition Y is bounded and closed and f is nonexpansive, then f_λ is asymptotically regular.*

(v) If f satisfies a (r, ψ) retraction-displacement condition, then f_λ satisfies the (r, θ) retraction-displacement condition with $\theta(t) = \psi\left(\frac{1}{\lambda}t\right)$, $t \in R_+$.

(vi) If X is an ordered Banach space, then f increasing implies f_λ increasing.

The following problem arises:

Problem 4. If f_λ is WPO, in which conditions on f, f_λ satisfies the (f_λ^∞, ψ) retraction-displacement condition?

Some results for this problem was given in Sect. 3, when f is a generalized contraction (see (v) in Theorem 12).

Remark 6. For the condition in which f_λ is WPO see [3, 10, 18, 50, 51], etc. For example, the following result is well known, see [3].

Definition 4. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. An operator $f : H \to H$ is said to be a *generalized pseudo-contraction* if there exists a constant $r > 0$ such that, for all x, y in H,

$$\| f(x) - f(y)\|^2 \le r^2 \| x - y \|^2 + \| f(x) - f(y) - r(x - y) \|^2 . \tag{4.5}$$

Remark 7. Condition 4.5 is equivalent to

$$\langle f(x) - f(y),\ x - y \rangle \ \le \ r \| x - y\|^2 , \quad \text{for all} \quad x, y \in H, \tag{4.6}$$

or to

$$\langle (I - f)\, x - (I - f)\, y \rangle \ \ge \ (1 - r) \| x - y\|^2 . \tag{4.7}$$

Remark 8. Note that any Lipschitzian operator f, that is, any operator for which there exists $s > 0$ such that

$$\| f(x) - f(y)\| \le s \cdot \| x - y\| , \quad x, y \in H, \tag{4.8}$$

is also a generalized pseudo-contractive operator, with $r = s$.

Theorem 13. *Let K be a ~~non-empty~~ nonempty closed convex subset of a real Hilbert space and let $f : K \to K$ be a generalized pseudocontractive and Lipschitzian operator with the corresponding constants r and s, respectively, such that*

$$0 < r < 1 \quad \text{and} \quad r \le s. \tag{4.9}$$

Then

(i) f has ~~an~~ a unique fixed point p;

(ii) *For each* $x_0 \in K$, *the Krasnoselskii iteration* $\{x_n\}_{n=0}^{\infty}$, *given by*

$$x_{n+1} = (1-\lambda)x_n + \lambda f(x_n), \quad n = 0, 1, 2, \ldots, \tag{4.10}$$

converges (strongly) to p, for all $\lambda \in (0,1)$ *satisfying*

$$0 < \lambda < 2(1-r)/(1-2r+s^2). \tag{4.11}$$

.

(iii) *The following retraction-displacement condition holds:*

$$d(x, f_\lambda^{\infty}(x)) \le \frac{\lambda}{1-\theta} d(x, f(x)), \ \forall x \in K,$$

where

$$\theta = \left((1-\lambda)^2 + 2\lambda(1-\lambda)\, r + \lambda^2 s^2\right)^{1/2}. \tag{4.12}$$

Proof. Denote

$$f_\lambda(x) = (1-\lambda)x + \lambda \cdot f(x), \quad x \in K, \tag{4.13}$$

for all $\lambda \in (0,1)$.

Since f is generalized pseudo-contractive and Lipschitzian, we have

$$\|f_\lambda(x) - f_\lambda(y)\|^2 = \|\,(1-\lambda)\,x + \lambda f(x) - (1-\lambda)\,y - \lambda f(y)\|^2 = \tag{4.14}$$

$$\|\,(1-\lambda)(x-y) + \lambda(f(x) - f(y)\|^2 = \tag{4.15}$$

$$(1-\lambda)^2 \cdot \|\,x-y\|^2 + 2\lambda(1-\lambda)\cdot\langle f(x) - f(y), x-y\rangle + \lambda^2 \cdot \|f(x) - f(y)\|^2 \le \tag{4.16}$$

$$\left((1-\lambda)^2 + 2\lambda(1-\lambda)r + \lambda^2 s^2\right)\cdot\|\,x-y\|^2, \tag{4.17}$$

which yields

$$\|f_\lambda(x) - f_\lambda(y)\| \le \theta \cdot \|\,x-y\|, \quad \text{for all} \quad x, y \in K. \tag{4.18}$$

In view of condition (4.12), we get that $0 < \theta < 1$, so f_λ is a θ-contraction. The conclusion now follows by Theorem 3.6 in [3] and Theorem 12.

A more general result can be similarly proven.

Theorem 14. *Let K be a ~~non-empty~~ nonempty closed convex subset of a Banach space and let* $f : K \to K$ *be a mapping satisfying the following assumptions:*

(i) $F_f \neq \emptyset$;

(ii) The Krasnoselskii iteration $\{x_n\}_{n=0}^{\infty}$ *converges to* $x^*(x) \in F_f$*, for any* $x \in K$*;*

(iii) There exist $0 \leq \delta < 1$ *and a function* $\theta : R_+ \to R_+$*, continuous at* 0 *with* $\theta(0) = 0$*, such that*

$$\|f(x) - x^*\| \leq \delta\|x - x^*\| + \theta(\|x - f(x)\|), \ \forall x \in K, x^* \in F_f. \tag{4.19}$$

Then the following retraction-displacement condition holds:

$$\|x - f_\lambda^\infty(x)\| \leq \frac{1}{1-\delta}\left(\|x - f(x)\| + \theta(\|x - f(x)\|)\right), \ \forall x \in K.$$

4.4.2 Mann Algorithm

Let us consider the Mann algorithm corresponding to f and Λ (see [3,10,51]):

$$x_0 \in Y, \ x_{n+1}(x_0) = f_{\lambda_n}(x_n(x_0)), \ n \in N.$$

We suppose that this algorithm is convergent, i.e.,

$$\text{for all } x_0 \in Y, \ x_n(x_0) \longrightarrow x^*(x_0) \in F_f \text{ as } n \to \infty.$$

In this condition we define the operator

$$f_\Lambda^\infty : Y \to F_f, \ x \mapsto x^*(x)$$

operator which is a set retraction.

By definition a convergent Mann algorithm satisfies a retraction-displacement condition if

$$\|x - f_\Lambda^\infty(x)\| \leq \psi(\|x - f(x)\|), \ \forall \ x \in Y$$

with ψ as in Definition 1.

The basic problem is the following:

Problem 5. If the Mann algorithm is convergent, in which conditions on f and Λ ~~is~~ it satisfies a retraction-displacement condition?

Remark 9. For the conditions in which the Mann algorithm is convergent see: [3, 10,51], etc. For example, we have the following ~~well known~~ well-known result, see [3], Chap. 4.

Theorem 15. *Let E be an arbitrary Banach space, K a closed convex subset of E, and* $f : K \to K$ *a Zamfirescu operator. Let* $\{x_n\}_{n=0}^{\infty}$ *be the Mann iteration defined by* $x_0 \in K$ *and*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n f(x_n)\,,\ \ n = 1, 2, \ldots \tag{4.20}$$

with $\{\alpha_n\} \subset [0, 1]$ *satisfying*

$$(iv) \qquad \sum_{n=0}^{\infty} \alpha_n = \infty\,.$$

Then

(i) f *is a Picard operator with* $F_f = \{p\}$*;*
(ii) $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point of* f*;*
(iii) f *satisfies the following retraction-displacement condition*

$$\|x-p\| \leq \frac{2\delta}{1-\delta}\|x-f(x)\|,\ \forall x \in K.$$

Proof. Remind that if f is a Zamfirescu mapping on K, then there exist the real numbers a, b, c satisfying $0 \leq a < 1,\ 0 \leq b < 0.5$ and $0 \leq c < 0.5$, such that, for each $x, y \in X$, at least one of the following is true:

(z_1) $\|f(x) - f(y)\| \leq a\ \|x - y\|$;
(z_2) $\|f(x) - f(y)\| \leq b[\|x - f(x)\| + \|y - f(y)\|]$;
(z_3) $\|f(x) - f(y)\| \leq c[\|x - f(y)\| + \|y - f(x)\|]$.

It is well known that f is a Picard operator (see, for example, Theorem 2.4 in [3]). By (z_1)–(z_3), we obtain that, for all $x, y \in K$, T satisfies

$$\|f(x) - f(y)\| \leq \delta\|x - y\| + 2\delta\|x - f(x)\| \tag{4.21}$$

where

$$\delta = \max\left\{a,\ \frac{b}{1-b}\,, \frac{c}{1-c}\right\} < 1. \tag{4.22}$$

Let $\{x_n\}_{n=0}^{\infty}$ be the Mann iteration given by 4.20, with $x_0 \in K$ arbitrary. Then

$$\begin{aligned}\|x_{n+1} - p\| &= \left\|(1-\alpha_n)x_n + \alpha_n f(x_n) - (1-\alpha_n+\alpha_n)p\right\| = \\ &= \left\|(1-\alpha_n)(x_n - p) + \alpha_n(f(x_n) - p)\right\| \leq \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|f(x_n) - p\|\,.\end{aligned} \tag{4.23}$$

Take $x := p$ and $y := x_n$ in 4.21 to obtain

$$\|f(x_n) - p\| \leq \delta \cdot \|x_n - p\|,$$

which together with 4.23 yields

$$\|x_{n+1} - p\| \le \left[1 - (1-\delta)\alpha_n\right] \|x_n - p\| \,, \quad n = 0, 1, 2, \ldots \,. \tag{4.24}$$

Inductively we get

$$\|x_{n+1} - p\| \le \prod_{k=0}^{n} \left[1 - (1-\delta)\alpha_k\right] \cdot \|x_0 - p\| \,, \quad n = 0, 1, 2, \ldots \,. \tag{4.25}$$

As $0 < \delta < 1, \alpha_k \in [0, 1]$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$, by a standard argument it results that

$$\lim_{n\to\infty} \prod_{k=0}^{n} \left[1 - (1-\delta)\alpha_k\right] = 0 \,,$$

which, together with the previous inequality, implies

$$\lim_{n\to\infty} \|x_{n+1} - p\| = 0 \,,$$

i.e., $\{x_n\}_{n=0}^{\infty}$ converges strongly to p. So, (i) and (ii) are proven. To prove (iii), we use the fact that

$$\|x - p\| \le \|x - f(x)\| + \|f(x) - f(p)\|.$$

So, by inequality 4.21,

$$\|f(x) - f(p)\| \le \delta\|x - p\| + 2\delta\|x - f(x)\|,$$

the desired estimate follows.

A more general result can be similarly proven.

Theorem 16. *Let K be a ~~non-empty~~ nonempty closed convex subset of a Banach space and let $f : K \to K$ be a mapping satisfying the following assumptions:*

(i) $F_f \neq \emptyset$;
(ii) The Mann iteration $\{x_n\}_{n=0}^{\infty}$ converges to $x^*(x) \in F_f$, for any $x \in K$;
(iii) There exist $0 \le \delta < 1$ and a function $\theta : R_+ \to R_+$, continuous at 0 with $\theta(0) = 0$, such that

$$\|f(x) - x^*\| \le \delta\|x - x^*\| + \theta(\|x - f(x)\|), \ \forall x \in K, x^* \in F_f. \tag{4.26}$$

Then, the following retraction-displacement condition holds:

$$\|x - f_\lambda^\infty(x)\| \le \frac{1}{1-\delta} \left(\|x - f(x)\| + \theta(\|x - f(x)\|)\right), \ \forall x \in K.$$

Remark 10. By Theorem 16, one can obtain a convergence theorem for Mann iteration by considering f an almost contraction with a unique fixed point, see~~for example~~ , for example, [3].

4.4.3 Halpern Algorithm

Now we consider the Halpern algorithm (see [3,10,41,50], etc.):

$$x_0 \in Y,\ x_{n+1}(x_0) = (1-\lambda_n)u + \lambda_n f(x_n(x_0)),\ n \in N,$$

where $u \in Y$ is a fixed anchor, see [47,54,55,59] for more details.

We suppose that this algorithm is convergent, i.e.,

$$\text{for all } x_0 \in Y,\ x_n(x_0) \to x^*(x_0) \in F_f \text{ as } n \to \infty.$$

So, if we denote $\Lambda = \{\{\lambda_n\}_{n=0}^{\infty} : \lambda_n \in [0,1]\}$, then we have the set retraction, $f_{\Lambda}^{\infty} : Y \to F_f, f_{\Lambda}^{\infty}(x) = x^*(x)$.

By definition, a convergent Halpern algorithm satisfies a retraction-displacement condition if

$$\|x - f_{\Lambda}^{\infty}(x)\| \le \psi(\|x - f(x)\|),\ \forall\, x \in Y,$$

with ψ as in Definition 1.

Similarly to the case of the previous algorithms, we have

Problem 6. If the Halpern algorithm is convergent, under which condition on f and Λ it satisfies a retraction-displacement condition?

Remark 11. For some conditions under which the Halpern algorithm is convergent, see [1,3,47,50,53–55,59], etc.

A general result similar to the ones established for Krasnoselskii and Mann algorithms can be easily proven for Halpern iteration, too.

Theorem 17. *Let K be a ~~non-empty~~ nonempty closed convex subset of a Banach space and let $f : K \to K$ be a mapping satisfying the following assumptions:*

(i) $F_f \neq \emptyset$;
(ii) The Halpern iteration $\{x_n(x)\}_{n=0}^{\infty}$ *converges to* $x^*(x) \in F_f$, *for any* $x \in K$;
(iii) There exist $0 \le \delta < 1$ *and a function* $\theta : R_+ \to R_+$, *continuous at* 0 *with* $\theta(0) = 0$, *such that*

$$\|f(x) - x^*\| \le \delta\|x - x^*\| + \theta(\|x - f(x)\|),\ \forall x \in K, x^* \in F_f. \qquad (4.27)$$

Then the following retraction-displacement condition holds:

$$\|x - f_{\Lambda}^{\infty}(x)\| \leq \frac{1}{1-\delta} \left(\|x - f(x)\| + \theta(\|x - f(x)\|)\right), \ \forall x \in K.$$

The next corollary provides an answer to Problem 6.

Corollary 3. *Let K be a ~~non-empty~~ nonempty closed convex subset of a Banach space and let $f : K \to K$ be a Zamfirescu mapping. Then*

(i) $F_f = \{x^*\}$;
(ii) *The Halpern iteration $\{x_n\}_{n=0}^{\infty}$ converges to $x^* \in F_f$, for any $x_0 \in K$, provided that $\{\lambda_n\}_{n=0}^{\infty} \subset [0, 1]$ satisfies the following condition:*

$$\lim_{n\to\infty} \lambda_n = 0. \tag{4.28}$$

(iii) *The following retraction-displacement condition holds:*

$$\|x - f_{\Lambda}^{\infty}(x)\| \leq \frac{1+2\delta}{1-\delta} \|x - f(x)\|, \ \forall x \in K.$$

Proof. (i) This follows by Theorem 2.4 in [3].

(ii) Let $\{x_n\}_{n=0}^{\infty}$ be the Halpern iteration, defined by $x_0 \in K$, the fixed anchor $u \in K$ and the parameter sequence $\{\lambda_n\}_{n=0}^{\infty} \subset [0, 1]$ satisfying 4.28. Then we have

$$\|x_{n+1} - x^*\| = \|\lambda_n u + (1-\lambda_n) f(x_n) - x^*\| = \|\lambda_n u - \lambda_n x^* + (1-\lambda_n)(f(x_n) - f(x^*)\|$$
$$\leq \lambda_n \|u - x^*\| + (1-\lambda_n)\|f(x_n) - f(x^*)\| \leq \lambda_n \|u - x^*\| + (1-\lambda_n)\delta\|x_n - x^*\|$$
$$\leq \delta\|x_n - x^*\| + \lambda_n\|u - x^*\|.$$

Thus

$$\|x_{n+1} - x^*\| \leq \delta\|x_n - x^*\| + \lambda_n\|u - x^*\|, \ n \geq 0,$$

which, by applying Lemma 1.6 in [3], yields the conclusion.

(iii) Since f is a Zamfirescu mapping, see Theorem 15, the inequality 4.21 holds and so, by Theorem 16, we get the estimate

$$\|x - f_{\Lambda}^{\infty}(x)\| \leq \frac{1+2\delta}{1-\delta} \|x - f(x)\|, \ \forall x \in K,$$

where δ is given by

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\} < 1 \tag{4.29}$$

and a, b, c are the constants appearing in Zamfirescu's conditions $(z_1) - (z_3)$.

4.4.4 Retraction-Displacement Condition and the Condition (I), in the Case $F_f = \{x^*\}$

In this section we briefly discuss, in the context of fixed point iterative algorithms, the connection between the retraction-displacement condition considered in the present paper and the condition (I), the latter introduced by Senter and Dotson in [48] (see also [58]) for the case of single-valued mappings, and used by many authors mainly in the case of multi-valued operators, to study the convergence of Mann and Ishikawa iterations for nonexpansive type mappings, see [25,48,49,56,57] and the references therein.

Definition 5 ([48]). Let (X, d) be a metric space. A mapping $f : X \to X$ is said to satisfy condition (I) if there is a nondecreasing function $\theta : [0, +\infty) \to [0, +\infty)$ with $\theta(0) = 0$ and $\theta(r) > 0$ for all $r > 0$ such that

$$d(x, Tx) \geq \theta(d(x, F_f)), \forall x \in X,$$

where F_f denotes, as usually, the set of fixed points of f.

In the particular case announced in the title of this section, i.e., when $F_f = \{x^*\}$, it is easy to see that condition (I) requires in fact that

$$d(x, Tx) \geq \theta(d(x, x^*)), \forall x \in X.$$

Example 2. Let (X, d) be a metric space and $f : X \to X$ a λ-contraction ($0 \leq \lambda < 1$). Then, f satisfies condition (I) with $\theta(r) = (1 - \lambda) \cdot r$, for all $r > 0$.

Example 3. Let (X, d) be a metric space and $f : X \to X$ a Kannan mapping, i.e., a mapping for which there exists $0 < \beta < 0.5$ such that

$$d(f(x), f(y)) \leq \beta[d(x, f(x)) + d(y, f(y))], \ \forall x, y \in X.$$

Then, f satisfies condition (I) with $\theta(r) = \dfrac{1 - 2\beta}{2\beta} r$, for all $r > 0$.

Example 4. Let (X, d) be a metric space and $f : X \to X$ a Zamfirescu mapping, see Theorem 16. Then, f satisfies condition (I) with $\theta(r) = \dfrac{1 - 2\delta}{2\delta} r$, for all $r > 0$, where

$$\delta = \max\left\{a, \frac{b}{1 - b}, \frac{c}{1 - c}\right\} < 1, \tag{4.30}$$

and a, b, c are the constants appearing in conditions (z_1)–(z_3).

Based on Examples 2–4, we can state the following generic result.

Proposition 1. *Let K be a ~~non-empty~~ nonempty closed convex subset of a Banach space and let $f : K \to K$ be a mapping satisfying the following assumptions:*

(i) $F_f = \{x^*\}$;
(ii) *A certain iterative algorithm* $f_P \equiv \{x_n(x_0)\}_{n=0}^{\infty}$ *converges to* x^*, *for any* $x_0 \in K$;
(iii) f *satisfies condition (I) with* θ *a bijection.*

Then the algorithm f_P *satisfies the following retraction-displacement condition:*

$$\|x - f_P^{\infty}(x)\| \leq \theta^{-1}\left(\|x - f(x)\|\right), \ \forall x \in K.$$

Remark 12. In the case of Krasnoselskii algorithm we have $P = \{\lambda\}$, $\lambda \in (0, 1)$, and hence $f_P \equiv f_\lambda$, while in the case of Mann iteration we have $P \equiv \Lambda = \{\lambda_n\}_{n=0}^{\infty}$, $\lambda_n \in (0, 1)$, and hence $f_P \equiv f_\Lambda$.

4.5 The Impact of Retraction-Displacement Condition on the Theory of Fixed Point Equations

Let (X, d) be a metric space and $f : X \to X$ an operator with $F_f \neq \emptyset$. We suppose that f satisfies a retraction-displacement condition as in Definition 1. In this section we consider the fixed point equation

$$x = f(x). \tag{4.31}$$

4.5.1 Data Dependence

Let us consider the fixed point equation (4.31) and let $g : X \to X$ be an operator such that $F_g \neq \emptyset$.

We have ~~:~~

Theorem 18. *We suppose that:*

(i) f *satisfies the* (r_1, ψ_1) *retraction-displacement condition;*
(ii) g *satisfies the* (r_2, ψ_2) *retraction-displacement condition;*
(iii) *there exists* $\eta > 0$ *such that*

$$d(f(x), g(x)) \leq \eta, \ \forall \ x \in X.$$

Then, $H_d(F_f, F_g) \leq \max(\psi_1(\eta), \psi_2(\eta))$.

Proof. Let $x^* \in F_f$. Then, $r_2(x^*) \in F_g$ and

$$d(x^*, r_2(x^*)) \leq \psi_2(d(x^*), g(x^*)) = \psi_2(d(f(x^*), g(x^*))) \leq \psi_2(\eta).$$

Let $y^* \in F_g$. Then $r_1(y^*) \in F_f$ and

$$d(y^*, r_1(y^*)) \leq \psi_1(d(y^*), f(y^*)) = \psi_1(d(g(y^*), f(y^*))) \leq \psi_1(\eta).$$

From a ~~well known~~ well-known property of the Pompeiu–Hausdorff functional (see, for example, [31], p. 76) it follows that

$$H_d(F_f, F_g) \leq \max(\psi_1(\eta), \psi_2(\eta)).$$

Theorem 19. *We suppose that* ~~:~~

(i) $F_f = \{x^*\}$ *and* f *satisfies a* (r, ψ) *retraction-displacement condition, where* $r(x) = x^*$, $\forall\, x \in X$;
(ii) there exists $\eta > 0$ *such that*

$$d(f(x), g(x)) \leq \eta, \ \forall\, x \in X.$$

Then, $d(y^*, x^*) \leq \psi(\eta)$, *for each* $y^* \in F_g$.

Proof. Let $y^* \in F_g$. Then

$$d(y^*, x^*) = d(y^*, r(y^*)) \leq \psi(d(y^*), f(y^*)) = \psi(d(g(y^*)), f(y^*)) \leq \psi(\eta).$$

Example 5. Let us consider the following functional integral equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds, \ \text{where } t \in R_+. \tag{4.32}$$

This equation is a mathematical model for epidemics and population growth (see, for example, [12, 14] and the references therein.

Let $0 < m < M$ and $) < \alpha < \beta$. We suppose:

(i) $f \in C(R \times [\alpha, \beta])$;
(ii) there exists $\omega > 0$ such that $f(t + \omega, u) = f(t, u)$, for all $t \in R$ and all $u \in [\alpha, \beta]$);
(iii) there exists $k > 0$ such that $k\tau < 1$ and $|f(t, u) - f(t, v)| \leq k|u - v|$, for all $t \in R$ and all $u, v \in [\alpha, \beta]$);
(iv) $m \leq f(t, u) \leq M$, for all $t \in R$ and all $u \in [\alpha, \beta]$);
(v) $\alpha \leq m\tau$ and $\beta \geq M\tau$.

If we define

$$X_\omega := \{x \in C(R, [\alpha, \beta]) \mid x(t + \omega) = x(t), \ \text{for each } t \in R\}$$

endowed with the metric

$$d(x,y) := \max_{0 \le t \le \omega} |x(t) - y(t)|,$$

then the operator A defined as

$$Ax(t) := \int_{t-\tau}^{t} f(s, x(s))ds, \text{ where } t \in R_+$$

has (using (i)–(iv)) the following properties:

(a) $A(X_\omega) \subset X_\omega$;
(b) A is a $k\tau$-contraction.

Then, by the Contraction Principle and Theorem 19, we obtain ~~:~~

Theorem 20. *Let us consider Eq. (4.32) and suppose that the assumptions (i)–(v) take place. Then:*

(1) Eq. (4.32) has in X_ω a unique solution x^*;
(2) $d(x, x^*) \le \frac{1}{1-k\tau} d(x, A(x))$, for all $x \in X_\omega$;
(3) let $g : R \times [\alpha, \beta] \to R$ be a function which satisfies the conditions (i),(ii),(iv) and (v) above. In addition, we suppose that there exists $\eta > 0$ such that

$$|f(t,u) - g(t,u)| \le \eta, \text{ for all } t \in R \text{ and } u \in [\alpha, \beta]);$$

If $y \in X_\omega$ is a solution of the integral equation

$$y(t) = \int_{t-\tau}^{t} g(s, y(s))ds, \text{ where } t \in R_+,$$

then

$$d(x,y) \le \frac{\tau\eta}{1-k\tau}.$$

4.5.2 Ulam Stability

We start our considerations with the following notions (see [44]).

Definition 6. By definition, the fixed point equation (4.31) is Ulam–Hyers stable if there exists a constant $c_f > 0$ such that: for each $\varepsilon > 0$ and each solution $y^* \in X$ of the inequation

$$d(y, f(y)) \le \varepsilon \tag{4.33}$$

there exists a solution x^* of Eq. (4.31) such that

$$d(y^*, x^*) \leq c_f \varepsilon.$$

Definition 7. By definition, Eq. (4.31) is generalized Ulam–Hyers stable if there exists $\theta : R_+ \to R_+$ increasing and continuous in 0 with $\theta(0) = 0$ such that: for each $\varepsilon > 0$ and for each solution y^* of (4.33) there exists a solution x^* of (4.31) such that

$$d(y^*, x^*) \leq \theta(\varepsilon).$$

We have

Theorem 21. *If f satisfies a (r, ψ) retraction-displacement condition, then Eq. (4.31) is generalized Ulam–Hyers stable.*

Proof. Let $y^* \in X$ be a solution of (4.33). Then $x^* = r(y^*) \in F_f$. Since f satisfies the (r, ψ) retraction-displacement condition we have

$$d(y^*, x^*) \leq \psi(d(y^*), f(y^*)) \leq \psi(\varepsilon).$$

Remark 13. If in the Theorem 21, the function $\psi(t) = c_f t$, $\forall t \in R_+$, then Eq. (4.31) is Ulam–Hyers stable.

Example 6. Let Ω be a bounded domain in R^m and let $X := C(\overline{\Omega}, R)$ endowed with the metric $d(x, y) := \max_{t \in \overline{\Omega}} |x(t) - y(t)|$.

We consider on X the following integral equation

$$x(t) = \int_{\Omega} K(t, s, x(s))ds + l(t), \ t \in \Omega. \tag{4.34}$$

With respect to the above equation, we suppose:

(i) $K \in C([\overline{\Omega} \times \overline{\Omega} \times R, R)$ and $l \in C(\overline{\Omega}, R)$;
(ii) there exists $k > 0$ such that

$$|K(t, s, u) - K(t, s, v)| \leq k|u - v|, \text{ for all } t, s \in \Omega \text{ and } u, v \in R;$$

(iii) $k \cdot mes(\Omega) < 1$.

Then, if we define $A : X \to X$ by

$$Ax(t) := \int_{\Omega} K(t, s, x(s))ds + l(t), \ t \in \Omega,$$

then, by (ii) and (iii), we obtain that A is a $k \cdot mes(\Omega)$-contraction. Applying the Contraction Principle and Theorem 21, we get

Theorem 22. *Consider Eq. (4.34) and suppose that the assumption (i)–(iv) take place. Then, Eq. (4.34) is Ulam–Hyers stable.*

Remark 14. For Ulam stability theory related to fixed point equations see [7,21,26,44], etc.

4.5.3 Well-Posedness for Fixed Point Problems

Let (X,d) be a metric space and $f : X \to X$ be an operator such that its fixed point set $F_f = \{x^*\}$. Following F.S. De Blasi and J. Myjak (see [39] p. 42, see also [43]), we say, by definition, that the fixed point problem

$$x = f(x), x \in X$$

is well-posed if the following implication holds:

$$(x_n)_{n\in N} \subset X \text{ and } d(x_n, f(x_n)) \to 0 \text{ as } n \to +\infty \;\Rightarrow\; \lim_{n\to\infty} x_n = x^*.$$

In this setting, we have the following general result.

Theorem 23. *Let (X,d) be a metric space and $f : X \to X$ be an operator such that its fixed point set $F_f = \{x^*\}$. If the operator f satisfies an (r,ψ) retraction-displacement condition, then the fixed point problem for f is well-posed.*

Proof. Let $(x_n)_{n\in N} \subset X$ such that $d(x_n, f(x_n)) \to 0$ as $n \to +\infty$. Then, we have ~~:~~

$$d(x_n, x^*) \le \psi(d(x_n, f(x_n)) \to 0 \text{ as } n \to \infty.$$

4.5.4 Ostrowski Stability

Let (X,d) be a metric space and $f : X \to X$ be an operator such that its fixed point set $F_f = \{x^*\}$. Let

$$x_{n+1} = f_n(x_n), n \in N$$

be an iterative algorithm with $f_n : X \to X$. By definition, this algorithm is said to be Ostrowski stable if the following implication holds:

$$(y_n)_{n\in N} \subset X \text{ and } d(y_{n+1}, f_n(y_n)) \to 0 \text{ as } n \to +\infty \;\Rightarrow\; \lim_{n\to\infty} y_n = x^*.$$

Some authors refer to the above property as the "limit shadowing property" (see [16,23,29,41,46], etc.).

The following open question seems to be a difficult one.

~~**Open Question.**~~

Open Question In which conditions a retraction-displacement condition on f implies that the fixed point problem for f is Ostrowski stable?

4.6 Some New Research Directions

4.6.1 Examples in R^m

We will present in this section some examples related to the following problem:

Problem 7. Which operators $f : R^n \to R^n$ with $F_f \neq \emptyset$ and $r : R^n \to F_f$ a retraction satisfies the retraction-displacement condition

$$d(x, r(x)) \leq \psi(d(x, f(x))), \quad \text{for all } x \in R^n \text{ ?}$$

Example 7. Let $f : R \to R$ such that $F_f = [a; b]$. Then f satisfies the retraction-displacement condition with $\psi(t) = ct$, $c > 0$ and $r : R \to F_f$

$$r(x) = \begin{cases} a, \; x < a \\ x, \; x \in [a; b] \\ b, \; x > b \end{cases}$$

if the following conditions are satisfied:

$$\frac{a}{c} + \frac{c-1}{c} x \leq f(x) \quad \text{or} \quad \frac{c+1}{c} x - \frac{a}{c} \geq f(x) \quad \text{for } x < a,$$

and

$$\frac{b}{c} + \frac{c-1}{c} x \geq f(x) \quad \text{or} \quad \frac{c+1}{c} x - \frac{b}{c} \leq f(x) \quad \text{for } x > b.$$

Example 8. Let $R > 0$ and

$$\overline{D_i} = \left\{(x, y) \in R^2 \mid x^2 + y^2 \leq R\right\},$$
$$D_e = \left\{(x, y) \in R^2 \mid x^2 + y^2 > R\right\}$$

Let $f : R^2 \to R^2$ be defined by

$$f(x, y) = \begin{cases} (x, y), & (x, y) \in \overline{D_i} \\ (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha), & (x, y) \in D_e \end{cases}$$

with $\alpha \in]0; 2\pi[$.

In this case we have that $F_f = \overline{D_i}$.

Let's consider

$$r(x,y) = \begin{cases} (x,y), & (x,y) \in \overline{D_i} \\ \left(R\frac{x}{\sqrt{x^2+y^2}}, R\frac{y}{\sqrt{x^2+y^2}}\right), & (x,y) \in D_e. \end{cases}$$

Then

$$\|(x,y) - r(x,y)\| \leq c \cdot \|(x,y) - f(x,y)\|, \ (x,y) \in R^2,$$

with $c = \frac{1}{\sqrt{2(1-\cos\alpha)}}$, i.e., f satisfies the retraction-displacement condition with $\psi(t) = ct$.

Proof. For $(x,y) \in \overline{D_i}$ the retraction-displacement condition with $\psi(t) = ct, c > 0$, and r is satisfied for any $c > 0$.

If $(x,y) \in D_e$, then

$$\|(x,y) - r(x,y)\| = \|(x,y)\| - R$$

and

$$\|(x,y) - f(x,y)\| = \|(x,y)\| \cdot \sqrt{2(1-\cos\alpha)}.$$

The function $\varphi : [R; +\infty[\to R$ defined by

$$\varphi(t) = \frac{t-R}{t\sqrt{2(1-\cos\alpha)}}$$

is an increasing bounded function with

$$\varphi(t) < \frac{1}{\sqrt{2(1-\cos\alpha)}}, \ t \in [R; +\infty[,$$

thus

$$\frac{\|(x,y) - r(x,y)\|}{\|(x,y) - f(x,y)\|} = \frac{\|(x,y)\| - R}{\|(x,y)\| \cdot \sqrt{2(1-\cos\alpha)}} < \frac{1}{\sqrt{2(1-\cos\alpha)}}, \ \forall (x,y) \in D_e,$$

and we get the conclusion.

4.6.2 The Case of R_+^m-Metric Spaces

Another research direction is the study of the retraction-displacement condition in the case of generalized metric spaces (see [18,23,39,44,44,45], etc.) For example, if we consider a vector-valued metric (i.e., $d(x, y) \in R_+^m$, then we can do the following commentaries:

(1) Let X be a nonempty set and let $d : X \times X \to R_+^m$ be a R_+^m-metric on X. Let $f : X \to X$ be an operator such that $F_f \neq \emptyset$ and let $r : X \to F_f$ be a set retraction. Then, by definition, f satisfies the (ψ, r) retraction-displacement condition if:

  (i) $\psi : R_+^m \to R_+^m$ is increasing, continuous at 0 with $\psi(0) = 0$;
  (ii) $d(x, r(x)) \leq \psi(d(x, f(x)))$, for all $x \in X$.

(2) For the weakly Picard operator theory in R_+^m-metric spaces, see [36,37,39,45], etc.
(3) For the Ulam stability in R_+^m-metric spaces, see [7,26,44], etc.

4.6.3 The Case of Nonself Operators

Let (X, d) be a metric space and Y be a nonempty subset of X. Then, by definition, the operator $f : Y \to X$ with $F_f \neq \emptyset$ satisfies the (ψ, r) retraction-displacement condition if:

(i) $\psi : R_+^m \to R_+^m$ is increasing, continuous at 0 with $\psi(0) = 0$;
(ii) $r : Y \to F_f$ is a set retraction;
(iii) $d(x, r(x)) \leq \psi(d(x, f(x)))$, for all $x \in Y$.

In this case, the problem is to study the fixed point equation $x = f(x)$ in terms of the (ψ, r) retraction-displacement condition.

Again some commentaries can be done:

(1) Let $\tilde{r} : X \to Y$ be a set retraction such that $F_f = F_{\tilde{r} \circ f}$. Then, the problem is to find a retraction $r : Y \to F_f$ as the limit operator of an iterative algorithm corresponding to the ~~self operator~~ self-operator $\tilde{r} \circ f$.
(2) For some results concerning this problem, see [5,11,29,43].

4.6.4 The Case of ~~Multivalued~~ Multi-valued Operators

We will present first a concept of set-retraction related to ~~multivalued~~ multi-valued operators. Recall first that, if $T : X \to P(X)$ is a ~~multivalued~~ multi-valued operator, then we denote by

$$Graph(T) := \{(x, y) \in X \times X : \ y \in T(x)\}, \text{ the graphic of } T$$

and by

$$F_T := \{x \in X | \ x \in T(x)\}, \text{ the fixed point set of } T.$$

Definition 8. Let X be a nonempty set, $Y \in P(X)$ and $T : X \to P(X)$ be a ~~multivalued~~ multi-valued operator. An operator $r : Graph(T) \to Y$ is called a strong set-retraction of X onto Y if $r(x, x) = x$, for all $x \in Y$.

Then, we define the retraction-displacement condition for ~~multivalued~~ multi-valued operators as follows.

Definition 9. Let (X, d) be a metric space and let $T : X \to P(X)$ be a ~~multivalued~~ multi-valued operator such that its fixed point set F_T is nonempty. Then, by definition, T satisfies the (ψ, r) retraction-displacement condition if there exist $\psi : R_+ \to R_+$ and a strong set retraction $r : Graph(T) \to P(F_T)$ such that:

(i) $\psi : R_+ \to R_+$ is increasing, continuous at 0 with $\psi(0) = 0$;
(ii) $d(x, r(x, y)) \leq \psi(d(x, y)), \text{ for all } (x, y) \in Graph(T)$.

In this case, the problem is to study the fixed point inclusion $x \in T(x)$ and the strict fixed point equation $\{x\} = T(x)$ in terms of the (ψ, r) retraction-displacement condition.

The case of nonself ~~multivalued~~ multi-valued operators can also be considered in a similar way.

Moreover, in particular, if $T : X \to P(X)$ is a ~~multivalued~~ multi-valued weakly Picard operator (i.e., for each $(x, y) \in Graph(T)$ there exists a sequence $(x_n)_{n \in N}$ such that:

(i) $x_0 = x, x_1 = y$;
(ii) $x_{n+1} \in T(x_n)$, for each $n \in N$;
(iii) $(x_n)_{n \in N}$ is convergent and its limit is a fixed point of T),
and we define the ~~multivalued~~ multi-valued operator $T^\infty : Graph(T) \to P(F_T)$ by the formula $T^\infty(x, y) := \{ \ z \in F_T \mid$ there exists a sequence $(x_n)_{n \in N}$ satisfying the assertions (i) and (ii) and convergent to z), then the strong set retraction r is any selection of T^∞ which satisfies the condition (ii) in Definition 9.

For the weakly Picard operator theory for ~~multivalued~~ multi-valued operators and related topics (data dependence, Ulam–Hyers stability, iterative algorithms) see [21, 26, 27, 29, 30, 36, 39, 44], etc. For retraction theory in the ~~multivalued operators context~~ multi-valued operators context, see also [8].

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