CHAPTER 2

Iterative Approximation of Fixed Points of Single-valued Almost Contractions

Vasile Berinde[a*†] and Mădălina Păcurar[‡]
[*] North University Center at Baia Mare, Department of Mathematics and Computer Science, Victoriei nr. 76, 430122 Baia Mare, Romania
† King Fahd University of Petroleum and Minerals, Department of Mathematics and Statistics, Dhahran, Saudi Arabia
‡ Babeş-Bolyai University of Cluj-Napoca, Department of Statistics, Analysis, Forecast and Mathematics Faculty of Economics and Bussiness Administration, 56–60 T. Mihali St., 400591 Cluj-Napoca, Romania
[*] Corresponding: vasile.berinde@gmail.com

Contents

Abstract

The aim of this chapter is to survey the most relevant developments done in the last decade around the concept of almost contraction, introduced in Berinde V, Approximating fixed points of weak contractions using the Picard iteration. Nonlinear Anal. Forum 2004;9(1):43–53.

2.1. Introduction

Metrical fixed point theory developed around Banach's contraction principle, which, in the case of a metric space setting, can be briefly stated as follows.

Theorem 2.1.1. *Let (X,d) be a complete metric space and $T: X \to X$ a strict contraction, i.e., a map satisfying*

$$d(Tx,Ty) \le a\,d(x,y), \quad \text{for all } x,y \in X, \tag{2.1.1}$$

http://dx.doi.org/10.1016/B978-0-12-804295-3.50002-4

where $0 \leq a < 1$ *is constant. Then*

(p1) T *has a unique fixed point* p *in* X *(i.e.,* $Tp = p$*);*

(p2) *The Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *defined by*

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots, \tag{2.1.2}$$

converges to p*, for any* $x_0 \in X$.

Remark 2.1.1. A map satisfying (p1) and (p2) in Theorem 2.1.1 is said to be a *Picard operator* (see Refs. [235, 236, 239, 240] for more details).

Theorem 2.1.1, which was established in a complete linear normed space in 1922 by Stefan Banach [49] (see also Ref. [50]), is in fact a formalization of the method of successive approximation that has previously been systematically used by Picard in 1890 [210] to study differential and integral equations.

Being a simple and versatile tool in establishing existence and uniqueness theorems for operator equations, Theorem 2.1.1 plays a very important role in nonlinear analysis. This fact motivated researchers to try to extend and generalize Theorem 2.1.1 in such a way that its area of applications should be enlarged as much as possible.

Most of these generalizations considered only continuous mappings, like in the original case of the contraction mapping in Theorem 2.1.1. It was natural to ask whether there exist or not alternative contractive conditions that ensure the conclusions of Theorem 2.1.1 but which do not force implicitly or explicitly that T is continuous.

This question was answered in the affirmative by Kannan in 1968 [149], who proved a fixed point theorem which extends Theorem 2.1.1 to mappings that need not be continuous. Kannan considered instead of (2.1.1) the following condition: there exists $b \in \left(0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)], \quad \text{for all } x, y \in X. \tag{2.1.3}$$

Following Kannan's theorem, many papers were devoted to obtaining fixed point theorems for various classes of contractive type conditions that do not require the continuity of T (see for example Refs. [60, 235, 239] and the references therein).

One of these results is actually a sort of dual of Kannan's fixed point theorem, and is due to Chatterjea [85]. It makes use of a condition similar to (2.1.3): there exists $c \in \left(0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq c\left[d(x, Ty) + d(y, Tx)\right], \quad \text{for all } x, y \in X. \tag{2.1.4}$$

Based on the fact (established later by Rhoades [226]) that the contractive conditions (2.1.1), (2.1.3), and (2.1.4) are independent, Zamfirescu [280] obtained a very interesting fixed point theorem, by combining (2.1.1), (2.1.3) and (2.1.4).

Theorem 2.1.2. *Let (X,d) be a complete metric space and $T: X \to X$ a map for which there exist the real numbers a,b and c satisfying $0 \le a < 1$, $0 < b$, $c < 1/2$ such that for each pair x,y in X, at least one of the following is true:*

(z_1) $d(Tx,Ty) \le a\, d(x,y)$;

(z_2) $d(Tx,Ty) \le b\big[d(x,Tx) + d(y,Ty)\big]$;

(z_3) $d(Tx,Ty) \le c\big[d(x,Ty) + d(y,Tx)\big]$.

Then T is a Picard operator.

The class of *almost contractions*, the central concept surveyed in this chapter, is closely related to Zamfirescu's contractions. Indeed, by conditions (z_2) and (z_3) one obtains that T satisfies the conditions:

$$d(Tx,Ty) \le \frac{b}{1-b}\, d(x,y) + \frac{2b}{1-b}\, d(y,Tx), \quad \text{for all } x,y \in X, \tag{2.1.5}$$

and

$$d(Tx,Ty) \le \frac{c}{1-c}\, d(x,y) + \frac{2c}{1-c}\, d(y,Tx), \quad \text{for all } x,y \in X, \tag{2.1.6}$$

respectively.

Thus, the birth of *almost contractions* could be dated to the very moment when we realized that (2.1.5) and (2.1.6) share the same property, i.e., $0 < \frac{b}{1-b} < 1$, $0 < \frac{c}{1-c} < 1$, and that (z_1), (z_2) and (z_3) could be unified within a single condition of the form:

$$d(Tx,Ty) \le \delta\, d(x,y) + L d(y,Tx), \tag{2.1.7}$$

with the constants δ and L satisfying $0 < \delta < 1$ and $L \ge 0$.

Soon after the publication of the first papers devoted to *almost contractions* [53–55], various researchers were attracted by the novelty that this class of mappings has brought to fixed point theory, see the rich list of references [2–16, 18–30, 40–48, 51, 60–76, 78–80, 84, 88–93, 102–104, 106, 107, 109–123, 125–143, 147, 148, 152–163, 166, 168–189, 194, 195, 197–199, 202–209, 216, 217, 219–225, 231, 232, 241–248, 250–258, 260–263, 266, 268–271, 276, 277].

It is therefore the main aim of this chapter to survey some of the most relevant developments in the last ten years or so on the concept of *almost contraction*.

The main aspects considered in the chapter are as follows:

- Fixed point theorems for single-valued self almost contractions
- Iterative approximation of the fixed point of implicit almost contractions
- Common fixed point theorems for almost contractions
- Almost contractive type mappings on product spaces
- Fixed point theorems for single-valued nonself almost contractions.

2.2. Fixed Point Theorems for Single-valued Self Almost Contractions

Definition 2.2.1 [55]. Let (X,d) be a metric space. A map $T : X \to X$ is called an *almost contraction* if there exist the constants $\delta \in (0,1)$ and $L \geq 0$ such that

$$d(Tx,Ty) \leq \delta \cdot d(x,y) + Ld(y,Tx), \quad \text{for all } x,y \in X. \tag{2.2.8}$$

In order to be more precise, we shall also call T *a* (δ,L)*-almost contraction*.

Remark 2.2.1. Note that the almost contraction condition (2.2.8) is not symmetric. But, due to the symmetry of the distance, (2.2.8) implicitly includes the following dual one:

$$d(Tx,Ty) \leq \delta \cdot d(x,y) + L \cdot d(x,Ty), \quad \text{for all } x,y \in X, \tag{2.2.9}$$

and so, by (2.2.8) and (2.2.9), we obtain the following *symmetric* condition:

$$d(Tx,Ty) \leq \delta \cdot d(x,y) + L_1 \cdot [d(x,Ty) + d(x,Ty)], \quad \text{for all } x,y \in X, \tag{2.2.10}$$

where $L_1 = L/2$. As shown in Ref. [73], (2.2.10) does not imply either (2.2.8) or (2.2.9).

Remark 2.2.2. Note that at the beginning (see Refs. [53–55]) and in some other subsequent papers, the author adopted the name *weak contraction* to designate an *almost contraction*. Soon after these papers had been published, we discovered that the term *weak contraction* had been used previously by other authors in different contexts.

Indeed, in 1967 Sz.-Nagy and Foiaş [267] used the concept of *weak contraction* in the context of the spectral theory of operators.

Later, Dugundji and Granas [124] also considered the concept of *weak contraction*, this time in the field of metrical fixed point theory. Dugundji and Granas called *weak contraction* a mapping $T : X \to X$ that satisfies the following condition:

$$d(Tx,Ty) \leq d(x,y) - \phi(d(x,y)), \quad \text{for all } x,y \in X, \tag{2.2.11}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a compactly positive function. They also obtained some applications of weak contractions, including a domain invariance theorem (see Ref. [124] for details).

Apparently not aware of the paper by Dugundji and Granas (the paper [124] is not cited in Ref. [17]), in 1997 Alber and Guerre-Delabriere [17] used exactly the same condition for a map T defined from a closed convex subset C of a Banach space X into C and called T *weakly contractive* if

$$\|Tx - Ty\| \leq \|x - y\| - \psi(\|x - y\|), \quad \text{for all } x,y \in C,$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing, ψ is positive on $\mathbb{R}_+ \setminus 0$, $\psi(0) = 0$ and $\lim\limits_{t\to\infty} \psi(t) = +\infty$.

The concept of weak contraction by Alber and Guerre-Delabriere has been extremely successful, as shown by Google Scholar, where we found more than 340 papers citing this reference.

So, the existence of different concepts of weak contraction in fixed point theory led us to change the name adopted in 2003 (see Refs. [53–55] and the subsequent papers) to that of *almost contraction*. This name has been adopted for the first time in our paper from 2008 [62], five years after the original use of almost contractions.

It is therefore not surprising that the authors that studied this class of mappings used various names to designate an almost contraction, depending on the source of documentation: weak contraction [3, 5, 25, 27, 44, 110, 137, 140, 147, 169, 177, 178, 186, 188, 205, 209, 224, 242, 248, 254, 255, 256], almost contraction [2, 6, 7, 11, 14, 16, 19, 20, 23, 24, 28, 40, 42, 43, 44, 91, 134, 183, 198, 199, 203, 204, 232, 246, 253, 257, 258, 260, 262, 271], (δ, L)-almost contraction [10, 128, 138, 277], Berinde mapping [9, 11, 22, 41, 47, 89, 90, 152, 202, 245, 251, 255, 266], etc.

Remark 2.2.3. Obviously, any classical contraction (2.1.1) will satisfy (2.2.8) with $\delta = a$ and $L = 0$, and hence the class of almost contractions (properly) includes the Banach contractions (see Examples 2.2.1 and 2.2.2).

Other examples of almost contractions are given in the following propositions.

Proposition 2.2.1. *Let (X,d) be a metric space. Any Kannan contraction, i.e., any mapping $T : X \to X$ satisfying the contractive condition (2.1.3), is an almost contraction.*

Proof. By condition (2.1.3) and the triangle inequality, we get

$$\begin{aligned} d(Tx,Ty) &\leq b\big[d(x,Tx) + d(y,Ty)\big] \\ &\leq b\Big\{\big[d(x,y) + d(y,Tx)\big] + \big[d(y,Tx) + d(Tx,Ty)\big]\Big\}, \end{aligned}$$

which yields

$$(1-b)d(Tx,Ty) \leq b\,d(x,y) + 2b \cdot d(y,Tx)$$

and this implies

$$d(Tx,Ty) \leq \frac{b}{1-b}d(x,y) + \frac{2b}{1-b}d(y,Tx), \quad \text{for all } x,y \in X.$$

Hence, in view of the condition $0 < b < \frac{1}{2}$, (2.2.8) holds with $\delta = \frac{b}{1-b}$ and $L = \frac{2b}{1-b}$. □

Proposition 2.2.2. *Let (X,d) be a metric space. Any Kannan contraction, i.e., any mapping $T : X \to X$ satisfying the contractive condition (2.1.4), is an almost contraction.*

Proof. Using $d(x,Ty) \le d(x,y) + d(y,Tx) + d(Tx,Ty)$ by (2.1.4) we get, after simple computations,

$$d(Tx,Ty) \le \frac{c}{1-c}\, d(x,y) + \frac{2c}{1-c}\, d(y,Tx),$$

which is (2.2.8), with $\delta = \frac{c}{1-c} < 1$ (since $c < 1/2$) and $L = \frac{2c}{1-c} \ge 0$. □

From Remark 2.2.3 and Propositions 2.2.1 and 2.2.2, we have the following proposition.

Proposition 2.2.3. *Let (X,d) be a metric space. Any Zamfirescu contraction, i.e., any mapping $T : X \to X$ satisfying the assumptions in Theorem 2.1.2, is an almost contraction.*

Proposition 2.2.4 [197]. *Let (X,d) be a metric space. Any Ćirić-Reich-Rus contraction, i.e., any mapping $T : X \to X$ satisfying the condition*

$$d(Tx,Ty) \le \alpha\, d(x,y) + \beta[d(x,Tx) + d(y,Ty)], \quad \textit{for all } x,y \in X,$$

where $\alpha, \beta \in \mathbb{R}_+$ and $\alpha + 2\beta < 1$, is an almost contraction.

Proposition 2.2.5. *Let (X,d) be a metric space. Let $T : X \to X$ be a quasi-contraction [96], i.e., an operator for which there exists $0 < h < 1$ such that*

$$d(Tx,Ty) \le h \cdot M(x,y), \quad \text{for all } x,y \in X, \tag{2.2.12}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right\}. \tag{2.2.13}$$

If $0 < h < 1/2$, then T is an almost contraction.

Proof. Let T satisfy (2.2.13) and let $x,y \in X$ be arbitrarily taken. We have to discuss five possible cases.

CASE 1. $M(x,y) = d(x,y)$. In this case, by virtue of (2.2.12), conditions (2.2.8) and (2.2.9) are obviously satisfied (with $\delta = h$ and $L = 0$).

CASE 2. $M(x,y) = d(x,Tx)$. In this case, by (2.2.12) and triangle inequality one obtains

$$d(Tx,Ty) \le h\,d(x,Tx) \le h\big[d(x,y)+d(y,Tx)\big],$$

and so (2.2.8) holds with $\delta = h$ and $L = h$.

Since $d(x,Tx) \le d(x,Ty)+d(Ty,Tx)$, we get

$$d(Tx,Ty) \le \frac{h}{1-h}d(x,Ty) \le \delta d(x,y) + \frac{h}{1-h}d(x,Ty), \quad \text{for all } \delta \in (0,1).$$

So (2.2.9) also holds.

Case 3. $M(x,y) = d(y,Ty)$, when (2.2.8) and (2.2.9) follow by Case 2, by virtue of the symmetry of $M(x,y)$.

Case 4. $M(x,y) = d(x,Ty)$, when (2.2.9) is obviously true and (2.2.8) is obtained only if $h < \frac{1}{2}$. Indeed, since by (2.2.12), $d(Tx,Ty) \le h \cdot d(x,Ty)$ and

$$d(x,Ty) \le d(x,y)+d(y,Tx)+d(Tx,Ty),$$

one obtains

$$d(Tx,Ty) \le \frac{h}{1-h}d(x,y) + \frac{h}{1-h}d(y,Tx),$$

which is (2.2.8) with $\delta = \frac{h}{1-h} < 1$ (since $h < \frac{1}{2}$) and $L = \frac{h}{1-h} > 0$.

Case 5. $M(x,y) = d(y,Tx)$, which reduces to Case 4. □

Remark 2.2.4. Proposition 2.2.5 shows that the quasi-contractions with $0 < h < 1/2$ are almost contractions. It appears then that $h < \frac{1}{2}$ is not a necessary condition for a quasi-contraction to be an almost contraction, as there exist quasi-contractions with $h \ge \frac{1}{2}$, which are still almost contractions, as shown by Example 2.2.2.

There are many other examples of contractive conditions which imply the almost contractiveness condition (see the list in Rhoades' classification [226]).

The first main result of this section is the following theorem.

Theorem 2.2.1 [55, Theorem 2]. *Let (X,d) be a complete metric space and let $T : X \to X$ be a (δ,L)-almost contraction. Then*

(a) *$Fix(T) = \{x \in X : Tx = x\} \neq \emptyset$;*

(b) *For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$, $x_n = T^n x_0$, converges to some $x^* \in Fix(T)$;*

(c) *The following estimate holds:*

$$d(x_{n+i-1},x^*) \le \frac{\delta^i}{1-\delta}d(x_n,x_{n-1}), \quad n = 0,1,2,\ldots; i = 1,2,\ldots. \tag{2.2.14}$$

Proof. We shall prove that T satisfying (2.2.8) has at least a fixed point in X. To this end, let $x_0 \in X$ be arbitrary but fixed and let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration defined by (2.1.2).

Take $x := x_{n-1}$, $y := x_n$ in (2.2.8) to obtain

$$d(Tx_{n-1}, Tx_n) \leq \delta \cdot d(x_{n-1}, x_n),$$

which shows that

$$d(x_n, x_{n+1}) \leq \delta \cdot d(x_{n-1}, x_n). \tag{2.2.15}$$

Using (2.2.15), we obtain by induction

$$d(x_n, x_{n+1}) \leq \delta^n d(x_0, x_1), \quad n = 0, 1, 2, \ldots,$$

and then

$$\begin{aligned} d(x_n, x_{n+p}) &\leq \delta^n \left(1 + \delta + \cdots + \delta^{p-1}\right) d(x_0, x_1) \\ &= \frac{\delta^n}{1-\delta}(1-\delta^p) \cdot d(x_0, x_1), \quad n, p \in \mathbb{N},\ p \neq 0. \end{aligned} \tag{2.2.16}$$

Since $0 < \delta < 1$, (2.2.16) shows that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and hence it is convergent. Let us denote

$$x^* = \lim_{n\to\infty} x_n. \tag{2.2.17}$$

Then

$$d(x^*, Tx^*) \leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) = d(x_{n+1}, x^*) + d(Tx_n, Tx^*).$$

By (2.2.8), we have

$$d(Tx_n, Tx^*) \leq \delta\, d(x_n, x^*) + L d(x^*, Tx_n)$$

and hence

$$d(x^*, Tx^*) \leq (1+L) d(x^*, x_{n+1}) + \delta \cdot d(x_n, x^*), \tag{2.2.18}$$

which is valid for all $n \geq 0$. Letting $n \to \infty$ in (2.2.18), we obtain

$$d(x^*, Tx^*) = 0,$$

i.e., x^* is a fixed point of T.

By letting $p \to \infty$ in (2.2.16), one obtains the *a priori* estimate

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), \quad n = 0, 1, 2, \ldots, \tag{2.2.19}$$

where δ is the constant appearing in (2.2.8).

On the other hand, let us observe that by (2.2.15) we inductively obtain

$$d(x_{n+k}, x_{n+k+1}) \leq \delta^{k+1} \cdot d(x_{n-1}, x_n), \quad k, n \in \mathbb{N},$$

and hence, similarly to deriving (2.2.16), we get

$$d(x_n, x_{n+p}) \leq \frac{\delta(1-\delta^p)}{1-\delta} d(x_{n-1}, x_n), \quad n \geq 1,\ p \in \mathbb{N}^*. \tag{2.2.20}$$

Now, by letting $p \to \infty$ in (2.2.20), the *a posteriori* estimate follows:

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n), \quad n = 1, 2, \ldots. \tag{2.2.21}$$

Now, (2.2.19) and (2.2.21) can be merged to obtain the estimate (2.2.14). □

Remark 2.2.5. (a) Theorem 2.2.1 is a significant extension of Theorem 2.1.1, Theorem 2.1.2 and many other related results in metrical fixed point theory.

(b) Note that for all the fixed point theorems mentioned in (a) as particular cases of Theorem 2.2.1, the fixed point is unique. However, in general, almost contractions need not have a unique fixed point, as shown by Examples 2.2.1 and 2.2.2.

(c) Recall (see Refs. [235, 236, 239]), that an operator $T : X \to X$ is said to be a *weakly Picard operator* if the sequence $\{T^n x_0\}_{n=0}^{\infty}$ converges for all $x_0 \in X$ and its limit is a fixed point of T. So, almost contractions are weakly Picard operators, while Banach contraction, Kannan contraction and Ćirić quasi-contractions are Picard operators.

(d) Note also that condition (2.2.8) implies the so-called Banach orbital condition or graphic contraction condition

$$d(Tx, T^2x) \leq a\, d(x, Tx), \quad \text{for all } x \in X, \tag{2.2.22}$$

studied by various authors (see Ref. [239, p. 39], for some historical remarks). As shown by the Graphic Contraction Principle (see Ref. [239, p. 35]), a graphic contraction that has closed graph is a weakly Picard operator.

So, in this context, the merit of condition (2.2.8) is that no other additional hypothesis is needed in order for an almost contraction to be a weakly Picard operator.

Remark 2.2.6. From a numerical point of view, a fixed point theorem is valuable if, apart from the conclusion regarding the existence (and, possibly, uniqueness) of the fixed point:

(a) it provides a method (generally, *iterative*) for constructing the fixed point(s);

(b) it is able to provide information on the error estimate or / and rate of convergence of the iterative process used to approximate the fixed point; and

(c) it can give concrete information on the stability of this procedure, that is, on the data dependence of the fixed point(s).

As shown above, Theorem 2.2.1 does possess all these features (see also Theorem 2.2.2), for the rate of convergence.

The next examples illustrate the diversity of almost contractions.

Example 2.2.1. Let $X = [0,1]$ be the unit interval with the usual norm and let $T : [0,1] \to [0,1]$ be given by $Tx = \frac{1}{2}$, for $x \in [0,2/3)$ and $Tx = 1$, for $x \in [2/3,1]$.

As T has two fixed points, that is, $Fix(T) = \left\{\frac{1}{2}, 1\right\}$, it does not satisfy either Banach contraction condition (2.1.1), Kannan contraction condition (2.1.3), Chatterjea contraction condition (2.1.4), Zamfirescu contractive conditions (z_1) and (z_2), or Ćirić's quasi-contraction condition (2.2.12), but T satisfies the almost contraction condition (2.2.8).

Indeed, for $x,y \in [0,2/3)$ or $x,y \in [2/3,1]$, (2.2.8) is obvious. For $x \in [0,2/3)$ and $y \in [2/3,1]$ or $y \in [0,2/3)$ and $x \in [2/3,1]$ we have $d(Tx,Ty) = 1/2$ and $d(y,Tx) = |y-1/2| \in [1/6,1/2]$, in the first case, and $d(y,Tx) = |y-1| \in [1/3,1]$, in the second case, which show that it suffices to take $L = 3$ in order to ensure that (2.2.8) holds for $0 < \delta < 1$ and $L \geq 0$ arbitrary and all $x,y \in X$.

Example 2.2.2. Let $X = [0,1]$ with the usual metric and let $T : [0,1] \to [0,1]$ be defined by

$$Tx = \frac{2}{3}x, \text{ if } 0 \leq x < \frac{1}{2} \text{ and } Tx = \frac{2}{3}x + \frac{1}{3}, \text{ if } \frac{1}{2} \leq x \leq 1.$$

Then:

(a) T is an almost contraction with constants $\delta = \frac{2}{3}$, $L = 6$ and $Fix(T) = \{0;1\}$;

(b) T does not satisfy any of the contraction conditions of Banach, Kannan, Chatterjea and Zamfirescu, and is not a Ćirić quasi-contraction, as T has two fixed points.

Proof. We have to discuss the following possible cases:

I. If $x,y \in [0,\frac{1}{2})$, then $T(x) = \frac{2}{3}x$ and $T(y) = \frac{2}{3}y$. Then condition (2.2.8) becomes

$$\left|\frac{2}{3}x - \frac{2}{3}y\right| \leq \delta\,|x-y| + L\left|y - \frac{2}{3}x\right|,$$

which obviously holds for $\delta \geq \frac{2}{3}$ and any $L \geq 0$.

II. If $x, y \in [\frac{1}{2}, 1]$, then $T(x) = \frac{2}{3}x + \frac{1}{3}$ and $T(y) = \frac{2}{3}y + \frac{1}{3}$. Similarly to case I, this holds for $\delta \geq \frac{2}{3}$ and any $L \geq 0$.

III. If $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1]$, then $T(x) = \frac{2}{3}x$ and $T(y) = \frac{2}{3}y + \frac{1}{3}$. Condition (2.2.8) becomes

$$\left|\frac{2}{3}x - \frac{2}{3}y - \frac{1}{3}\right| \leq \delta\,|x-y| + L\left|y - \frac{2}{3}x\right|. \tag{2.2.23}$$

On the left-hand side we have that:

$$-1 \leq \frac{2}{3}x - \frac{2}{3}y - \frac{1}{3} < -\frac{1}{3} \Rightarrow \left|\frac{2}{3}x - \frac{2}{3}y - \frac{1}{3}\right| \in \left(\frac{1}{3}, 1\right],$$

while on the right-hand side we have that:

$$\frac{1}{6} < y - \frac{2}{3}x \leq 1 \Rightarrow \left|y - \frac{2}{3}x\right| \in \left(\frac{1}{6}, 1\right].$$

Then (2.2.23) holds for $\delta \in [0,1)$ and $L \geq 6$.

IV. If $x \in [\frac{1}{2}, 1]$ and $y \in [0, \frac{1}{2})$, then $f(x) = \frac{2}{3}x + \frac{1}{3}$ and $f(y) = \frac{2}{3}y$.
Condition (2.2.8) becomes

$$\left|\frac{2}{3}x - \frac{2}{3}y + \frac{1}{3}\right| \leq \delta\,|x-y| + L\left|y - \frac{2}{3}x - \frac{1}{3}\right|. \tag{2.2.24}$$

On the left-hand side we have that:

$$\left|\frac{2}{3}x - \frac{2}{3}y - \frac{1}{3}\right| \in \left(\frac{1}{3}, 1\right],$$

while on the right-hand side we have that:

$$-1 < y - \frac{2}{3}x - \frac{1}{3} \leq -\frac{1}{6} \Rightarrow \left|y - \frac{2}{3}x - \frac{1}{3}\right| \in \left[\frac{1}{6}, 1\right).$$

Then (2.2.24) holds for $\delta \in [0,1)$ and $L \geq 6$.

The conclusion is that T satisfies (2.2.8) for any $x, y \in X$ if $\delta \in [\frac{2}{3}, 1)$ and $L \geq 6$. Taking $\delta = \frac{2}{3}$ and $L = 6$, notice that $\delta + L > 1$. □

Example 2.2.3. Let $[0,1]$ be the unit interval with the usual norm. Let $T : [0,1] \to [0,1]$ be the identity map, i.e., $Tx = x$, for all $x \in [0,1]$. Then

(a) T does not satisfy Ciric's contractive condition (2.2.12);

(b) T satisfies the almost contraction condition (2.2.8) with $\delta \in (0,1)$ arbitrary and

$L \geq 1-\delta$. Indeed, conditions (2.2.8) and (2.2.9) lead to

$$|x-y| \leq \delta|x-y| + L\cdot|y-x|,$$

which is true for all $x,y \in [0,1]$ if we take $\delta \in (0,1)$ arbitrary and $L \geq 1-\delta$.

(c) The set of fixed points of T is the interval $[0,1]$, i.e., $F(T) = [0,1]$.

It is possible to force an almost contraction to be a Picard operator by imposing an additional contractive condition, quite similar to (2.2.8), as shown by the next theorem.

Theorem 2.2.2 [55, Theorem 2.2]. *Let (X,d) be a complete metric space and $T : X \to X$ be a (δ,L)- almost contraction for which there exist $\delta_u \in (0,1)$ and $L_u \geq 0$ such that*

$$d(Tx,Ty) \leq \delta_u \cdot d(x,y) + L_u \cdot d(x,Tx), \quad \text{for all } x,y \in X. \tag{2.2.25}$$

Then

(a) *T has a unique fixed point, i.e., $F(T) = \{x^*\}$;*

(b) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (2.1.2) converges to x^*, for any $x_0 \in X$;*

(c) *The following estimate holds:*

$$d(x_{n+i-1},x^*) \leq \frac{\delta^i}{1-\delta} d(x_n,x_{n-1}), \quad n = 0,1,2,\ldots; i = 1,2,\ldots; \tag{2.2.26}$$

(d) *The rate of convergence of the Picard iteration is given by*

$$d(x_n,x^*) \leq \delta_u\, d(x_{n-1},x^*), \quad n = 1,2,\ldots. \tag{2.2.27}$$

Proof. Assume T has two distinct fixed points $x^*, y^* \in X$. Then, by (2.2.25), with $x := x^*$, $y := y^*$ we get

$$d(x^*,y^*) \leq \delta_u \cdot d(x^*,y^*) \quad \Leftrightarrow \quad (1-\delta_u)\, d(x^*,y^*) \leq 0,$$

so contradicting $d(x^*,y^*) > 0$.

Letting $y := x_n$, $x := x^*$ in (2.2.25), we obtain the estimate (2.2.27).

The rest of the proof follows by Theorem 2.2.1. □

Remark 2.2.7. (a) Note that the uniqueness condition (2.2.25) has been used by Osilike [190 – 192] to prove stability results for certain fixed point iteration procedures. It is shown there (see Refs. [190, 191]), that condition (2.2.25) alone does not imply T has a fixed point. But if a mapping T satisfying (2.2.25) has a fixed point, then this fixed point is certainly unique.

(b) It is easy to check that any operator T satisfying one of the conditions (2.1.1), (2.1.3), (2.1.4), or the conditions in Theorem 2.1.2, also satisfies the uniqueness conditions (2.2.25). Therefore, in view of Examples 2.2.1–2.2.3, Theorem 2.2.1 and Theorem 2.2.2 properly generalize Theorem 2.1.1, Kannan's fixed point theorem [149], Theorem 2.1.2, and many other related results.

(c) Rus [236] has shown that, if T is a weakly Picard operator, then there exists a partition of X,

$$X = \bigcup_{\lambda \in \Lambda} X_\lambda$$

such that $T|X_\lambda$ is a Banach contraction. In the case of the almost contractions in Examples 2.2.1–2.2.3, we have the following partitions.

For T in Example 2.2.1, we have

$$[0,1] = [0,2/3) \cup [2/3,1];$$

for T in Example 2.2.2, we have

$$[0,1] = [0,1/2) \cup [1/2,1];$$

while for T in Example 2.2.3, we have

$$[0,1] = \bigcup_{\lambda \in [0,1]} \{\lambda\}.$$

(d) As can easily be seen, Theorem 2.2.2 (as well as Theorem 2.2.1, except for the uniqueness of the fixed point) preserves all conclusions in the Banach contraction principle in its complete form [60, Theorem 2.1] under significantly weaker contractive conditions.

Indeed, the metrical contractive conditions known in the literature (see Ref. [226]) that involve on the right-hand side the displacements

$$d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)$$

with nonnegative coefficients, say

$$a(x,y), b(x,y), c(x,y), d(x,y), e(x,y),$$

respectively, are commonly based on the very restrictive assumption

$$0 < a(x,y) + b(x,y) + c(x,y) + d(x,y) + e(x,y) \le 1,$$

while, in condition (2.2.8), which involves only the displacements

$$d(x,y), d(y,Tx)$$

the constant coefficients δ and L are not required to satisfy

$$\delta + L \leq 1.$$

This is obvious for T in Example 2.2.2, where we have

$$\delta + L = 2/3 + 6.$$

It is possible to extend significantly Theorems 2.2.1 and 2.2.2, by replacing $d(x,y)$ in (2.2.8) by a certain expression of the displacements $d(x,y)$, $d(x,Tx)$, $d(y,Ty)$, $d(x,Ty)$, $d(y,Tx)$, first used by Ćirić [95]:

$$M_1(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\left[d(x,Ty) + d(y,Tx)\right]\right\}. \quad (2.2.28)$$

We thus have the following result, taken from Ref. [64].

Theorem 2.2.3. *Let (X,d) be a complete metric space and let $T : X \to X$ be a strong Ćirić almost contraction, that is, a mapping for which there exist two constants $\alpha \in [0,1)$ and $L \geq 0$ such that*

$$d(Tx,Ty) \leq \alpha \cdot M_1(x,y) + Ld(y,Tx), \quad \text{for all } x,y \in X, \quad (2.2.29)$$

where $M_1(x,y)$ is given by (2.2.28). Then

(a) *$Fix(T) = \{x \in X : Tx = x\} \neq \emptyset$;*

(b) *For any $x_0 = x \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (2.1.2) converges to some $x^* \in Fix(T)$;*

(c) *The following estimate holds:*

$$d(x_{n+i-1}, x^*) \leq \frac{\alpha^i}{1-\alpha} d(x_n, x_{n-1}), \quad n = 0,1,2,\ldots; i = 1,2,\ldots. \quad (2.2.30)$$

Proof. Let $x \in X$ be arbitrary and let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration defined by (2.1.2) with $x_0 = x$. By taking $x := x_{n-1}$, $y := x_n$ in (2.2.29), we obtain

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha \cdot M_1(x_{n-1}, x_n),$$

that is,

$$d(x_n, x_{n+1}) \leq \alpha \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}\left[d(x_{n-1}, x_{n+1}) + 0\right]\right\},$$

since $d(x_n, Tx_{n-1}) = 0$. Now, by the triangle inequality

$$d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1})$$

and using the inequality $\frac{a+b}{2} \leq \max\{a,b\}$, we deduce that either

$$\max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{1}{2}d(x_{n-1},x_{n+1})\right\} = d(x_{n-1},x_n) \tag{2.2.31}$$

or

$$\max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{1}{2}d(x_{n-1},x_{n+1})\right\} = d(x_n,x_{n+1}). \tag{2.2.32}$$

The case (2.2.32) cannot hold because it would lead to the contradiction

$$d(x_n,x_{n+1}) \leq h\,d(x_n,x_{n+1}).$$

Hence, (2.2.31) must always hold, and this leads to

$$d(x_n,x_{n+1}) \leq h\,d(x_{n-1},x_n).$$

The rest of the proof is similar to that of Theorem 2.2.1. □

Like in the case of Theorem 2.2.1, it is possible to force the uniqueness of the fixed point of a Ćirić strong almost contraction by imposing an additional contractive condition, quite similar to (2.2.29), as shown by the next theorem.

Theorem 2.2.4. *Let (X,d) be a complete metric space and let $T : X \to X$ be a Ćirić strong almost contraction for which there exist $\theta \in [0,1)$ and some $L_1 \geq 0$ such that*

$$d(Tx,Ty) \leq \theta \cdot d(x,y) + L_1 \cdot d(x,Tx), \quad \text{for all } x,y \in X. \tag{2.2.33}$$

Then

(a) *T has a unique fixed point, i.e., $Fix(T) = \{x^*\}$;*

(b) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (2.1.2) converges to x^*, for any $x_0 \in X$;*

(c) *The error estimate (2.2.30) holds.*

(d) *The rate of convergence of the Picard iteration is given by*

$$d(x_n,x^*) \leq \theta\, d(x_{n-1},x^*), \quad n = 1,2,\ldots. \tag{2.2.34}$$

Proof. Assume T has two distinct fixed points, say $x^*, y^* \in X$. Then by (2.2.25), with $x := x^*$, $y := y^*$ we get

$$d(x^*,y^*) \leq \theta \cdot d(x^*,y^*) \;\;\Leftrightarrow\;\; (1-\theta)\,d(x^*,y^*) \leq 0,$$

so contradicting $d(x^*,y^*) > 0$.

Now letting $y := x_n$, $x := x^*$ in (2.2.33), we obtain the estimate (2.2.34).

The rest of the proof follows by Theorem 2.2.3. □

B978-0-12-804295-3.50002-4, 50002

Remark 2.2.8. Note that $M_1(x,y)$ given by (2.2.28) cannot be replaced by $M(x,y)$ appearing on the right-hand side of quasi-contraction condition (2.2.12), i.e.,

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right\}.$$

This would lead to the contraction condition

$$d(Tx,Ty) \leq \alpha \cdot M(x,y) + L d(y,Tx), \quad \text{for all } x,y \in X, \tag{2.2.35}$$

with $\alpha \in [0,1)$ and $L \geq 0$, which is too weak to ensure the existence of a fixed point (see the next example, taken from Ref. [63]; see also Ref. [62]).

Example 2.2.4. Let $X = \mathbb{N} = \{0,1,2,\ldots\}$ with the usual norm and let T be defined by $T(n) = n+1$. Then T does satisfy (2.2.35) with $\alpha = \frac{1}{2}$ and $L = 2$ but T is fixed point free. Indeed, if we take $x = n$, $y = m$, $m > n$, then $d(Tx,Ty) = m-n$, $M(x,y) = m-n+1$, $d(y,Tx) = m-n-1$. Thus condition (2.2.35) reduces to

$$m-n \leq \alpha(m-n+1) + 2(m-n-1) = \frac{5}{2}(m-n) - \frac{3}{2},$$

which is true, since $m-n \geq 1$.

An equivalent (see Ref. [199]) contractive condition that ensures the uniqueness of the fixed point has been obtained by Babu et al. [48] for almost contractions.

We state the fixed point theorem corresponding to this uniqueness condition in the case of Ćirić strong almost contractions.

Theorem 2.2.5. *Let (X,d) be a complete metric space and let $T: X \to X$ be a mapping for which there exist $\alpha \in [0,1)$ and some $L \geq 0$ such that for all $x,y \in X$*

$$d(Tx,Ty) \leq \alpha \cdot M_1(x,y) + L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}. \tag{2.2.36}$$

Then

(a) *T has a unique fixed point, i.e., $Fix(T) = \{x^*\}$;*

(b) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (2.1.2) converges to x^*, for any $x_0 \in X$;*

(c) *The error estimate (2.2.30) holds.*

For more details and results on Ćirić strong almost contractions, see Refs. [62, 63].

Starting from the fact that φ-contractions are natural generalizations of Banach contractions, we can extend the previous results from almost contractions to the more general class of almost φ-contractions. The same extension can be done for Ćirić strong almost contractions.

B978-0-12-804295-3.50002-4, 50002

To do this, the central concept is that of the comparison function (see Ref. [52] and references therein for more details, results and proofs).

Definition 2.2.2. A map $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *comparison function* if it satisfies:

(i_φ) φ is monotone increasing, i.e., $t_1 < t_2 \Rightarrow \varphi(t_1) \leq \varphi(t_2)$;

(ii_φ) the sequence $\{\varphi^n(t)\}_{n=0}^{\infty}$ converges to zero, for all $t \in \mathbb{R}_+$, where φ^n stands for the n^{th} iterate of φ.

If φ satisfies (i_φ) and

(iii_φ) $\sum\limits_{k=0}^{\infty} \varphi^k(t)$ converges for all $t \in \mathbb{R}_+$, then φ is said to be a *(c)-comparison function*.

It has been shown (see for example Ref. [60]), that φ satisfies (iii_φ) if and only if there exist $0 < c < 1$ and a convergent series of positive terms, $\sum\limits_{n=0}^{\infty} u_n$, such that

$$\varphi^{k+1}(t) \leq c\varphi^k(t) + u_k, \quad \text{for all } t \in \mathbb{R}_+ \quad \text{and } k \geq k_0 \text{ (fixed).}$$

It is also known that if φ is a (c)-comparison function, then the sum of the comparison series, i.e.,

$$s(t) = \sum_{k=0}^{\infty} \varphi^k(t), \quad t \in \mathbb{R}_+, \tag{2.2.37}$$

is monotone increasing and continuous at zero, and that any (c)-comparison function is a comparison function.

A prototype for comparison functions is

$$\varphi(t) = at, \quad t \in \mathbb{R}_+ \quad (0 \leq a < 1)$$

but, as shown by Example 2.2.5, the comparison functions need not be either linear or continuous.

Note however that any comparison function is continuous at zero.

Example 2.2.5. Let $\varphi_1(t) = \frac{t}{t+1}$, $t \in \mathbb{R}_+$ and $\varphi_2(t) = \frac{1}{2}t$, if $0 \leq t < 1$ and $\varphi_2(t) = t - \frac{1}{3}$, if $t \geq 1$. Then φ_1 is a nonlinear comparison function, which is not a (c)-comparison function, while φ_2 is a discontinuous (c)-comparison function.

By replacing the well-known strict contractive condition (2.1.1) appearing in Banach's fixed point theorem, i.e.,

$$d(Tx, Ty) \leq a\, d(x, y), \quad \text{for all } x, y \in X,$$

by a more general one

$$d(Tx,Ty) \leq \varphi\big(d(x,y)\big), \quad \text{for all } x,y \in X, \tag{2.2.38}$$

where φ is a certain comparison function, several fixed point theorems have been obtained (see Ref. [235] and references therein).

Recall that an operator T which satisfies a condition of the form (2.2.38) is commonly named a φ-*contraction*.

Following the way in which the Banach contractions were extended to φ-contractions, in the following we would like to extend Theorems 2.2.1 and 2.2.2 from almost contractions to almost φ-contractions.

Definition 2.2.3. Let (X,d) be a metric space. A mapping $T: X \to X$ is said to be an *almost φ-contraction* or a (φ,L)-*almost contraction* provided that there exist a comparison function φ and some $L \geq 0$, such that

$$d(Tx,Ty) \leq \varphi\big(d(x,y)\big) + Ld(y,Tx), \quad \text{for all } x,y \in X. \tag{2.2.39}$$

Remark 2.2.9. Clearly, any almost contraction is an almost φ-contraction, with $\varphi(t) = \delta t$, $t \in \mathbb{R}_+$ and $0 < \delta < 1$. Also, all φ-contractions are almost φ-contractions with $L \equiv 0$ in (2.2.39).

The next theorem, taken from Ref. [54], extends Theorem 2.2.1 from almost contractions to almost φ-contractions.

Theorem 2.2.6. *Let (X,d) be a complete metric space and $T: X \to X$ an almost φ-contraction with φ a (c)-comparison function. Then*

(a) $F(T) = \{x \in X : Tx = x\} \neq \phi$*;*

(b) *For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_0 \in X$ and*

$$x_{n+1} = Tx_n, \quad n = 0,1,2,\ldots, \tag{2.2.40}$$

converges to a fixed point x^* of T;

(c) *The following estimate:*

$$d(x_n,x^*) \leq s\big(d(x_n,x_{n+1})\big), \quad n = 0,1,2,\ldots, \tag{2.2.41}$$

holds, where $s(t)$ is given by (2.2.37).

Proof. We shall prove that T has at least one fixed point in X. To this end, let $x_0 \in X$ be arbitrary and $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration defined by (2.2.40).

Since T is an almost φ-contraction, there exist a (c)-comparison function φ and some $L \geq 0$, such that

$$d(Tx,Ty) \leq \varphi\big(d(x,y)\big) + L \cdot d(y,Tx) \tag{2.2.42}$$

holds, for all $x, y \in X$.

Take $x := x_{n-1}$, $y := x_n$ in (2.2.42). We get

$$d(x_n,x_{n+1}) \leq \varphi\big(d(x_{n-1},x_n)\big), \quad \text{for all } n = 1,2,\ldots. \tag{2.2.43}$$

Since φ is not decreasing, by (2.2.43) we have

$$d(x_{n+1},x_{n+2}) \leq \varphi\big(d(x_n,x_{n+1})\big),$$

which inductively yields

$$d(x_{n+k},x_{n+k+1}) \leq \varphi^k\big(d(x_n,x_{n+1})\big), \quad k = 0,1,2,\ldots.$$

By the triangle rule, we have

$$\begin{aligned} d(x_n,x_{n+p}) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{n+p-1},x_{n+p}) \\ &\leq r + \varphi(r) + \cdots + \varphi^{n+p-1}(r), \end{aligned} \tag{2.2.44}$$

where $r := d(x_n,x_{n+1})$.

Again by (2.2.43), we find

$$d(x_n,x_{n+1}) \leq \varphi^n\big(d(x_0,x_1)\big), \quad n = 0,1,2,\ldots, \tag{2.2.45}$$

which, by property (ii_φ) of a comparison function, implies

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{2.2.46}$$

As φ is positive, it is obvious that

$$r + \varphi(r) + \cdots + \varphi^{n+p-1}(r) < s(r), \tag{2.2.47}$$

where $s(r)$ is the sum of the series $\sum_{k=0}^{\infty} \varphi^k(r)$.

Then by (2.2.44) and (2.2.47), we get

$$d(x_n,x_{n+p}) \leq s\big(d(x_n,x_{n+1})\big), \quad n \in \mathbb{N},\ p \in \mathbb{N}. \tag{2.2.48}$$

Since s is continuous at zero, (2.2.46) and (2.2.47) imply that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. As X is complete, $\{x_n\}_{n=0}^{\infty}$ is convergent.

Let $x^* = \lim_{n\to\infty} x_n$. We shall prove that x^* is a fixed point of T. Indeed,

$$d(x^*,Tx^*) \leq d(x^*,x_{n+1}) + d(x_{n+1},Tx^*) = d(x_{n+1},x^*) + d(Tx_n,Tx^*).$$

By (2.2.42), we have

$$d(Tx_n, Tx^*) \le \varphi(d(x_n, x^*)) + L d(x^*, Tx_n),$$

and hence

$$d(x^*, Tx^*) \le (1+L)\, d(x_{n+1}, x^*) + \varphi(\, d(x_n, x^*)), \quad \text{for all } n \ge 0. \tag{2.2.49}$$

Now letting $n \to \infty$ in (2.2.49) and using the continuity of φ at zero, we obtain

$$d(x^*, Tx^*) = 0,$$

i.e., x^* is a fixed point of T. The estimate (2.2.41) follows by (2.2.44) by letting $p \to \infty$. □

Remark 2.2.10. (a) Using the *a posteriori* error estimate (2.2.41) and (2.2.45) we easily obtain

$$d(x_n, x^*) \le s\Big(\varphi^n\big(d(x_0, x_1)\big)\Big), \quad n = 0, 1, 2, \ldots,$$

which is the *a priori* estimate for the Picard iteration $\{x_n\}_{n=0}^{\infty}$.

(b) If we take $\varphi(t) = \delta \cdot t$, $t \in \mathbb{R}_+$, $0 < \delta < 1$, by Theorem 2.2.6 we obtain the corresponding result for almost contractions, i.e., Theorem 2.2.1.

Like in the case of almost contractions, in order to guarantee the uniqueness of the fixed point of T, we have to consider an additional contractive type condition, as in the next theorem.

Theorem 2.2.7. *Let X and T be as in Theorem 2.2.6. Suppose T also satisfies the following condition: there exist a comparison function ψ and some $L_1 \ge 0$ such that*

$$d(Tx, Ty) \le \psi\big(d(x,y)\big) + L_1 d(x, Tx), \quad \text{for all } x, y \in X. \tag{2.2.50}$$

Then

(a) *T has a unique fixed point, i.e., $F(T) = \{x^*\}$;*

(b) *The estimate (2.2.41) holds;*

(c) *The rate of convergence of the Picard iteration is expressed by*

$$d(x_n, x^*) \le \varphi\big(d(x_{n-1}, x^*)\big), \quad n = 1, 2, \ldots . \tag{2.2.51}$$

Proof. Assume there exist two distinct fixed points $x^*, y^* \in X$. Then by (2.2.50) with $x := x^*$ and $y := y^*$, we get

$$d(x^*, y^*) \le \psi\big(d(x^*, y^*)\big),$$

which by induction yields

$$d(x^*,y^*) \le \psi^n\big(d(x^*,y^*)\big), \quad n=1,2,\ldots. \tag{2.2.52}$$

Now, letting $n \to \infty$ in (2.2.52) we get

$$d(x^*,y^*)=0,$$

i.e., $x^*=y^*$, a contradiction. Therefore, T has a unique fixed point.

To obtain (2.2.51), we let $x := x^*$, $y := x_n$ in (2.2.50). □

The next Maia type extension of Theorem 2.2.7 is very natural (see Ref. [54]).

Theorem 2.2.8. *Let X be a nonempty set and d,ρ be two metrics on X such that (X,d) is complete. Let $T: X \to X$ be a self operator satisfying the following.*

(i) *There exists a (c)-comparison function φ and $L \ge 0$ such that*

$$d(Tx,Ty) \le \varphi\big(d(x,y)\big) + Ld(y,Tx), \quad \text{for all } x,y \in X.$$

(ii) *There exists a comparison function ψ and $L_1 \ge 0$ such that*

$$\rho(Tx,Ty) \le \psi\big(\rho(x,y)\big) + L_1\rho(x,Tx), \quad \text{for all } x,y \in X.$$

Then

(a) *T has a unique fixed point x^*;*

(b) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$, $x_{n+1}=Tx_n$, $n \ge 0$, converges to x^*, for all $x_0 \in X$;*

(c) *The* a posteriori *error estimate*

$$d(x_n,x^*) \le s\big(d(x_n,x_{n+1})\big), \quad n=0,1,2,\ldots,$$

holds, where $s(t)=\sum\limits_{k=0}^{\infty}\varphi^k(t)$;

(d) *The rate of convergence of the Picard iteration is given by*

$$\rho(x_n,x^*) \le \psi\big(\rho(x_{n-1},x^*)\big), \quad n \ge 1.$$

For other extensions of single-valued almost contractions, see Refs. [6–9, 11, 12, 14, 18, 20, 22, 28, 30, 41, 42, 44, 45, 47, 51, 89, 91, 125, 128, 138, 140, 161, 169, 181, 198, 199, 202, 203, 208, 216, 217, 243–248, 251, 252, 254, 257, 260, 262–266, 271, 277].

Final note. In Ref. [261] the authors tried to prove that Theorems 2.2.1 and 2.2.2 are false. Their claims were shown to be false and based on some wrong calculations (see the reply paper [73]).

Another attempt to diminish the merits of almost contractions is contained in a series of two recent papers [274, 275], where the author claims that "the almost contraction condition is *almost covered*" by a contractive condition studied in Ref. [155].

To be more specific, we present the statement of the main result in the paper [157], and then show that the above claim is not valid either. The authors of Ref. [157] established the following result (Theorem 2).

Theorem 2.2.9. *Let (X,d) be a complete metric space, T a self map of X, and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing and continuous function with the property $\varphi(t) = 0$ if and only if $t = 0$. Furthermore, let a, b, c be three decreasing functions from $\mathbb{R}_+ \setminus \{0\}$ into $[0,1)$ such that $a(t) + 2b(t) + c(t) < 1$ for every $t > 0$. Suppose that T satisfies the following condition:*

$$\begin{aligned}\varphi(d(Tx,Ty)) \quad \leq \quad & a(x,y)\varphi(d(x,y)) \\ & +b(x,y)[\varphi(d(x,Tx)) + \varphi(d(y,Ty))] \\ & +c(d(x,y))\min\{\varphi(d(x,Ty)), \varphi(d(y,Tx))\},\end{aligned} \tag{2.2.53}$$

where $x, y \in X$ and $x \neq y$. Then T has a unique fixed point.

Because the coefficients $a(t)$, $b(t)$, $c(t)$ are subject to the strong condition $a(t) + 2b(t) + c(t) < 1$ (see Remark 2.2.7), the class of mappings satisfying (2.2.53) cannot cover the class of almost contractions, for which the corresponding nonnegative coefficient $c(t)$ should be free of any restriction.

2.3. Implicit Almost Contractions

A simple and natural way to unify and prove in a simple manner several metrical fixed point theorems is to consider an implicit contraction type condition instead of the usual explicit contractive conditions.

It appears that Turinici [272] was the first to consider fixed point theorems for contractions defined by implicit relations. If (X,d) is a metric space, in Ref. [272] there are considered mappings $T : X \to X$, satisfying the implicit contraction condition

$$d(Tx,Ty) \leq f(d(x,y), d(x,Tx), d(y,Ty)), \quad \text{for all } x, y \in X,$$

where $f : \mathbb{R}_+^3 \to \mathbb{R}_+$.

Later, Popa [211, 212], initiated a systematic study of the contractions defined by implicit relations of the form:

$$F\,(d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)) \leq 0,$$

where $F : \mathbb{R}_+^6 \to \mathbb{R}_+$.

This direction of research led to a comprehensive literature (that cannot be completely cited here) on fixed point, common fixed point and coincidence point theorems, both for single-valued and multivalued mappings, in various ambient spaces.

So, the aim of this section is to obtain general fixed point theorems for implicit almost contractions, largely by following Ref. [69].

Let $\mathscr{F}$ be the set of all continuous real functions $F:\mathbb{R}_+^6 \to \mathbb{R}_+$, for which we consider the following conditions:

(F_{1a}) F is nonincreasing in the fifth variable and $F(u,v,v,u,u+v,0) \le 0$ for $u,v \ge 0$ $\Rightarrow \exists h \in [0,1)$ such that $u \le hv$;

(F_{1b}) F is nonincreasing in the fourth variable and $F(u,v,0,u+v,u,v) \le 0$ for $u,v \ge 0 \Rightarrow \exists h \in [0,1)$ such that $u \le hv$;

(F_{1c}) F is nonincreasing in the third variable and $F(u,v,u+v,0,v,u) \le 0$ for $u,v \ge 0$ $\Rightarrow \exists h \in [0,1)$ such that $u \le hv$;

(F_2) $F(u,u,0,0,u,u) > 0$, for all $u > 0$.

The following functions are related to well-known fixed point theorems and satisfy most of the conditions (F_{1a})–(F_2) above.

Example 2.3.1. The function $F \in \mathscr{F}$, given by

$$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2,$$

where $a \in [0,1)$, satisfies (F_2) and (F_{1a})–(F_{1c}), with $h = a$.

Example 2.3.2. Let $b \in \left[0,\frac{1}{2}\right)$. Then the function $F \in \mathscr{F}$, given by

$$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - b(t_3+t_4),$$

satisfies (F_2) and (F_{1a})–(F_{1c}), with $h = \frac{b}{1-b} < 1$.

Example 2.3.3. Let $c \in \left[0,\frac{1}{2}\right)$. Then the function $F \in \mathscr{F}$, given by

$$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - c(t_5+t_6),$$

satisfies (F_2) and (F_{1a})–(F_{1c}), with $h = \frac{c}{1-c} < 1$.

Example 2.3.4. The function $F \in \mathscr{F}$, given by

$$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\},$$

where $a \in [0,1)$, satisfies (F_2) and (F_{1a})–(F_{1c}), with $h = a$.

Example 2.3.5. The function $F \in \mathscr{F}$, given by

$$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2 - b(t_3+t_4) - c(t_5+t_6),$$

where $a, b, c \in [0,1)$ and $a + 2b + 2c < 1$, satisfies (F_2) and (F_{1a})–(F_{1c}), with $h = \frac{a+b+c}{1-b-c} < 1$.

Example 2.3.6. The function $F \in \mathscr{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \max\left\{t_2, \frac{t_3 + t_4}{2}, t_5, t_6\right\},$$

where $a \in [0,1)$, satisfies (F_2) and (F_{1b}), (F_{1c}), with $h = a$ and (F_{1a}), with $h = \frac{a}{1-a} < 1$, if $a < 1/2$.

Example 2.3.7. The function $F \in \mathscr{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - Lt_3,$$

where $a \in [0,1)$ and $L \geq 0$, satisfies (F_2) and (F_{1b}), with $h = a$, but, in general, does not satisfy (F_{1a}) and (F_{1c}).

Example 2.3.8. The function $F \in \mathscr{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - Lt_6,$$

where $a \in [0,1)$ and $L \geq 0$, satisfies (F_{1a}), with $h = a$, but, in general, does not satisfy (F_{1b}), (F_{1c}) and (F_2).

The following theorem, which is an enriched version of Theorem 3 of Popa [211] and unifies the most important metrical fixed point theorems for contractive mappings in Rhoades' classification [226], was given in Ref. [69].

Theorem 2.3.1. *Let (X, d) be a complete metric space, $T : X \to X$ a self mapping for which there exists $F \in \mathscr{F}$ such that for all $x, y \in X$,*

$$F\left(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right) \leq 0. \tag{2.3.54}$$

If F satisfies (F_{1a}) and (F_2) then:

(p1) *T has a unique fixed point $\overline{x}$ in X;*

(p2) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by*

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots, \tag{2.3.55}$$

converges to $\overline{x}$, for any $x_0 \in X$.

(p3) *The following estimate holds:*

$$d(x_{n+i-1},\overline{x}) \leq \frac{h^i}{1-h} d(x_n,x_{n-1}), \quad n = 0,1,2,\ldots; i = 1,2,\ldots, \tag{2.3.56}$$

where h is the constant appearing in (F_{1a}).

(p4) *If, additionally, F satisfies* (F_{1c})*, then the rate of convergence of Picard iteration is given by:*

$$d(x_{n+1},\overline{x}) \leq hd(x_n,\overline{x}), \quad n = 0,1,2,\ldots. \tag{2.3.57}$$

Remark 2.3.1. (a) If F is the function in Example 2.3.1, then by Theorem 2.3.1 we obtain the Banach contraction mapping principle, in its complete form (see Theorem 2.1.1).

(b) If F is the function in Example 2.3.2, then by Theorem 2.3.1 we obtain Theorem 1 in Ref. [58], that extends the well-known Kannan fixed point theorem [149].

(c) If F is the function in Example 2.3.3, then by Theorem 2.3.1 we obtain a fixed point theorem that extends the Chatterjea fixed point theorem [85].

(d) If F is the function in Example 2.3.4, then by Theorem 2.3.1 we obtain Theorem 2 in Ref. [58], that extends the well-known Zamfirescu fixed point theorem [280].

(e) If F is the function in Example 2.3.5, then by Theorem 2.3.1 we obtain a fixed point theorem that extends the Reich-Rus fixed point theorem [239].

For other important particular cases of Theorem 2.3.1, see Refs. [53, 69] and the references therein.

The first main result of this section extends Theorem 2.3.1 in such a way to also include some known fixed point theorems for explicit almost contractions.

Theorem 2.3.2. *Let* (X,d) *be a complete metric space,* $T: X \to X$ *a self mapping for which there exists* $F \in \mathscr{F}$*, satisfying* (F_{1a})*, such that for all* $x,y \in X$*,*

$$F\left(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\right) \leq 0. \tag{2.3.58}$$

Then

(p1) $Fix(T) \neq \emptyset$;

(p2) *For any* $x_0 \in X$*, the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *converges to a fixed point* $\overline{x}$ *of* T.

(p3) *The following estimate holds:*

$$d(x_{n+i-1},\overline{x}) \le \frac{h^i}{1-h} d(x_n,x_{n-1}), \quad n=0,1,2,\dots; i=1,2,\dots, \tag{2.3.59}$$

where h is the constant appearing in (F_{1a}).

(p4) *If, additionally, F satisfies* (F_{1c})*, then the rate of convergence of Picard iteration is given by:*

$$d(x_{n+1},\overline{x}) \le h\, d(x_n,\overline{x}), \quad n=0,1,2,\dots. \tag{2.3.60}$$

Proof. (p1) Let x_0 be an arbitrary point in X and $x_{n+1}=Tx_n$, $n=0,1,\dots$, be the Picard iteration. If we take $x:=x_{n-1}$ and $y:=x_n$ in (2.3.58) and denote $u:=d(x_n,x_{n+1})$, $v:=d(x_{n-1},x_n)$ we get

$$F(u,v,v,u,d(x_{n-1},x_{n+1}),0) \le 0.$$

By the triangle inequality, $d(x_{n-1},x_{n+1}) \le d(x_{n-1},x_n)+d(x_n,x_{n+1}) = u+v$ and, since F is nonincreasing in the fifth variable, we have

$$F(u,v,v,u,u+v,0) \le F(u,v,v,u,d(x_{n-1},x_{n+1}),0) \le 0$$

and hence, in view of assumption (F_{1a}), there exists $h\in[0,1)$ such that $u\le hv$, that is,

$$d(x_n,x_{n+1}) \le h\, d(x_{n-1},x_n), \tag{2.3.61}$$

which, in a straightforward way, leads to the conclusion that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

Since (X,d) is complete, there exists a $\overline{x}$ in X such that

$$\lim_{n\to\infty} x_n = \overline{x}. \tag{2.3.62}$$

By taking $x:=x_n$ and $y:=\overline{x}$ in (2.3.58) we get

$$F(d(Tx_n,T\overline{x}),d(x_n,\overline{x}),d(x_n,Tx_n),d(\overline{x},T\overline{x}),d(x_n,T\overline{x}),d(\overline{x},Tx_n)) \le 0. \tag{2.3.63}$$

As F is continuous, by letting $n\to\infty$ in (2.3.63), we obtain

$$F(d(\overline{x},T\overline{x}),0,0,d(\overline{x},T\overline{x}),d(\overline{x},T\overline{x}),0)) \le 0,$$

which, by assumption (F_{1a}), yields $d(\overline{x},T\overline{x}) \le 0$, that is, $\overline{x}=T\overline{x}$.

(p2) This follows by the proof of (p1).

(p3) This follows by a double inductive process by means of (2.3.61).

(p4) By taking $x:=x_n$ and $y:=\overline{x}$ in (2.3.58) we get

$$F(d(Tx_n,\overline{x}),d(x_n,\overline{x}),d(x_n,Tx_n),d(\overline{x},\overline{x}),d(x_n,\overline{x}),d(\overline{x},Tx_n)) \le 0,$$

that is,

$$F\left(d(x_{n+1},\overline{x}),d(x_n,\overline{x}),d(x_n,x_{n+1}),0,d(x_n,\overline{x}),d(\overline{x},x_{n+1})\right)\leq 0. \tag{2.3.64}$$

Denote $u:=d(x_{n+1},\overline{x})$, $v:=d(x_n,\overline{x})$. Then, by the triangle inequality we have $d(x_n,x_{n+1})\leq d(x_n,\overline{x})+d(x_{n+1},\overline{x})=u+v$ and hence, in view of assumption (F_{1c}), by (2.3.64) we obtain

$$F\left(u,v,u+v,0,v,u\right))\leq F\left(u,v,d(x_n,x_{n+1}),0,v,u\right)\leq 0,$$

which again by (F_{1c}) implies the existence of $h\in[0,1)$ such that $u\leq hv$, which is exactly the desired estimate (2.3.60). □

Remark 2.3.2. (a) If F in Theorem 2.3.2 also satisfies (F_2), then by Theorem 2.3.2 we obtain Theorem 2.3.1.

(b) If F is the function in Example 2.3.7, then by Theorem 2.3.2 (but not by Theorem 2.3.1) we obtain Theorem 2.2.1, i.e., the existence theorem [55, Theorem 2.1].

(c) If F is the function in Example 2.3.8, then by Theorem 2.3.2 (or by Theorem 2.3.1) we obtain Theorem 2.2.2, i.e., the existence and uniqueness theorem [55, Theorem 2.2].

Remark 2.3.3. From the unifying error estimates (2.3.59), inspired by Ref. [278], we get both the *a priori* estimate

$$d(x_n,\overline{x})\leq\frac{h^n}{1-h}d(x_0,x_1),\quad n=0,1,2,\ldots,$$

and the *a posteriori* estimate

$$d(x_n,\overline{x})\leq\frac{h}{1-h}d(x_n,x_{n-1}),\quad n=1,2,\ldots,$$

which are extremely important in applications, especially when approximating the solutions of nonlinear equations.

One can also obtain an existence and uniqueness fixed point theorem, corresponding to Theorem 2.3.2.

Theorem 2.3.3. *Let (X,d) be a complete metric space, $T:X\to X$ a self mapping for which there exists $F\in\mathscr{F}$, satisfying (F_{1a}), such that for all $x,y\in X$, (2.3.58) holds, and there exists $G\in\mathscr{F}$, satisfying (F_2), such that for all $x,y\in X$,*

$$G\left(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\right)\leq 0. \tag{2.3.65}$$

Then

(p1) *T has a unique fixed point $\overline{x}$ in X;*

(p2) *For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ converges to $\overline{x}$.*

(p3) *The error estimate (2.3.59) holds;*

(p4) *If, additionally, F or G satisfies (F_{1c}), then the rate of convergence of the Picard iteration is given by:*

$$d(x_{n+1},\overline{x}) \le hd(x_n,\overline{x}), \quad n = 0,1,2,\ldots. \tag{2.3.66}$$

Proof. The existence of the fixed point as well as the estimates (2.3.59) and (2.3.66) follow as in the proof of Theorem 2.3.2.

In order to prove the uniqueness of $\overline{x}$, assume the contrary, i.e., there exists $\overline{y} \in Fix(T), \overline{x} \neq \overline{y}$. Then by taking $x := \overline{x}$ and $y := \overline{y}$ in (2.3.65) and by denoting $\delta := d(\overline{x},\overline{y}) > 0$ we get

$$G(\delta,\delta,0,0,\delta,\delta) \le 0,$$

which contradicts (F_2).

This proves that T has a unique fixed point. □

Remark 2.3.4. (a) If F is the function in Example 2.3.8 and G is the function in Example 2.3.7, then by Theorem 2.3.3 we obtain Theorem 2.2.2, i.e., Theorem 2.2 in Ref. [55].

(b) If $F \equiv G$ is the function in Example 2.3.9, then by Theorem 2.3.3 one obtains the main result (Theorem 2.3) in Ref. [47].

(c) If F is the function in Example 2.3.11, then by Theorem 2.3.3 one obtains the second uniqueness result [63, Theorem 2.4].

Theorem 2.3.3 can now be significantly extended by considering two metrics on the set X, similarly to Theorem 5 in Ref. [54].

Theorem 2.3.4. *Let X be a nonempty set and d,ρ two metrics on X such that (X,d) is complete. Let $T : X \to X$ be a self operator for which*

(i) *there exists $F \in \mathscr{F}$ satisfying (F_{1a}) such that for all $x,y \in X$,*

$$F\left(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\right) \le 0;$$

(ii) *there exists $G \in \mathscr{F}$ satisfying (F_{1c}) and (F_2) such that for all $x,y \in X$,*

$$G\left(\rho(Tx,Ty),\rho(x,y),\rho(x,Tx),\rho(y,Ty),\rho(x,Ty),\rho(y,Tx)\right) \le 0.$$

Then

(a) *T has a unique fixed point $\overline{x}$;*

(b) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$, $x_{n+1} = Tx_n$, $n \geq 0$, converges to $\overline{x}$, for all $x_0 \in X$;*

(c) *The error estimate (2.3.59) holds;*

(d) *The rate of convergence of the Picard iteration is given by*

$$\rho(x_n, x^*) \leq h\rho(x_{n-1}, x^*), \quad n \geq 1. \tag{2.3.67}$$

Proof. The existence of the fixed point as well as the estimates (2.3.59) and (2.3.67) follow as in the proof of Theorem 2.3.2.

In order to prove the uniqueness of $\overline{x}$, assume the contrary, i.e., there exists $\overline{y} \in Fix(T), \overline{x} \neq \overline{y}$. Then by taking $x := \overline{x}$ and $y := \overline{y}$ in (2.3.65) and by denoting $\delta := \rho(\overline{x}, \overline{y}) > 0$ we get

$$G(\delta, \delta, 0, 0, \delta, \delta) \leq 0,$$

which contradicts (F_2). This proves that T has a unique fixed point. □

In order to illustrate the generality of Theorems 2.3.3 and 2.3.4, we consider three more examples of functions $F \in \mathscr{F}$.

Example 2.3.9. The function $F \in \mathscr{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - L\min\{t_3, t_4, t_5, t_6\},$$

where $a \in [0, 1)$ and $L \geq 0$, satisfies (F_2) and (F_{1a})–(F_{1c}), with $h = a$.

Example 2.3.10. The function $F \in \mathscr{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a\max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\} - Lt_6,$$

where $a \in [0, 1)$ and $L \geq 0$, satisfies (F_{1a}), with $h = a$, but, in general, does not satisfy (F_{1b}), (F_{1c}) and (F_2). To prove (F_{1a}) let us observe that with $F(u, v, v, u, u + v, 0) \leq 0$ one obtains

$$u - a\max\left\{v, v, u, \frac{u + v}{2}\right\} \leq 0.$$

If one admits that $u > v$, then by the previous inequality one obtains $u - au \leq 0 \Leftrightarrow (1 - a)u \leq 0$, a contradiction. Hence $u \leq v$ and thus (F_{1a}) is satisfied with $h = a$.

Example 2.3.11. The function $F \in \mathscr{F}$, given by

$$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\} - L\min\{t_3,t_4,t_5,t_6\},$$

where $a \in [0,1)$ and $L \geq 0$, satisfies (F_2) and (F_{1a})–(F_{1c}), with $h = a$.

Remark 2.3.5. (a) If we set $d \equiv \rho$, by Theorem 2.3.4 we obtain Theorem 2.3.3.

(b) If F is the function in Example 2.3.8 and G is the function in Example 2.3.7, then by Theorem 2.3.3 we obtain Theorem 2.2.2, i.e., Theorem 2.2 in Ref. [55].

(c) If F is the function given by

$$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\} - Lt_6,$$

where $a \in [0,1)$ and $L \geq 0$, and G is as in Example 2.3.7, then F satisfies (F_2) and (F_{1a})–(F_{1c}), with $h = a$, and hence by Theorem 2.3.3 one obtains the first uniqueness result [63, Theorem 2.3].

All contractive conditions considered in this chapter are defined by *linear* functions $F \in \mathscr{F}$ (see Examples 2.3.1–2.3.11 and Remark 2.3.5), but generally, in Theorems 2.3.2, 2.3.3, and 2.3.4, neither F nor G is assumed to be linear.

This ensures a great generality to the results obtained in the present section. Several *nonlinear* contractive conditions associated with similar fixed point theorems can be found, for example, in Refs. [211, 212].

It is very important to note that, to our best knowledge, the contraction conditions defined by the functions in Examples 2.3.7–2.3.11 in this chapter have not been considered in any other papers devoted to fixed point theorems for mappings defined by implicit relations.

2.4. Common Fixed Point Theorems for Almost Contractions

The Banach contraction mapping principle (Theorem 2.1.1) has been extended in another direction than the ones illustrated in the previous sections by Jungck [144] to obtain the following common fixed point theorem.

Theorem 2.4.1 [144]. *Let (X,d) be a complete metric space. Let S be a continuous self map on X and T be any self map on X that commutes with S. Further let S and T satisfy $T(X) \subset S(X)$ and there exists a constant $\lambda \in (0,1)$ such that for every $x,y \in X$,*

$$d(Tx,Ty) \leq \lambda\, d(Sx,Sy), \quad \text{for all } x,y \in X. \tag{2.4.68}$$

Then S and T have a unique common fixed point.

Note that Theorem 2.4.1 reduces to Theorem 2.1.1 in the case $S = I$ (the identity map on X).

Other common fixed point results for the Kannan, Chatterjea and Zamfirescu contractive conditions have been recently obtained in Refs. [1] and [63] respectively, while the corresponding common fixed point result for Ciric's fixed point theorem has been derived by Das and Naik [105].

Theorem 2.4.2 [105]. *Let (X,d) be a complete metric space. Let S be a continuous self map on X and T be any self map on X that commutes with S. Further let S and T satisfy $T(X) \subset S(X)$. If there exists a constant $h \in (0,1)$ such that for every $x,y \in X$,*

$$d(Tx,Ty) \leq hM(x,y), \tag{2.4.69}$$

where

$$M(x,y) = \max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), d(Sx,Ty), d(Sy,Tx)\right\},$$

then S and T have a unique common fixed point.

Due to the fact that Theorem 2.4.2 requires S and T to be commuting mappings, an extension of this result to weakly commuting generalized quasi-contractions has been given in Ref. [56].

As all Banach, Kannan, Chatterjea and Zamfirescu contractive conditions imply the almost contractive condition (2.2.8), it is the main purpose of the present section to extend Theorem 1 in Ref. [56], and thus all its subsequent results, to the case of a pair of mappings (S,T) satisfying an almost contraction condition.

To this end we need some notions and results from Refs. [1] and [145].

Definition 2.4.1 [1]. Let S and T be self maps of a nonempty set X. If there exists $x \in X$ such that $Sx = Tx$ then x is called a *coincidence point* of S and T, while $y = Sx = Tx$ is called a *point of coincidence* (or *coincidence value*) of S and T. If $Sx = Tx = x$, then x is called a *common fixed point* of S and T.

Definition 2.4.2 [145]. Let S and T be self maps of a nonempty set X. The pair of mappings S and T is said to be *weakly compatible* if they commute at their coincidence points.

The next proposition will be needed to prove the last part in our main results.

Proposition 2.4.1 [1, Proposition 1.4]. *Let S and T be weakly compatible self maps of a nonempty set X. If S and T have a unique coincidence point x, then x is the unique* *common fixed point of S and T.*

For some other recent related results, see Refs. [72, 146].

We start this section by presenting a coincidence point theorem for almost contraction type mappings.

Theorem 2.4.3 [63]. *Let (X,d) be a metric space and let $T,S:X\to X$ be two mappings for which there exist a constant $\delta\in(0,1)$ and some $L\geq 0$ such that*

$$d(Tx,Ty)\leq\delta\cdot d(Sx,Sy)+Ld(Sy,Tx),\quad \text{for all } x,y\in X. \tag{2.4.70}$$

If the range of S contains the range of T and $S(X)$ is a complete subspace of X, then T and S have a coincidence point in X.

Moreover, for any $x_0\in X$, the iteration $\{Sx_n\}$ defined by (2.4.72) converges to some coincidence point x^* of T and S, with the following error estimate:

$$d(Sx_{n+i-1},x^*)\leq\frac{\delta^i}{1-\delta}d(Sx_n,Sx_{n-1}),\quad n=0,1,2,\ldots;i=1,2,\ldots. \tag{2.4.71}$$

Proof. Let x_0 be an arbitrary point in X. Since $T(X)\subset S(X)$, we can choose a point x_1 in X such that $Tx_0=Sx_1$. Continuing in this way, for a value x_n in X, we can find $x_{n+1}\in X$ such that

$$Sx_{n+1}=Tx_n,\quad n=0,1,\ldots. \tag{2.4.72}$$

If $x:=x_n$, $y:=x_{n-1}$ are two successive terms of the sequence defined by (2.4.72), then by (2.4.70) we have

$$d(Sx_n,Sx_{n+1})=d(Tx_{n-1},Tx_n)\leq L\cdot d(Sx_n,Tx_{n-1})+\delta\cdot d(Sx_{n-1},Sx_n),$$

which in view of (2.4.72) yields

$$d(Sx_{n+1},Sx_n)\leq\delta\cdot d(Sx_n,Sx_{n-1}),\quad n=0,1,2,\ldots. \tag{2.4.73}$$

Now by induction, from (2.4.73) we obtain

$$d(Sx_{n+k},Sx_{n+k-1})\leq\delta^k\cdot d(Sx_n,Sx_{n-1}),\, n,k=0,1,\ldots\ (k\neq 0),$$

and then, for $p>i$, we get after straightforward calculations

$$d(Sx_{n+p},Sx_{n+i-1})\leq\frac{\delta^i(1-\delta^{p-i+1})}{1-\delta}\cdot d(Sx_n,Sx_{n-1}),\quad n\geq 0;\, i\geq 1. \tag{2.4.74}$$

Take $i=1$ in (2.4.74) and then, by an inductive process, we get

$$d(Sx_{n+p},Sx_n)\leq\frac{\delta}{1-\delta}\cdot d(Sx_n,Sx_{n-1})\leq\frac{\delta^n}{1-\delta}\cdot d(Sx_1,Sx_0),\quad n=0,1,2\ldots,$$

which shows that $\{Sx_n\}$ is a Cauchy sequence.

Since $S(X)$ is complete, there exists a value of x^* in $S(X)$ such that

$$\lim_{n\to\infty} Sx_{n+1} = x^*. \tag{2.4.75}$$

We can find $p \in X$ such that $Sp = x^*$. By (2.4.72) and (2.4.73) we further have

$$d(Sx_n, Tp) \le \delta d(Sx_{n-1}, Sp) \le \delta^{n-1} d(Sx_1, Sp),$$

which shows that we also have

$$\lim_{n\to\infty} Sx_n = Tp. \tag{2.4.76}$$

Now by (2.4.75) and (2.4.76) we find that $Tp = Sp$, that is, p is a coincidence point of T and S (or x^* is a point of coincidence of T and S). The estimate (2.4.71) is obtained from (2.4.74) by letting $p \to \infty$. □

Remark 2.4.1. Let us note that the coincidence point ensured by Theorem 2.4.3 is not generally unique (see Ref. [55, Example 1]).

In order to obtain a common fixed point theorem from the coincidence Theorem 2.4.3, we need the uniqueness of the coincidence point, which can be obtained by imposing an additional contractive condition, similar to (2.4.70).

Theorem 2.4.4. *Let (X,d) be a metric space and let $T, S : X \to X$ be two mappings satisfying (2.4.70) for which there exist a constant $\theta \in (0,1)$ and some $L_1 \ge 0$ such that*

$$d(Tx, Ty) \le \theta \cdot d(Sx, Sy) + L_1 d(Sx, Tx), \quad \text{for all } x, y \in X. \tag{2.4.77}$$

If the range of S contains the range of T and $S(X)$ is a complete subspace of X, then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

In both cases, for any $x_0 \in X$, the iteration $\{Sx_n\}$ defined by (2.4.72) converges to the unique common fixed point (coincidence point) x^* of S and T, with the error estimate (2.4.71).

The convergence rate of the iteration $\{Sx_n\}$ is given by

$$d(Sx_n, x^*) \le \theta \cdot d(Sx_{n-1}, x^*), \quad n = 1, 2, \ldots. \tag{2.4.78}$$

Proof. By the proof of Theorem 2.4.3, we have that T and S have at least one point of coincidence. Now let us show that T and S actually have a unique point of coincidence.

Assume there exists $q \in X$ such that $Tq = Sq$. Then, by (2.4.77) we get

$$d(Sq, Sp) = d(Tq, Tp) \leq 2\delta d(Sq, Tq) + \delta d(Sq, Tp) = \delta d(Sq, Sp),$$

which shows that $Sq = Sp = x^*$, that is, T and S have a unique point of coincidence, x^*.

Now if T and S are weakly compatible, by Proposition 2.4.1 it follows that x^* is their unique common fixed point. The estimate (2.4.78) is obtained from (2.4.77) by taking $x = x_n$ and $y = x^*$. □

An equivalent (see Ref. [199]) but simpler contractive condition that ensures the uniqueness of the coincidence point and which actually unifies (2.4.70) and (2.4.77) has been very recently obtained by Babu et al. [48]. We state in the following the common fixed point theorem corresponding to this fixed point result.

Theorem 2.4.5. *Let (X,d) be a metric space and let $T, S : X \to X$ be two mappings for which there exist a constant $\delta \in (0,1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta \cdot d(Sx, Sy) + L \min\{d(Sx, Tx) + d(Sy, Ty) + d(Sx, Ty) + d(Sy, Tx)\}, \quad (2.4.79)$$

for all $x, y \in X$. If the range of S contains the range of T and $S(X)$ is a complete subspace of X, then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

In both cases, for any $x_0 \in X$, the iteration $\{Sx_n\}$ defined by (2.4.72) converges to the unique common fixed point (coincidence point) x^* of S and T, with the error estimate (2.4.71) and convergence rate given by (2.4.78).

Proof. If $x := x_n$, $y := x_{n-1}$ are two successive terms of the sequence defined by (2.4.72), then by (2.4.79) we have

$$d(Sx_n, Sx_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \delta \cdot d(Sx_{n-1}, Sx_n) + L \cdot M,$$

where

$$M = \min\{d(Sx_n, Tx_n) + d(Sx_{n-1}, Tx_{n-1}) + d(Sx_n, Tx_{n-1}) + d(Sx_{n-1}, Tx_n)\} = 0,$$

since $d(Sx_n, Tx_{n-1}) = 0$. The rest of the proof follows from that of Theorem 2.4.4. □

Remark 2.4.2. (a) If $S = I$ (the identity map on X), then by Theorem 2.4.3 we obtain the existence fixed point theorem given in Ref. [55] for almost contractions (Theorem 2.2.1).

If $S = I$, then by Theorem 2.4.4 we obtain the existence and uniqueness fixed point theorem given in Ref. [55] for almost contractions (Theorem 2.2.2).

B978-0-12-804295-3.50002-4, 50002

If $S = I$, then by Theorem 2.4.5 we obtain the existence and uniqueness fixed point theorem given in Ref. [48] for strict almost contractions.

(b) If $S = I$ and $L = 0$ in condition (2.4.70), then by Theorem 2.4.3 we obtain a result that extends Jungck's common fixed point theorem [144] from commuting mappings to weakly compatible mappings.

Three other particular cases that are obtained from our main results are given in the following as corollaries.

Corollary 2.4.1. *Let (X,d) be a metric space and let $T,S : X \to X$ be two mappings for which there exist $b \in [0,\frac{1}{2})$ such that, for all $x,y \in X$,*

$$(z_2) \qquad d(Tx,Ty) \le b\big[d(Sx,Tx) + d(Sy,Ty)\big].$$

If the range of S contains the range of T and $S(X)$ is a complete subspace of X, then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

In both cases, the iteration $\{Sx_n\}$ defined by (2.4.72) converges to the unique (coincidence) common fixed point x^* of S and T, for any $x_0 \in X$, with the following error estimate:

$$d(Sx_{n+i-1},x^*) \le \frac{\delta^i}{1-\delta}\, d(Sx_n,Sx_{n-1}), \quad n,i = 0,1,2,\dots\ (i \ne 0), \tag{2.4.80}$$

where $\delta = \frac{b}{1-b}$.

The convergence rate of the iteration $\{Sx_n\}$ is given by

$$d(Sx_n,x^*) \le \delta \cdot d(Sx_{n-1},x^*), \quad n = 1,2,\dots. \tag{2.4.81}$$

Proof. By condition (z_2) and the triangle rule, we get

$$\begin{aligned} d(Tx,Ty) &\le b\big[d(x,Tx) + d(y,Ty)\big] \le \\ &\le b\Big\{\big[d(x,y) + d(y,Tx)\big] + \big[d(y,Tx) + d(Tx,Ty)\big]\Big\}, \end{aligned}$$

which yields

$$(1-b)d(Tx,Ty) \le bd(x,y) + 2b \cdot d(y,Tx)$$

and which implies

$$d(Tx,Ty) \le \frac{b}{1-b}\, d(x,y) + \frac{2b}{1-b}\, d(y,Tx), \quad \text{for all } x,y \in X.$$

B978-0-12-804295-3.50002-4, 50002

Now, in view of $0 < b < \frac{1}{2}$, (2.4.70) holds with $\delta = \frac{b}{1-b}$ and $L = \frac{2b}{1-b}$. The uniqueness condition (2.4.81) follows similarly. To obtain the conclusion, apply Theorem 2.4.4. □

Corollary 2.4.2. *Let (X,d) be a metric space and let $T,S : X \to X$ be two mappings for which there exist $c \in [0, \frac{1}{2})$ such that, for all $x,y \in X$,*

$$(z_3) \qquad d(Tx,Ty) \le c\big[d(Sx,Ty) + d(Sy,Tx)\big].$$

If the range of S contains the range of T and $S(X)$ is a complete subspace of X, then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

In both cases, the iteration $\{Sx_n\}$ defined by (2.4.72) converges to the unique (coincidence) common fixed point x^* of S and T, for any $x_0 \in X$, with the following error estimate:

$$d(Sx_{n+i-1},x^*) \le \frac{\delta^i}{1-\delta} d(Sx_n,Sx_{n-1}), \quad n,i = 0,1,2,\dots\ (i \ne 0), \tag{2.4.82}$$

where $\delta = \frac{c}{1-c}$.

The convergence rate of the iteration $\{Sx_n\}$ is given by

$$d(Sx_n,x^*) \le \delta \cdot d(Sx_{n-1},x^*), \quad n = 1,2,\dots. \tag{2.4.83}$$

Proof. By condition (z_3) and the triangle rule, we get

$$d(Tx,Ty) \le \frac{c}{1-c} d(x,y) + \frac{2c}{1-c} d(y,Tx),$$

which is (2.4.70), with $\delta = \frac{c}{1-c} < 1$ and $L = \frac{2c}{1-c} \ge 0$.

The uniqueness condition (2.4.81) follows similarly. Now apply Theorem 2.4.4 to obtain the conclusion. □

Since any Banach contraction condition implies (2.4.70) (with L=0), by Corollaries 2.4.1 and 2.4.2 we obtain in particular the main result in Ref. [66].

Corollary 2.4.3. *Let (X,d) be a metric space and let $T,S : X \to X$ be two mappings for which there exist $a \in [0,1)$, b, $c \in [0, \frac{1}{2})$ such that, for all $x,y \in X$, at least one of the following conditions is true:*

(z_1) $d(Tx,Ty) \le a\,d(Sx,Sy)$;

(z_2) $d(Tx,Ty) \le b\big[d(Sx,Tx) + d(Sy,Ty)\big]$;

(z_3) $d(Tx,Ty) \le c\big[d(Sx,Ty) + d(Sy,Tx)\big]$.

If the range of S contains the range of T and $S(X)$ is a complete subspace of X, then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

In both cases, the iteration $\{Sx_n\}$ defined by (2.4.72) converges to the unique (coincidence) common fixed point x^* of S and T, for any $x_0 \in X$, with the following error estimate:

$$d(Sx_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta} d(Sx_n, Sx_{n-1}), \quad n = 0, 1, 2, \ldots; i = 1, 2, \ldots,$$

where $\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$.

The convergence rate of the iteration $\{Sx_n\}$ is given by

$$d(Sx_n, x^*) \leq \delta \cdot d(Sx_{n-1}, x^*), \quad n = 1, 2, \ldots.$$

Remark 2.4.3. It is important to note that all our results established here are important from a computational point of view, due to the fact that they offer a method for computing the common fixed points (the coincidence points, respectively).

Moreover, for the iterative method thus obtained, we have *a priori* and *a posteriori* error estimates, both contained in the unified estimates of the form (2.4.71). Note that in (2.2.19) and (2.4.70) we can have $\delta = 0$, provided that in this case we also have $L = 0$, which ensures that Theorems 2.2.1 and 2.4.3 also include the Banach contraction mapping principle.

Several other fixed point results can be obtained as particular cases of our main results in this section (see Refs. [85, 149]).

For common fixed point theorems of almost contractive mappings in cone metric spaces, see Ref. [67].

For other developments concerning common fixed point theorems or coincidence theorems for almost contractive type mappings, see Refs. [13, 46, 123, 137, 146, 166, 185, 196, 204, 222–224, 232, 253, 255, 256, 258, 268, 276], etc.

2.5. Almost Contractive type Mappings on Product Spaces

Banach's contraction principle (Theorem 2.1.1) for a self mapping $T : X \to X$ has been generalized by Prešić [218] to the case of self mappings defined on a product space, $f : X^k \to X$. This generalization has a direct connection with a dynamic field of research devoted today to the study of nonlinear difference equations, with applications in economics, biology, ecology, genetics, psychology, sociology, probability theory and others (see for example Refs. [81, 86, 108, 127, 164, 165, 167, 193, 234, 264, 265] and others). Some examples of such equations (see Refs. [234, 264] and the papers referred to therein) are:

- the generalized Beddington-Holt stock recruitment model:

$$x_{n+1} = ax_n + \frac{bx_{n-1}}{1+cx_{n-1}+dx_n}, \quad x_0, x_1 > 0, n \in \mathbb{N},$$

  where $a \in (0,1)$, $b \in \mathbb{R}_+^*$ and $c,d \in \mathbb{R}_+$, with $c+d>0$;
- the delay model of a perennial grass:

$$x_{n+1} = ax_n + (b+cx_{n-1})\mathrm{e}^{x_n}, \quad n \in \mathbb{N},$$

  where $a,c \in (0,1)$ and $b \in \mathbb{R}_+$;
- the flour beetle population model:

$$x_{n+3} = ax_{n+2} + bx_n\mathrm{e}^{-(cx_{n+2}+dx_n)}, \quad n \in \mathbb{N},$$

  where $a,b,c,d \geq 0$ and $c+d>0$.

These suggest considering the following kth order nonlinear difference equation, corresponding to a k-step iteration method:

$$x_{n+k} = f(x_n, \ldots, x_{n+k-1}), \quad n \in \mathbb{N}, \tag{2.5.84}$$

with the initial values $x_0, \ldots, x_k \in X$, where (X,d) is a metric space, $k \in \mathbb{N}$, $k \geq 1$ and $f : X^k \to X$.

Equation (2.5.84) can be studied by means of fixed point theory in view of the fact that $x^* \in X$ is a solution of (2.5.84) if and only if x^* is a *fixed point of* f, that is,

$$x^* = f(x^*, \ldots, x^*).$$

Remark 2.5.1. For any operator $f : X^k \to X$, k a positive integer, we can define its *associate operator* $F : X \to X$ (see Refs. [234, 249]) by

$$F(x) = f(x, \ldots, x), \quad x \in X.$$

Obviously, $x \in X$ is a fixed point of $f : X^k \to X$ if and only if it is a fixed point of its associate operator F. This enables the study of f by means of the operator F.

One of the pioneering results in this direction is due to Prešić [218] (see Refs. [279]). We begin by defining the following concept.

Definition 2.5.1. Let (X,d) be a metric space, k a positive integer, $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}_+$, $\sum_{i=1}^{k} \alpha_i = \alpha < 1$ and $f : X^k \to X$ a mapping satisfying

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \leq \sum_{i=1}^{k} \alpha_i d(x_{i-1}, x_i), \tag{2.5.85}$$

for all $x_0, \dots, x_k \in X$. Then f is called a *Prešić operator*.

It is obvious that for $k = 1$ the above definition reduces to the definition of classical Banach contractions.

The result of Prešić [218], enriched with some quantitative information regarding the rate of convergence of the k-step iterative method, is given below.

Theorem 2.5.1. *Let (X,d) be a complete metric space, k a positive integer and $f : X^k \to X$ a Prešić operator. Then*

(a) *f has a unique fixed point x^*;*

(b) *the sequence $\{y_n\}_{n\geq 0}$,*

$$y_{n+1} = f(y_n, y_n, \dots, y_n), \quad n \geq 0, \tag{2.5.86}$$

converges to x^*;

(c) *the sequence $\{x_n\}_{n\geq 0}$ with $x_0, \dots, x_{k-1} \in X$ and*

$$x_n = f(x_{n-k}, x_{n-k+1}, \dots, x_{n-1}), \quad n \geq k, \tag{2.5.87}$$

also converges to x^*, with a rate estimated by

$$d(x_{n+1}, x^*) \leq \alpha d(x_n, x^*) + M \cdot \theta^n, \quad n \geq 0, \tag{2.5.88}$$

where $M > 0$ and $\theta \in (0,1)$ are constants.

As already said, Prešić's result is a generalization of the contraction Banach principle (Theorem 2.1.1), by considering contractions defined on product spaces. It was then natural to search for similar Prešić type extensions also for other classes of generalized contractions. This has been done for Prešić-Rus operators in Ref. [233] (see also Ref. [234]), for Ćirić-Prešić operators in Ref. [101], for Prešić-Kannan operators in Ref. [201], for almost Prešić operators in Ref. [203] and so on. The study of Prešić type results has been very dynamic lately, as the great number of very recent papers on the topic shows (see references in Ref. [76] for a few of them).

In the following we shall briefly present only the results strictly related to almost contractions, i.e., those concerning almost Prešić operators. The name of this class suggests a Prešić type extension starting from the class of almost contractions. Actually only a subclass is referred to, namely that of *strict almost contractions*, by this meaning those almost contractions, as defined in the first section of this chapter, which satisfy the additional condition which ensures the uniqueness of the fixed point.

Definition 2.5.2. Let (X,d) be a metric space. An operator $f : X \to X$ is called a *strict almost contraction* if it satisfies both conditions

$$d(f(x), f(y)) \leq \delta\, d(x,y) + L d(y, f(x)), \quad \text{for any } x, y \in X, \tag{2.5.89}$$

and

$$d(f(x), f(y)) \leq \delta_u d(x,y) + L_u d(x, f(x)), \quad \text{for any } x, y \in X, \tag{2.5.90}$$

with some real constants $\delta \in [0,1)$, $L \geq 0$ and $\delta_u \in [0,1)$, $L_u \geq 0$, respectively.

Having in view Definition 2.5.2, we can restate (part of) Theorem 2.2.2 as follows (see also Ref. [197]):

Theorem 2.5.2. *Let (X,d) be a complete metric space and $f : X \to X$ a strict almost contraction with constants $\delta \in [0,1)$, $L \geq 0$ and $\delta_u \in [0,1)$, $L_u \geq 0$, respectively. Then f has a unique fixed point, say x^*, that can be approximated by means of the Picard iteration $\{x_n\}_{n\geq 0}$ of f, starting from any $x_0 \in X$.*

Remark 2.5.2. An equivalent definition of strict almost contractions, first introduced in Ref. [47], is studied in Ref. [199], where it is shown that an operator $f : X \to X$ is a strict almost contraction if and only if there exist two constants $\delta_B \in [0,1)$ and $L_B \geq 0$ such that

$$\begin{aligned} d(f(x), f(y)) \leq{}& \delta_B d(x,y) \\ &+ L_B \min\{d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}, \end{aligned} \tag{2.5.91}$$

for any $x, y \in X$.

Having in view this equivalent definition of strict almost contractions, we introduce:

Definition 2.5.3. Let (X,d) be a metric space and k a positive integer. An operator $f : X^k \to X$ for which there exist some real constants $\delta_1, \ldots, \delta_k \in \mathbb{R}_+$ with $\sum_{i=1}^{k} \delta_i = \delta < 1$ and $L \geq 0$ such that

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \leq \sum_{i=1}^{k} \delta_i d(x_{i-1}, x_i) + \overline{M}(x_0, x_k), \tag{2.5.92}$$

where

$$\begin{aligned} \overline{M}(x_0,x_k) \;=\;& L \min\{d(x_0, f(x_0, \ldots, x_0)), d(x_k, f(x_k, \ldots, x_k)), \\ & d(x_0, f(x_k, \ldots, x_k)), d(x_k, f(x_0, \ldots, x_0)), d(x_k, f(x_0, x_1, \ldots, x_{k-1}))\}, \end{aligned}$$

for any $x_0,\dots,x_k \in X$, is called an *almost Prešić operator*.

It is easy to check that for $k=1$ the terms $d(x_k, f(x_0,\dots,x_0))$ and $d(x_k, f(x_0,x_1,\dots,x_{k-1}))$ actually coincide, and Definition 2.5.3 reduces to the equivalent definition of strict almost contractions mentioned above. Also for $L=0$, from the above condition (2.5.92), we obtain condition (2.5.85) which defines Prešić operators.

Remark 2.5.3. Considering f as in Definition 2.5.3 and its associate operator F, for any $x_0,\dots,x_k \in X$, we have

$$\begin{aligned}
\overline{M}(x_0,x_k) &= L\min\{d(x_0,f(x_0,\dots,x_0)), d(x_k,f(x_k,\dots,x_k)),\\
&\quad d(x_0,f(x_k,\dots,x_k)), d(x_k,f(x_0,\dots,x_0)), d(x_k,f(x_0,x_1,\dots,x_{k-1}))\}\\
&\le L\min\{d(x_0,f(x_0,\dots,x_0)), d(x_k,f(x_k,\dots,x_k)), d(x_0,f(x_k,\dots,x_k)),\\
&\quad d(x_k,f(x_0,\dots,x_0))\}\\
&= L\min\{d(x_0,F(x_0)), d(x_k,F(x_k)), d(x_0,F(x_k)), d(x_k,F(x_0))\}.
\end{aligned}$$

In order to prove our main result we shall also need the following lemmas:

Lemma 2.5.1 [218]. *Let $k\in\mathbb{N}, k\neq 0$ and $\alpha_1,\alpha_2,\dots,\alpha_k \in \mathbb{R}_+$ such that $\sum_{i=1}^{k}\alpha_i=\alpha<1$. If $\{\Delta_n\}_{n\ge 1}$ is a sequence of positive numbers satisfying*

$$\Delta_{n+k} \le \alpha_1\Delta_n + \alpha_2\Delta_{n+1} + \ldots + \alpha_k\Delta_{n+k-1}, \quad n\ge 1, \tag{2.5.93}$$

then there exist $L>0$ and $\theta\in(0,1)$ such that $\Delta_n \le L\cdot\theta^n$ for all $n\ge 1$.

Lemma 2.5.2 [52]. *Let $\{a_n\}_{n\ge 0}$, $\{b_n\}_{n\ge 0}$ be two sequences of positive real numbers and $q\in(0,1)$ such that $a_{n+1}\le qa_n+b_n, n\ge 0$ and $b_n\to 0$ as $n\to\infty$. Then $\lim\limits_{n\to\infty} a_n = 0$.*

In the following we shall prove the convergence of the Prešić type method constructed by means of almost Prešić operators, also providing the rate of convergence for this iterative procedure.

Theorem 2.5.3. *Let (X,d) be a complete metric space, k a positive integer and $f: X^k\to X$ an almost Prešić operator with constants $\delta_1,\dots,\delta_k\in\mathbb{R}_+$, $\sum_{i=1}^{k}\delta_i=\delta<1$ and $L\ge 0$. Then*

(a) *f has a unique fixed point, say $x^*\in X$;*

(b) *the sequence* $\{y_n\}_{n\geq 0}$ *defined by*

$$y_n = f(y_{n-1}, \ldots, y_{n-1}), \quad n \geq 1,$$

converges to x^* *for any starting point* $y_0 \in X$*;*

(c) *the sequence* $\{x_n\}_{n\geq 0}$ *defined by* $x_0, \ldots, x_{k-1} \in X$ *and (2.5.84) also converges to* x^**, with a rate estimated by*

$$d(x_n, x^*) \leq E_{n-k} + \delta d(x_{n-1}, x^*), \quad n \geq 0, \tag{2.5.94}$$

where

$$E_{n-k} := \delta_1 d(x_{n-k}, x_{n-k+1}) + (\delta_1 + \delta_2) d(x_{n-k+1}, x_{n-k+2}) + \cdots + (\delta_1 + \cdots + \delta_{k-1}) d(x_{n-2}, x_{n-1}). \tag{2.5.95}$$

Proof. (a) and (b) We consider the associate operator $F : X \to X$ defined by $F(x) = f(x, \ldots, x)$, $x \in X$. For any $x, y \in X$ we have that:

$$\begin{aligned} d(F(x), F(y)) &= d(f(x, \ldots, x), f(y, \ldots, y)) \\ &\leq d(f(x, \ldots, x), f(x, \ldots, x, y)) + \cdots + d(f(x, y, \ldots, y), f(y, \ldots, y)). \end{aligned}$$

By (2.5.92) and Remark 2.5.3, this implies:

$$\begin{aligned} d(F(x), F(y)) &\leq \delta_k d(x, y) + L \min\{d(x, F(x)), d(y, F(y)), d(x, F(y)), d(y, F(x)) \\ &+ \delta_{k-1} d(x, y) + L \min\{d(x, F(x)), d(y, F(y)), d(x, F(y)), d(y, F(x)) \\ &+ \cdots + \\ &+ \delta_1 d(x, y) + L \min\{d(x, F(x)), d(y, F(y)), d(x, F(y)), d(y, F(x)), \end{aligned}$$

which is equivalent to

$$d(F(x), F(y)) \leq \delta d(x, y) + + kL \min\{d(x, F(x)), d(y, F(y)), d(x, F(y)), d(y, F(x)),$$

that is, F satisfies condition (2.5.91) above, with constants $\delta \in [0, 1)$ and $kL \geq 0$, so by Theorem 2.5.2 and Remark 2.5.2 above it has a unique fixed point, say $x^* \in X$, that can be obtained as the limit of the successive approximations of F starting from any $x \in X$.

Having in view the definition of F and considering the sequence of successive approximations of F, $\{y_n\}_{n\geq 0}$ defined by

$$y_n = F(y_{n-1}) = f(y_{n-1}, y_{n-1}, \ldots, y_{n-1}), \quad n \geq 1,$$

this leads exactly to conclusions (a) and (b).

(c) Now let us prove that the k-step iterative method $\{x_n\}_{n\geq 0}$ given by (2.5.84) converges to x^* as well.

Let $x_0,\dots,x_{k-1}\in X$ and $x_n=f(x_{n-k},\dots,x_{n-1})$, $n\geq k$. Then

$$\begin{aligned} d(x_k,x^*) &= d(f(x_0,\dots,x_{k-1}),f(x^*,\dots,x^*)) \\ &\leq d(f(x_0,\dots,x_{k-1}),f(x_1,\dots,x_{k-1},x^*)) \\ &\quad +\cdots+d(f(x_{k-1},x^*,\dots,x^*),f(x^*,\dots,x^*)). \end{aligned} \tag{2.5.96}$$

When applying (2.5.92) and Remark 2.5.3 for each term of the sum on the right-hand side of (2.5.96), we get

$$\begin{aligned} &d(f(x_0,\dots,x_{k-1}),f(x_1,\dots,x_{k-1},x^*)) \\ &\leq \delta_1 d(x_0,x_1)+\cdots+\delta_{k-1}d(x_{k-2},x_{k-1})+\delta_k d(x_{k-1},x^*) \\ &+L\min\{d(x_0,F(x_0)),d(x^*,F(x^*)),d(x_0,F(x^*)),d(x^*,F(x_0))\} \end{aligned}$$

and so on,

$$\begin{aligned} &d(f(x_{k-2},x_{k-1},x^*,\dots,x^*),f(x_{k-1},x^*,\dots,x^*)) \\ &\leq \delta_1 d(x_{k-2},x_{k-1})+\delta_2 d(x_{k-1},x^*)+ \\ &+L\min\{d(x_{k-2},F(x_{k-2})),d(x^*,F(x^*)),d(x_{k-2},F(x^*)),d(x^*,F(x_{k-2}))\}, \end{aligned}$$

respectively

$$\begin{aligned} &d(f(x_{k-1},x^*,\dots,x^*),f(x^*,\dots,x^*)) \leq \delta_1 d(x_{k-1},x^*) \\ &\quad +L\min\{d(x_{k-1},F(x_{k-1})),d(x^*,F(x^*)),d(x_{k-1},F(x^*)),d(x^*,F(x_{k-1}))\}. \end{aligned} \tag{2.5.97}$$

As $d(x^*,F(x^*))=0$, (2.5.96) finally leads to

$$\begin{aligned} d(x_k,x^*) &\leq \delta_1 d(x_0,x_1)+(\delta_1+\delta_2)d(x_1,x_2)+\cdots+ \\ &\quad +(\delta_1+\cdots+\delta_{k-1})d(x_{k-2},x_{k-1})+\delta d(x_{k-1},x^*). \end{aligned} \tag{2.5.98}$$

Since k is a fixed positive integer, so are the coefficients δ_1, $\delta_1+\delta_2$, $\dots$, $\delta_1+\cdots+\delta_{k-1}$, δ. Therefore we may denote

$$E_0:=\delta_1 d(x_0,x_1)+(\delta_1+\delta_2)d(x_1,x_2)+\cdots+(\delta_1+\cdots+\delta_{k-1})d(x_{k-2},x_{k-1}),$$

so (2.5.98) can be written as

$$d(x_k,x^*)\leq E_0+\delta d(x_{k-1},x^*). \tag{2.5.99}$$

Similarly, we get

$$\begin{aligned} d(x_{k+1},x^*) &\leq \delta_1 d(x_1,x_2)+(\delta_1+\delta_2)d(x_2,x_3)+\cdots+ \\ &\quad +(\delta_1+\cdots+\delta_{k-1})d(x_{k-1},x_k)+\delta d(x_k,x^*). \end{aligned} \tag{2.5.100}$$

Denoting

$$E_1:=\delta_1 d(x_1,x_2)+(\delta_1+\delta_2)d(x_2,x_3)+\cdots+(\delta_1+\cdots+\delta_{k-1})d(x_{k-1},x_k),$$

inequality (2.5.100) can be written as

$$d(x_{k+1},x^*) \le E_1 + \delta d(x_k,x^*). \tag{2.5.101}$$

In this manner we obtain, for $n \ge k$, that

$$d(x_n,x^*) \le \delta_1 d(x_{n-k},x_{n-k+1}) + (\delta_1+\delta_2) d(x_{n-k+1},x_{n-k+2}) + \cdots + (\delta_1+\cdots+\delta_{k-1}) d(x_{n-2},x_{n-1}) + \delta d(x_{n-1},x^*).$$

Denoting

$$E_{n-k} := \delta_1 d(x_{n-k},x_{n-k+1}) + (\delta_1+\delta_2) d(x_{n-k+1},x_{n-k+2}) + \cdots + (\delta_1+\cdots+\delta_{k-1}) d(x_{n-2},x_{n-1}), \tag{2.5.102}$$

inequality (2.5.98) becomes

$$d(x_n,x^*) \le E_{n-k} + \delta d(x_{n-1},x^*), \text{ for } n \ge k. \tag{2.5.103}$$

In order to apply Lemma 2.5.2, we still have to prove that the sequence $\{E_n\}_{n\ge 0}$ given by

$$E_n = \delta_1 d(x_n,x_{n+1}) + (\delta_1+\delta_2) d(x_{n+1},x_{n+2}) + \cdots + + (\delta_1+\cdots+\delta_{k-1}) d(x_{n+k-2},x_{n+k-1}), \quad n \ge 0,$$

converges to 0 as $n \to \infty$.

For $n \ge k$, we have

$$d(x_n,x_{n+1}) = d(f(x_{n-k},\ldots,x_{n-1}), f(x_{n-k+1},\ldots,x_n)). \tag{2.5.104}$$

By (2.5.92) this yields

$$d(x_n,x_{n+1}) \le \delta_1 d(x_{n-k},x_{n-k+1}) + \cdots + \delta_k d(x_{n-1},x_n) + + L\min\{d(x_{n-k},F(x_{n-k})), d(x_n,F(x_n)), d(x_{n-k},F(x_n)), d(x_n,F(x_{n-k})), d(x_n, f(x_{n-k},\ldots,x_{n-1}))\}. \tag{2.5.105}$$

As $d(x_n, f(x_{n-k},\ldots,x_{n-1})) = 0$, (2.5.104) finally leads to

$$d(x_n,x_{n+1}) \le \delta_1 d(x_{n-k},x_{n-k+1}) + \cdots + \delta_k d(x_{n-1},x_n), \text{ for } n \ge k. \tag{2.5.106}$$

According to Lemma 2.5.1, this implies the existence of $\theta \in (0,1)$ and $K \ge 0$ such that

$$d(x_n,x_{n+1}) \le K\theta^{n+k}, n \ge 0.$$

Since k is fixed, it is evident that the sequence $\{E_n\}_{n\ge 0}$ converges to 0 as $n \to \infty$.

Denoting $\overline{E}_n := E_{n-k}$, (2.5.103) is written as:

$$d(x_n,x^*) \leq \overline{E}_n + \delta d(x_{n-1},x^*). \tag{2.5.107}$$

Now taking $a_n = d(x_n,x^*)$, $n \geq k$ and $b_n = \overline{E}_n$, $n \geq k$ in Lemma 2.5.2, by (2.5.107) it follows that $d(x_n,x^*) \to 0$ as $n \to \infty$, that is, the multistep iterative method $\{x_n\}_{n\geq 0}$ converges to x^*, the unique fixed point of f. □

Remark 2.5.4. Note that for $L = 0$, from Theorem 2.5.3, we get the result due to Prešić [218], while for $k = 1$, Theorem 2.2.2 for strict almost contractions in metric spaces is obtained. For $k = 1$ and $L = 0$, Theorem 2.5.3 reduces to the contraction mapping principle (Theorem 2.1.1).

Other results and remarks concerning almost Presić operators can be found in Ref. [203].

In Section 2.4 of this chapter the existence of coincindence points and common fixed points for single-valued almost contractions was discussed. In the following we shall present some common fixed point results for pairs of mappings where at least one of them is an almost Prešić operator. These were introduced in Ref. [204].

We shall start with the case where only one of the mappings is defined on a product space. In this respect the following extensions of classical notions and results, given in Section 2.4 above (see also Ref. [204]), have to be considered:

Definition 2.5.4. Let X be a nonempty set, k a positive integer and $f : X^k \to X$, $g : X \to X$ two operators.

An element $p \in X$ is called a *coincidence point* of f and g if it is a coincidence point of F and g, where F is defined by $F(x) = f(x,x,\ldots,x)$.

Similarly, $s \in X$ is a *coincidence value* of f and g if it is a coincidence value of F and g.

An element $p \in X$ is a *common fixed point* of f and g if it is a common fixed point of F and g.

Definition 2.5.5. Let X be a nonempty set, k a positive integer and $f : X^k \to X$, $g : X \to X$. The operators f and g are said to be *weakly compatible* if F and g are weakly compatible.

The following result is a generalization of Proposition 1.4 in Ref. [1], included in the previous section as Proposition 2.4.1. For its proof see Ref. [204].

Lemma 2.5.3. *Let X be a nonempty set, k a positive integer and $f : X^k \to X$, $g : X \to X$*

two weakly compatible operators. If f and g have a unique coincidence value $x^* = f(p,\dots,p) = g(p)$, then x^* is the unique common fixed point of f and g.

Definition 2.5.2 can be extended as follows:

Definition 2.5.6. Let (X,d) be a metric space, k a positive integer, $\delta_1, \dots, \delta_k \in \mathbb{R}_+$, with $\sum_{i=1}^{k} \delta_i = \delta < 1$ and $L \geq 0$ constants and $f : X^k \to X$, $g : X \to X$ two operators satisfying:

$$d(f(x_0,\dots,x_{k-1}), f(x_1,\dots,x_k)) \leq \sum_{i=1}^{k} \delta_i d(g(x_{i-1}), g(x_i)) + \overline{M_g}(x_0,x_k), \qquad (2.5.108)$$

for any $x_0, x_1, \dots, x_k \in X$, where

$$\begin{aligned} \overline{M_g}(x_0,x_k) &= L\min\{d(g(x_0), f(x_0,\dots,x_0)), d(g(x_k)), f(x_k,\dots,x_k), \\ &\quad d(g(x_k)), f(x_0,\dots,x_0)), d(g(x_0), f(x_k,\dots,x_k)), \\ &\quad d(g(x_k), f(x_0,\dots,x_{k-1}))\}. \end{aligned}$$

Then f is said to be an *almost Prešić operator w. r. t. g*.

Remark 2.5.5. For any $x_0, x_1, \dots, x_k \in X$ we have that

$$\begin{aligned} \overline{M_g}(x_0,x_k) &\leq L\min\{d(g(x_0), f(x_0,\dots,x_0)), d(g(x_k)), f(x_k,\dots,x_k), \\ &\quad d(g(x_k)), f(x_0,\dots,x_0)), d(g(x_0), f(x_k,\dots,x_k))\}, \end{aligned}$$

that is,

$$\begin{aligned} \overline{M_g}(x_0,x_k) &\leq L\min\{d(g(x_0), F(x_0)), d(g(x_k)), F(x_k)), \\ &\quad d(g(x_k)), F(x_0)), d(g(x_0), F(x_k))\}. \end{aligned}$$

Remark 2.5.6. Considering Remarks 2.5.2 and 2.5.5 above, it is easy to see that for $k = 1$ Definition 2.5.6 reduces to the definition of strict almost contractions. Besides, for $g = 1_X$, f is an almost Prešić operator – see Ref. [197] for more results.

A general common fixed point result regarding almost Prešić operators is presented below:

Theorem 2.5.4. *Let (X,d) be a metric space, k a positive integer and $f : X^k \to X$, $g : X \to X$ two operators such that:*

(i) *f is an almost Prešić operator with respect to g;*

(ii) *there exists a complete subspace* $Y \subseteq X$ *such that* $f(X^k) \subseteq Y \subseteq g(X)$.

Then

(a) f *and* g *have a unique coincidence value, say* $x^* \in X$*;*

(b) *the sequence* $\{g(z_n)\}_{n\geq 0}$ *defined by* $z_0 \in X$ *and*

$$g(z_n) = f(z_{n-1}, \dots, z_{n-1}), \quad n \geq 1, \tag{2.5.109}$$

converges to x^**;*

(c) *the sequence* $\{g(x_n)\}_{n\geq 0}$ *defined by* $x_0, \dots, x_{k-1} \in X$ *and*

$$g(x_n) = f(x_{n-k}, \dots, x_{n-1}), \quad n \geq k, \tag{2.5.110}$$

converges to x^* *as well, with a rate estimated by*

$$d(g(x_n), x^*) \leq E_{n-k} + \delta d(g(x_{n-1}), x^*), \tag{2.5.111}$$

where E_{n-k} *is given by (2.5.127);*

(d) *if in addition* f *and* g *are weakly compatible, then* x^* *is their unique common fixed point.*

Proof. (a) and (b) Let $z_0 \in X$. Then $f(z_0, \dots, z_0) \in f(X^k) \subset g(X)$, so there exists $z_1 \in X$ such that

$$f(z_0, \dots, z_0) = g(z_1).$$

Similarly, $f(z_1, \dots, z_1) \in f(X^k) \subset g(X)$, so there exists $z_2 \in X$ such that

$$f(z_1, \dots, z_1) = g(z_2).$$

In this manner we construct the sequence $\{g(z_n)\}_{n\geq 0}$ with $z_0 \in X$ and

$$g(z_n) = f(z_{n-1}, \dots, z_{n-1}), n \geq 1. \tag{2.5.112}$$

Due to the manner $\{g(z_n)\}_{n\geq 0}$ was constructed, it is easy to recognize that

$$\{g(z_n)\}_{n\geq 0} \subseteq f(X^k) \subseteq Y \subseteq g(X). \tag{2.5.113}$$

For $n \geq 1$, we have

$$\begin{aligned} d(g(z_n), g(z_{n+1})) &= d(f(z_{n-1}, \dots, z_{n-1}), f(z_n, \dots, z_n)) \\ &\leq d(f(z_{n-1}, \dots, z_{n-1}), f(z_{n-1}, \dots, z_{n-1}, z_n)) \\ &\quad + \cdots + d(f(z_{n-1}, z_n, \dots, z_n), f(z_n, \dots, z_n)). \end{aligned} \tag{2.5.114}$$

B978-0-12-804295-3.50002-4, 50002

Applying relation (2.5.108) and then Remark 2.5.5 to each of the distances on the right-hand side of (2.5.114), we obtain

$$\begin{aligned} &d(f(z_{n-1},\dots,z_{n-1}),f(z_{n-1},\dots,z_{n-1},z_n)) \\ &\leq \delta_k d(g(z_{n-1}),g(z_n)) + L\min\{d(g(z_{n-1}),F(z_{n-1})),d(g(z_n),F(z_n)), \\ &d(g(z_{n-1}),F(z_n)),d(g(z_n),F(z_{n-1}))\} \end{aligned} \tag{2.5.115}$$

and so on,

$$\begin{aligned} &d(f(z_{n-1},z_n,\dots,z_n),f(z_n,\dots,z_n)) \\ &\leq \delta_1 d(g(z_{n-1}),g(z_n)) + L\min\{d(g(z_{n-1}),F(z_{n-1})),d(g(z_n),F(z_n)), \\ &d(g(z_{n-1}),F(z_n)),d(g(z_n),F(z_{n-1}))\}. \end{aligned} \tag{2.5.116}$$

As $d(g(z_{n-1}),F(z_n)) = d(g(z_{n-1}),f(z_n,\dots,z_n)) = 0$, (2.5.114) finally becomes

$$d(g(z_n),g(z_{n+1})) \leq \delta d(g(z_{n-1}),g(z_n)). \tag{2.5.117}$$

By induction we get that

$$d(g(z_n),g(z_{n+1})) \leq \delta^n d(g(z_0),g(z_1)), \quad n \geq 0.$$

Using the triangle inequality, for $p \geq 1$ we obtain that:

$$d(g(z_n),g(z_{n+p})) \leq \delta^n \frac{1-\delta^p}{1-\delta} d(g(z_0),g(z_1)), \quad n \geq 0, \tag{2.5.118}$$

where $\delta \in [0,1)$.

Letting $n \to \infty$ in (2.5.118), we find that $\{g(z_n)\}_{n\geq 0}$ is a Cauchy sequence included, by (2.5.113), in the complete subspace Y. Consequently, there exists $x^* \in Y$ such that $x^* = \lim_{n\longrightarrow\infty} g(z_n)$ and, since $Y \subset g(X)$, there exists $p \in X$ such that

$$g(p) = x^* = \lim_{n\longrightarrow\infty} g(z_n). \tag{2.5.119}$$

Now we shall prove that $f(p,\dots,p) = x^*$. We have that

$$\begin{aligned} &d(g(z_{n+1}),f(p,\dots,p)) = d(f(z_n,\dots,z_n),f(p,\dots,p)) \\ &\leq d(f(z_n,\dots,z_n),f(z_n,\dots,z_n,p)) + \cdots + d(f(z_n,p,\dots,p),f(p,\dots,p)). \end{aligned} \tag{2.5.120}$$

It is obvious that the minimum among five quantities is less or equal to any of these quantities, which we may conveniently choose.

Thus, when applying (2.5.108) to the distances on the right-hand side of (2.5.120), each time we choose this quantity to be $d(g(p),f(z_n,\dots,z_n))$. In this manner (2.5.120) becomes:

$$d(g(z_{n+1}),f(p,\dots,p)) \leq \delta\, d(g(z_n),g(p)) + kL d(g(p),f(z_n,\dots,z_n)),$$

i.e.,

$$d(g(z_{n+1}), f(p,\dots,p)) \le \delta\, d(g(z_n), x^*) + kL d(x^*, g(z_{n+1})), \quad n \ge 0.$$

Letting $n \longrightarrow \infty$, by (2.5.119) it immediately follows that

$$f(p,\dots,p) = x^* = g(p),$$

that is, x^* is a coincidence value for f and g.

In order to prove its uniqueness, we suppose there would be some $q \in X$ such that

$$f(q,\dots,q) = g(q) \neq x^*.$$

Then

$$\begin{aligned} d(g(p), g(q)) &= d(f(p,\dots,p), f(q,\dots,q)) \\ &\le d(f(p,\dots,p), f(p,\dots,p,q)) + \cdots + d(f(p,q,\dots,q), f(q,\dots,q)). \end{aligned} \quad (2.5.121)$$

We now use a similar reasoning as before, applying (2.5.108) to each of the distances on the right-hand side of inequality (2.5.121). This time we choose the minimum less or equal to $d(g(p), f(p,\dots,p)) = d(g(q), f(q,\dots,q)) = 0$. Thus (2.5.121) becomes:

$$d(g(p), g(q)) \le \delta\, d(g(p), g(q)),$$

or

$$(1-\delta) d(g(p), g(q)) \le 0.$$

As $\delta \in [0,1)$, this implies that $d(g(p), g(q)) = 0$, so x^* is the unique coincidence value of f and g.

(c) Let $x_0,\dots,x_{k-1} \in X$. Then $f(x_0,\dots,x_{k-1}) \in f(X^k) \subset g(X)$, so there exists $x_k \in X$ such that

$$f(x_0,\dots,x_{k-1}) = g(x_k).$$

Similarly, $f(x_1,\dots,x_k) \in f(X^k) \subset g(X)$, so there exists $x_{k+1} \in X$ such that

$$f(x_1,\dots,x_k) = g(x_{k+1}).$$

In this manner we construct the sequence $\{g(x_n)\}_{n\ge 0}$ defined by $x_0,\dots,x_{k-1} \in X$ and

$$g(x_n) = f(x_{n-k},\dots,x_{n-1}), \quad n \ge k.$$

Again we notice that, by construction,

$$\{g(x_n)\}_{n\ge 0} \subset f(x^k) \subset g(X). \quad (2.5.122)$$

B978-0-12-804295-3.50002-4, 50002

We shall prove that $\{g(x_n)\}_{n\geq 0}$ converges to x^* as well. We have that:

$$\begin{aligned} d(g(x_k),x^*) &= d(f(x_0,\dots,x_{k-1}),f(p,\dots,p)) \\ &\leq d(f(x_0,\dots,x_{k-1}),f(x_1,\dots,x_{k-1},p))+\dots+ \\ &\quad +d(f(x_{k-1},p,\dots,p),f(p,\dots,p)). \end{aligned} \tag{2.5.123}$$

As $d(g(p),f(p,\dots,p))=0$, by applying (2.5.108) to each of the distances on the right-hand side of the above inequality (2.5.123), it follows that

$$\begin{aligned} d(g(x_k),x^*) &\leq \delta_1 d(g(x_0),g(x_1))+(\delta_1+\delta_2)d(g(x_1),g(x_2))+\ldots+ \\ &\quad +(\delta_1+\ldots+\delta_{k-1})d(g(x_{k-2}),g(x_{k-1}))+\delta d(g(x_{k-1}),g(p)). \end{aligned} \tag{2.5.124}$$

Since k is a fixed positive integer, the coefficients δ_1, $\delta_1+\delta_2$, $\dots$, $\delta_1+\dots+\delta_{k-1}$, δ are also fixed, so we may denote

$$\begin{aligned} E_0 := \delta_1 d(g(x_0),g(x_1))+(\delta_1+\delta_2)d(g(x_1),g(x_2))+\ldots+ \\ +(\delta_1+\ldots+\delta_{k-1})d(g(x_{k-2}),g(x_{k-1})), \end{aligned}$$

and (2.5.124) can be written as

$$d(g(x_k),x^*) \leq E_0+\delta\, d(g(x_{k-1}),x^*). \tag{2.5.125}$$

In the same manner, we obtain

$$d(g(x_{k+1}),x^*) \leq E_1+\delta\, d(g(x_k),x^*),$$

where

$$\begin{aligned} E_1 := \delta_1 d(g(x_1),g(x_2))+(\delta_1+\delta_2)d(g(x_2),g(x_3))+\ldots+ \\ +(\delta_1+\ldots+\delta_{k-1})d(g(x_{k-1}),g(x_k)). \end{aligned}$$

By induction, for $n\geq k$ we obtain:

$$d(g(x_n),x^*) \leq E_{n-k}+\delta\, d(g(x_{n-1}),x^*), \tag{2.5.126}$$

where

$$\begin{aligned} E_{n-k} := \delta_1 d(g(x_{n-k}),g(x_{n-k+1}))+(\delta_1+\delta_2)d(g(x_{n-k+1}),g(x_{n-k+2}))+ \\ +\ldots+(\delta_1+\ldots+\delta_{k-1})d(g(x_{n-2}),g(x_{n-1})), \quad n\geq k. \end{aligned} \tag{2.5.127}$$

Inequality (2.5.126) leads us to the estimation of the rate of convergence (2.5.111).

In order to apply Lemma 2.5.2, we have to prove that $\{E_n\}_{n\geq 0}$, defined by

$$\begin{aligned} E_n := \delta_1 d(g(x_n),g(x_{n+1}))+(\delta_1+\delta_2)d(g(x_{n+1}),g(x_{n+2})) \\ +\ldots+(\delta_1+\ldots+\delta_{k-1})d(g(x_{n+k-2}),g(x_{n+k-1})), \quad n\geq 0, \end{aligned} \tag{2.5.128}$$

converges to 0. For $n \geq k$ we have that

$$d(g(x_n), g(x_{n+1})) = d(f(x_{n-k}, \ldots, x_{n-1}), f(x_{n-k+1}, \ldots, x_n)),$$

which by (2.5.108) yields

$$\begin{aligned} d(g(x_n), g(x_{n+1})) \quad \leq \quad & \delta_1 d(g(x_{n-k}), g(x_{n-k+1})) + \cdots + \delta_k d(g(x_{n-1}), g(x_n)) \\ & + L \min\{d(g(x_{n-k}), f(x_{n-k}, \ldots, x_{n-k})), d(g(x_n), f(x_n, \ldots, x_n)), \\ & d(g(x_{n-k}), f(x_n, \ldots, x_n)), d(g(x_n), f(x_{n-k}, \ldots, x_{n-k})), \\ & d(g(x_n), f(x_{n-k}, x_{n-k+1} \ldots, x_{n-1}))\}. \end{aligned}$$

Since $d(g(x_n), f(x_{n-k}, \ldots, x_{n-1})) = 0$, it follows that

$$d(g(x_n), g(x_{n+1})) \leq \delta_1 d(g(x_{n-k}), g(x_{n-k+1})) + \cdots + \delta_k d(g(x_{n-1}), g(x_n)).$$

By Lemma 2.5.1 this implies the existence of $\theta \in (0, 1)$ and $K \geq 0$ such that

$$d(g(x_n), g(x_{n+1})) \leq K\theta^n, \quad n \geq k.$$

It is then immediate that the sequence $\{E_n\}_{n \geq 0}$ converges to 0 as $n \to \infty$. Denoting $\overline{E}_n = E_{n-k}$, from (2.5.126) we get:

$$d(g(x_n), x^*) \leq \overline{E}_n + \delta\, d(g(x_{n-1}), x^*). \tag{2.5.129}$$

Now taking $a_n = d(g(x_n), x^*)$, $n \geq k$, and $b_n = \overline{E}_n$, $n \geq k$, by (2.5.129) and Lemma 2.5.2 it follows that

$$d(g(x_n), x^*) \to 0, \text{ as } n \to \infty,$$

so $\{g(x_n)\}_{n \geq 0}$ converges to the unique coincidence value x^* as well.

(d) If f and g are weakly compatible, then by Lemma 2.5.3 their unique coincidence value is actually their unique common fixed point. □

Going further, we can establish common fixed point results for the more general case $f : X^k \to X$ and $g : X^l \to X$, with k and l positive integers. In this respect we have to extend the notions mentioned above (see also Refs. [196, 200, 204]).

Definition 2.5.7. Let X be a metric space, k, l positive integers and $f : X^k \to X$, $g : X^l \to X$ two operators.

An element $p \in X$ is called a *coincidence point* of f and g if it is a coincidence point of F and G, where $F, G : X \to X$ are the associate operators of f and g respectively (see Remark 2.5.1).

An element $s \in X$ is called a *coincidence value* of f and g if it is a coincidence value of F and G.

An element $p \in X$ is called a *common fixed point* of f and g if it is a common fixed

B978-0-12-804295-3.50002-4, 50002

point of F and G.

Definition 2.5.8. Let (X,d) be a metric space, k,l positive integers and $f : X^k \to X$, $g : X^l \to X$. The operators f and g are said to be *weakly compatible* if F and G are weakly compatible.

The following extends Definition 2.5.6:

Definition 2.5.9. Let (X,d) be a metric space, k,l positive integers and $f : X^k \to X$, $g : X^l \to X$ two operators such that f is an almost Prešić operator w.r.t. $G : X \to X$, the associated operator of g. Then f is said to be an *almost Prešić operator w.r.t.* g.

Using this definition one can prove the following extension of Theorem 2.5.4:

Theorem 2.5.5. *Let (X,d) be a metric space, k and l positive integers and $f : X^k \to X$, $g : X^l \to X$ two operators such that:*

(i) *f is an almost Prešić operator with respect to g;*

(ii) *there exists a complete subspace $Y \subseteq X$ such that $f(X^k) \subseteq Y \subseteq G(X)$, where $G : X \to X$ is the associated operator of g.*

Then

(a) *f and g have a unique coincidence value, say x^*, in X;*

(b) *the sequence $\{G(z_n)\}_{n\geq 0}$ defined by $z_0 \in X$ and*

$$G(z_n) = f(z_{n-1},\ldots,z_{n-1}), \quad n \geq 1,$$

converges to x^*;

(c) *the sequence $\{G(x_n)\}_{n\geq 0}$ defined by $x_0,\ldots,x_{k-1} \in X$ and*

$$G(x_n) = f(x_{n-k},\ldots,x_{n-1}), \quad n \geq k,$$

converges to x^* as well, with a rate estimated by

$$d(G(x_n),x^*) \leq \overline{E}_n + \delta d(G(x_{n-1}),x^*),$$

where

$$\overline{E}_n := \delta_1 d(G(x_{n-k}),G(x_{n-k+1})) + (\delta_1+\delta_2)d(G(x_{n-k+1}),G(x_{n-k+2})) + \ldots + (\delta_1+\ldots+\delta_{k-1})d(G(x_{n-2}),G(x_{n-1})), \quad n \geq k;$$

(d) *if in addition f and g are weakly compatible, then x^* is their unique common fixed point.*

B978-0-12-804295-3.50002-4, 50002

For the study of stability of k-step fixed point iterative schemes associated with contractive type mappings defined on product spaces, see Ref. [75]. For some other developments on Prešić operators, see a partial list in Ref. [76].

2.6. Fixed Point Theorems for Single-valued Nonself Almost Contractions

In a natural continuation and completion of the abundant fixed point theory for self mappings, produced in the last five decades, it was also an important and challenging research topic to obtain fixed point theorems for *nonself mappings*.

In 1972, Assad and Kirk [38] extended the Banach contraction mapping principle to nonself multivalued contraction mappings $T : K \to \mathscr{P}(X)$ in the case where (X,d) is a convex metric space in the sense of Menger and K is a nonempty closed subset of X such that T maps ∂K (the boundary of K) into K. In 1976, by using an alternative and weaker condition, i.e., T is metrically inward, Caristi [82] has shown that any nonself single-valued contraction has a fixed point.

Next, in 1978, Rhoades [228] proved a fixed point result in Banach spaces for single-valued nonself mapping satisfying the following contraction condition:

$$d(Tx,Ty) \le \lambda \max\left\{\frac{d(x,y)}{2}, d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{1+2\lambda}\right\}, \tag{2.6.130}$$

for all $x,y \in K$, where $0 < \lambda < 1$.

Rhoades' result [228] has been slightly extended by Ćirić [98]. Note that although the class of mappings satisfying (2.6.130) is large enough to include some discontinuous mappings, it does not include contraction mappings satisfying (2.1.1) for $\frac{1}{2} \le \lambda < 1$.

A more general result, which also solved a very difficult problem that was open for more than 20 years, has been obtained by Ćirić [99], who considered instead of (2.6.130) the quasi-contraction condition that he previously introduced and studied in Ref. [96]:

$$d(Tx,Ty) \le \lambda \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}, \tag{2.6.131}$$

for all $x,y \in K$, where $0 < \lambda < 1$. More recently, Ćirić, Ume, Khan and Pathak [100] have considered a contraction condition which is more general than (2.6.130) and (2.6.131), i.e.,

$$d(Tx,Ty) \le \max\{\varphi(d(x,y)), \varphi(d(x,Tx)), \varphi(d(y,Ty)), \varphi(d(x,Ty)), \varphi(d(y,Tx))\}, \tag{2.6.132}$$

for all $x,y \in K$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a certain comparison function.

For some other fixed point results for nonself mappings, see also Refs. [33–37, 71] and Problem 5 in Ref. [238].

As shown in Section 2.2, quasi-contractions and almost contractions are independent classes of mappings as the latter have a unique fixed point, while the former do not.

Starting from these facts, the aim of the present section is to obtain fixed point theorems for nonself almost contractions. Thus, we shall give a solution to Problem 5 in Ref. [238] in the case of almost contractions. The material is adapted from Ref. [74].

In order to do so, we first present some aspects and results related to self almost contractions and then we extend them to nonself almost contractions.

Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a nonself mapping. If $x \in K$ is such that $Tx \notin K$, then we can always choose an $y \in \partial K$ (the boundary of K) such that $y = (1-\lambda)x + \lambda Tx (0 < \lambda < 1)$, which actually expresses the fact that

$$d(x,Tx) = d(x,y) + d(y,Tx), \quad \text{for } y \in \partial K, \tag{2.6.133}$$

where we denoted $d(x,y) = \|x-y\|$.

In general, the set Y of points y satisfying condition (2.6.133) above may contain more than one element.

In this context we shall need the following concept.

Definition 2.6.1. Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a nonself mapping. Let $x \in K$ with $Tx \notin K$ and let $y \in \partial K$ be the corresponding elements given by (2.6.133). If, for any such elements x, we have

$$d(y,Ty) \leq d(x,Tx), \tag{2.6.134}$$

for all corresponding $y \in Y$, then we say that T has property (M).

Very recently we found that a condition quite similar to (2.6.134) had been used in Ref. [126] (see also Ref. [83]).

Note also that the nonself mapping T in the next example has property (M).

Example 2.6.1. Let X be the set of real numbers with the usual metric, $K = [0,1]$, and let $T : K \to X$ be defined (see Ref. [100, Remark 1.3]) by $Tx = -0.1$ if $x = 0.9$ and $Tx = \frac{x}{x+1}$ if $x \neq 0.9$.

Then T satisfies condition (2.6.132), T is discontinuous, 0 is the unique fixed point of T and T is continuous at 0, T has property (M) but T does not satisfy the almost contraction condition (2.6.135) below. Indeed, the only $x \in K$ with $Tx \notin K$ is $x = 0.9$ and the corresponding $y \in \partial K$ is $y = 0$. It is now easy to check that (2.6.134) holds.

To prove the last claim take $x \neq 0.9$ and $y = \frac{x}{1+x}$ in (2.6.135) to get, for any $x > 0$,

$$\frac{1+x}{1+2x} \leq \delta < 1, \quad x > 0.$$

If we now let $x \to 0$ in the previous double inequality, we get the contradiction

$$1 \leq \delta < 1.$$

We now state and prove our main result in this section, which is taken from Ref. [74].

Theorem 2.6.1. *Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a nonself almost contraction, that is, a mapping for which there exist two constants $\delta \in [0,1)$ and $L \geq 0$ such that*

$$d(Tx,Ty) \leq \delta \cdot d(x,y) + Ld(y,Tx), \quad \textit{for all } x,y \in K. \tag{2.6.135}$$

If T has property (M) and satisfies Rothe's boundary condition

$$T(\partial K) \subset K, \tag{2.6.136}$$

then T has a fixed point in K.

Proof. If $T(K) \subset K$, then T is actually a self mapping on the closed set K and the conclusion follows by Theorem 2.2.1 for $X = K$. Therefore, we consider the case $T(K) \not\subset K$. Let $x_0 \in \partial K$. By (2.6.136) we know that $Tx_0 \in K$. Denote $x_1 = Tx_0$. Now, if $Tx_1 \in K$, set $x_2 = Tx_1$. If $Tx_1 \notin K$, we can choose an element x_2 on the segment $[x_1, Tx_1]$ which also belongs to ∂K, that is,

$$x_2 = (1-\lambda)x_1 + \lambda Tx_1 \quad (0 < \lambda < 1).$$

Continuing in this way we obtain a sequence $\{x_n\}$ whose terms satisfy one of the following properties:

(i) $x_n = Tx_{n-1}$, if $Tx_{n-1} \in K$;

(ii) $x_n = (1-\lambda)x_{n-1} + \lambda Tx_{n-1} \in \partial K \, (0 < \lambda < 1)$, if $Tx_{n-1} \notin K$.

To simplify the argumentation in the proof, let us denote

$$P = \{x_k \in \{x_n\} : x_k = Tx_{k-1}\}$$

and

$$Q = \{x_k \in \{x_n\} : x_k \neq Tx_{k-1}\}.$$

Note that $\{x_n\} \subset K$ and that, if $x_k \in Q$, then both x_{k-1} and x_{k+1} belong to the set P. Moreover, by virtue of (2.6.136), we cannot have two consecutive terms of $\{x_n\}$ in the set Q (but we can have two consecutive terms of $\{x_n\}$ in the set P) .

We claim that $\{x_n\}$ is a Cauchy sequence. To prove this, we must discuss three different cases:

CASE I. $x_n, x_{n+1} \in P$;
CASE II. $x_n \in P,\ x_{n+1} \in Q$;
CASE III. $x_n \in Q,\ x_{n+1} \in P$.

CASE I. $x_n, x_{n+1} \in P$.

In this case we have $x_n = Tx_{n-1}$, $x_{n+1} = Tx_n$ and by (2.6.135), we get

$$d(x_{n+1},x_n) = d(Tx_n,Tx_{n-1}) \leq \delta d(x_n,x_{n-1}) + Ld(x_n,Tx_{n-1}),$$

that is,

$$d(x_{n+1},x_n) \leq \delta d(x_n,x_{n-1}), \tag{2.6.137}$$

since $x_n = Tx_{n-1}$.

CASE II. $x_n \in P,\ x_{n+1} \in Q$.

In this case we have $x_n = Tx_{n-1}$ but $x_{n+1} \neq Tx_n$ and

$$d(x_n,x_{n+1}) + d(x_{n+1},Tx_n) = d(x_n,Tx_n).$$

Hence

$$d(x_n,x_{n+1}) \leq d(x_n,Tx_n) = d(Tx_{n-1},Tx_n)$$

and so by using (2.6.135), we get

$$d(x_n,x_{n+1}) \leq \delta d(x_n,x_{n-1}) + Ld(x_n,Tx_{n-1}) = \delta d(x_n,x_{n-1}),$$

which yields again inequality (2.6.137).

CASE III. $x_n \in Q,\ x_{n+1} \in P$.

In this situation, we have $x_{n-1} \in P$. Having in view that T has property (M), it follows that

$$d(x_n,x_{n+1}) = d(x_n,Tx_n) \leq d(x_{n-1},Tx_{n-1}).$$

Since $x_{n-1} \in P$ we have $x_{n-1} = Tx_{n-2}$ and by (2.6.135) we get

$$d(Tx_{n-2},Tx_{n-1}) \leq \delta d(x_{n-2},x_{n-1}) + Ld(x_{n-1},Tx_{n-2}) = \delta d(x_{n-2},x_{n-1}),$$

which shows that

$$d(x_n,x_{n+1}) \leq \delta d(x_{n-2},x_{n-1}). \tag{2.6.138}$$

Therefore, by summarizing all three cases and using (2.6.137) and (2.6.138), it follows that the sequence $\{x_n\}$ satisfies the inequality

$$d(x_n,x_{n+1}) \leq \delta \max\{d(x_{n-2},x_{n-1}),\, d(x_{n-1},x_n)\}, \quad \text{for all } n \geq 2. \tag{2.6.139}$$

Now, by induction for $n \geq 2$, from (2.6.139) one obtains

$$d(x_n, x_{n+1}) \leq \delta^{[n/2]} \max\{d(x_0, x_1), d(x_1, x_2)\},$$

where $[n/2]$ denotes the greatest integer not exceeding $n/2$.

Further, for $m > n > N$,

$$d(x_n, x_m) \leq \sum_{i=N}^{\infty} d(x_i, x_{i-1}) \leq 2\frac{\delta^{[N/2]}}{1-\delta} \max\{d(x_0, x_1), d(x_1, x_2)\},$$

which shows that $\{x_n\}$ is a Cauchy sequence.

Since $\{x_n\} \subset K$ and K is closed, $\{x_n\}$ converges to some point in K.

Denote

$$x^* = \lim_{n\to\infty} x_n, \tag{2.6.140}$$

and let $\{x_{n_k}\} \subset P$ be an infinite subsequence of $\{x_n\}$ (such a subsequence always exists) that we denote in the following for simplicity by $\{x_n\}$ too.

Then

$$d(x^*, Tx^*) \leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) = d(x_{n+1}, x^*) + d(Tx_n, Tx^*).$$

By (2.6.135), we have

$$d(Tx_n, Tx^*) \leq \delta\, d(x_n, x^*) + L d(x^*, Tx_n),$$

and hence

$$d(x^*, Tx^*) \leq (1+L) d(x^*, x_{n+1}) + \delta \cdot d(x_n, x^*), \quad \text{for all } n \geq 0. \tag{2.6.141}$$

Letting $n \to \infty$ in (2.6.141) we obtain

$$d(x^*, Tx^*) = 0,$$

which shows that x^* is a fixed point of T. □

Remark 2.6.1. Note that although T satisfying (2.6.135) may be discontinuous (see Example 2.6.2), T is also continuous at the fixed point. Indeed, if $\{y_n\}$ is a sequence in K convergent to $x^* = Tx^*$, then by (2.6.135) we have

$$d(Ty_n, x^*) = d(Tx^*, Ty_n) \leq \delta d(x^*, y_n) + L d(y_n, Tx^*),$$

and letting $n \to \infty$ in the previous inequality, we get exactly the continuity of T at the fixed point x^*:

$$d(Ty_n, x^*) \to 0 \text{ as } n \to \infty, \text{ that is, } Ty_n \to x^*.$$

Example 2.6.2. Let X be the set of real numbers with the usual norm, $K = [0, 1]$ be the unit interval and let $T:[0,1]\to\mathbb{R}$ be given by $Tx=\frac{2}{3}x$ for $x\in[0,1/2)$, $T\left(\frac{1}{2}\right)=-1$, and $Tx=\frac{2}{3}x+\frac{1}{3}$, for $x\in(1/2,1]$.

As T has two fixed points, that is, $Fix(T)=\{0,1\}$, it does not satisfy either of Ćirić's conditions (2.6.131) and (2.6.132), or the Banach, Kannan, Chatterjea, Zamfirescu or Ćirić [94] contractive conditions in the corresponding nonself form, but T satisfies the contraction condition (2.6.135).

Indeed, for the cases
(1) $x\in[0,1/2)$, $y\in(1/2,1]$;
(2) $y\in[0,1/2)$, $x\in(1/2,1]$;
(3) $x,y\in[0,1/2)$; and
(4) $x,y\in(1/2,1]$,

we have by Example 1.3.10 in Ref. [197, pp. 28–29] that (2.6.135) is satisfied with $\delta=2/3$ and $L\geq 6$.

We have to cover the remaining four cases:
(5) $x=1/2$, $y\in[0,1/2)$;
(6) $x\in[0,1/2)$, $y=1/2$;
(7) $x=1/2$, $y\in(1/2,1]$; and
(8) $x\in(1/2,1]$, $y=1/2$.

Case (5), $x=1/2$, $y\in[0,1/2)$. In this case, (2.6.135) reduces to

$$\left|-1-\frac{2}{3}y\right|\leq\delta\left|\frac{1}{2}-y\right|+L|y+1|,\quad y\in[0,1/2).$$

Since $\left|-1-\frac{2}{3}y\right|\leq\frac{4}{3}$ and $1\leq|y+1|$, in order to have the previous inequality satisfied, we simply need to take $L\geq\frac{4}{3}$.

Case (6), $x\in[0,1/2)$, $y=1/2$. In this case, (2.6.135) reduces to

$$\left|\frac{2}{3}x+1\right|\leq\delta\left|x-\frac{1}{2}\right|+L\left|\frac{1}{2}-\frac{2}{3}x\right|,\quad x\in[0,1/2).$$

Since $\left|\frac{2}{3}x+1\right|\leq\frac{4}{3}$ and $\left|\frac{1}{2}-\frac{2}{3}x\right|\geq\frac{1}{6}$, to have the previous inequality satisfied, it is enough to take $L\geq 8$.

Case (7), $x=1/2$, $y\in(1/2,1]$. In this case, (2.6.135) reduces to

$$\left|-1-\frac{2}{3}y-\frac{1}{3}\right|\leq\delta\left|\frac{1}{2}-y\right|+L|y+1|,\quad y\in(1/2,1].$$

Since $\left|1+\frac{2}{3}y+\frac{1}{3}\right|\leq 2$ and $|y+1|>\frac{3}{2}$, to have the previous inequality satisfied, it is enough to take $L\geq\frac{4}{3}$.

Case (8), $x\in(1/2,1]$, $y=1/2$. Similarly, we find that (2.6.135) holds with $L\geq 8$ and $0<\delta<1$ arbitrary.

By summarizing all possible cases, we conclude that T satisfies (2.6.135) with $\delta=2/3$ and $L=8$.

As we have shown in Section 2.2, it is possible to force the uniqueness of the fixed point of an almost contraction by imposing an additional contractive condition, quite similar to (2.6.135), as shown by the next theorem.

Theorem 2.6.2. *Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a nonself almost contraction for which there exist $\theta \in (0,1)$ and some $L_1 \geq 0$ such that*

$$d(Tx,Ty) \leq \theta \cdot d(x,y) + L_1 \cdot d(x,Tx), \quad \text{for all } x,y \in K. \tag{2.6.142}$$

If T has property (M) and satisfies Rothe's boundary condition

$$T(\partial K) \subset K,$$

then T has a unique fixed point in K.

Remark 2.6.2. By the considerations in the first part of this section we could immediately obtain various fixed point results as corollaries of Theorems 2.6.1 and 2.6.2, for T satisfying one of the conditions of the type (2.6.130).

REFERENCES

1. Abbas M, Jungck G, Common fixed point results for noncommuting mappings without continuity in cone metric spaces. J. Math. Anal. Appl. 2008;341:416–420.
2. Abbas M, Coincidence points of multivalued f-almost nonexpansive mappings. Fixed Point Theory 2012;13:3–10.
3. Abbas M, Babu GVR, Alemayehu GN, On common fixed points of weakly compatible mappings satisfying 'generalized condition (B)'. Filomat 2011;25:9–19.
4. Abbas M, Damjanović B, Lazović R, Fuzzy common fixed point theorems for generalized contractive mappings. Appl. Math. Lett. 2010;23:1326–1330.
5. Abbas M, Hussain N, Rhoades BE, Coincidence point theorems for multivalued f-weak contraction mappings and applications. Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM 2011;105:261–272.
6. Abbas M, Ilić D, Common fixed points of generalized almost nonexpansive mappings. Filomat 2010;24:11–18.
7. Abbas M, Kim JK, Nazir T, Common fixed point of mappings satisfying almost generalized contractive condition in partially ordered G-metric spaces. J. Comput. Anal. Appl. 2015;19:928–938.
8. Abbas M, Turkoglu D, Fixed point theorem for a generalized contractive fuzzy mapping. J. Intell. Fuzzy Systems 2014;26:33–36.
9. Abbas M, Vetro P, Khan SH, On fixed points of Berinde's contractive mappings in cone metric spaces. Carpathian J. Math. 2010;26:121–133.
10. Abbas M, Ali B, Romaguera S, Coincidence points of generalized multivalued (f,L)-almost F-contraction with applications. J. Nonlinear Sci. Appl. 2015;8:919–934.
11. Acar Ö, Berinde V, Altun I, Fixed point theorems for Ćirić-type strong almost contractions on partial metric spaces. J. Fixed Point Theory Appl. 2012;12:247–259.
12. Acar Ö, Altun I, Durmaz G, A fixed point theorem for new type contractions on weak partial metric spaces. Vietnam J. Math. DOI 10.1007/s10013-014-0112-0.
13. Aghajani A, Radenović S, Roshan JR, Common fixed point results for four mappings satisfying almost generalized (S,T)-contractive condition in partially ordered metric spaces. Appl. Math. Comput. 2012;218:5665–5670.

14. Akinbo G, Mewomo O, Fixed point theorems for a general class of almost contractions in metric spaces. Acta Math. Acad. Paedagog. Nyházi. (N.S.) 2011;27:299–305.
15. Al-Badarneh AA, On bi-shadowing of subclasses of almost contractive type mappings. Canadian J. Pure Appl. Sci. 2015;9:3449–3453.
16. Al-Badarneh AA, Bi-shadowing of some classes of single-valued almost contractions. Appl. Math. Sci. 2015;9:2859–2869.
17. Alber YI, Guerre-Delabriere S, Principle of weakly contractive maps in Hilbert spaces. In: Gohberg I, Lyubich Y, editors, New results in operator theory and its applications, Basel: Birkhäuser; 1997;7–22.
18. Allahyari R, Arab R, Shole Haghighi A, A generalization on weak contractions in partially ordered *b*-metric spaces and its application to quadratic integral equations. J. Inequal. Appl. 2014;2014:Article ID 355.
19. Alghamdi MA, Berinde V, Shahzad N, Fixed points of multivalued nonself almost contractions. J. Appl. Math. 2013;2013:Article ID 621614.
20. Alghamdi MA, Berinde V, Shahzad N, Fixed points of non-self almost contractions. Carpathian J. Math. 2014;30:7–14.
21. Ali MU, Kamran T, Hybrid generalized contractions. Math. Sci. (Springer) 2013;7:5 pp.
22. Altun I, Acar Ö, Fixed point theorems for weak contractions in the sense of Berinde on partial metric spaces. Topology Appl. 2012;159:2642–2648.
23. Altun I, Acar Ö, Multivalued almost contractions in metric space endowed with a graph. Creat. Math. Inform. 2015;24:1–8.
24. Altun I, Durmaz G, Minak G, Romaguera S, Multivalued almost F-contractions on complete metric spaces. Filomat (in press).
25. Altun I, Hancer HA, Minak G, On a general class of weakly Picard operators. Miskolc Math. Notes (in press).
26. Altun I, Minak G, Dağ H, Multivalued F-contractions on complete metric spaces. J. Nonlinear Convex Anal. 2015;16:659–666.
27. Altun I, Olgun M, Minak G, On a new class of multivalued weakly Picard operators on complete metric spaces. Taiwanese J. Math. 2015;19:659–672.
28. Altun I, Sadarangani K, Fixed point theorems for generalized almost contractions in partial metric spaces. Math. Sci. (Springer) 2014;8:6 pp.
29. Amini-Harandi A, Fakhar M, Hajisharifi HR, Hussain N, Some new results on fixed and best proximity points in preordered metric spaces. Fixed Point Theory Appl. 2013;2013:Article ID 263.
30. Ariza-Ruiz D, Convergence and stability of some iterative processes for a class of quasinonexpansive type mappings. J. Nonlinear Sci. Appl. 2012;5:93–103.
31. Assad NA, On a fixed point theorem of Iséki. Tamkang J. Math. 1976;7:19–22.
32. Assad NA, On a fixed point theorem of Kannan in Banach spaces. Tamkang J. Math. 1976;7:91–94.
33. Assad NA, On some nonself nonlinear contractions. Math. Japon. 1988;33:17–26.
34. Assad NA, On some nonself mappings in Banach spaces. Math. Japon. 1988;33:501–515.
35. Assad NA, Approximation for fixed points of multivalued contractive mappings. Math. Nachr. 1988;139:207–213.
36. Assad NA, A fixed point theorem in Banach space. Publ. Inst. Math. (Beograd) (N.S.) 1990;47(61):137–140.
37. Assad NA, A fixed point theorem for some non-self-mappings. Tamkang J. Math. 1990;21:387–393.
38. Assad NA, Kirk WA, Fixed point theorems for set-valued mappings of contractive type. Pacific J. Math. 1972;43:553–562.
39. Assad NA, Sessa S, Common fixed points for nonself compatible maps on compacta. Southeast Asian Bull. Math. 1992;16:91–95.
40. Aydi H, Felhi A, Sahmim S, Fixed points of multivalued nonself almost contractions in metric-like spaces. Math. Sci. 2015;9:103–108.
41. Aydi H, Hadj Amor S, Karapinar E, Berinde-type generalized contractions on partial metric spaces. Abstr. Appl. Anal. 2013;2013:Article ID 312479.
42. Aydi H, Hadj Amor S, Karapinar E, Some almost generalized (ψ, ϕ)-contractions in G-metric spaces. Abstr. Appl. Anal. 2013;2013:Article ID 165420.

43. Aydi H, Karapinar E, Mustafa Z, Some tripled coincidence point theorems for almost generalized contractions in ordered metric spaces. Tamkang J. Math. 2013;44:233–251.
44. Babu GVR, Babu DR, Rao KN, Kumar BVS, Fixed points of (ψ,ϕ)-almost weakly contractive maps in G-metric spaces. Appl. Math. E-Notes 2014;14:69–85.
45. Babu GVR, Kidane KT, Fixed points of almost generalized $\alpha - \Psi$-contractive maps. Int. J. Math. Sci. Comput. 2013;3:30–38.
46. Babu GVR, Sailaja PD, Existence of common fixed points of generalized almost weakly contractive maps. Proc. Jangjeon Math. Soc. 2013;16:71–86.
47. Babu GVR, Sandhya ML, Kameswari MVR, A note on a fixed point theorem of Berinde on weak contractions. Carpathian J. Math. 2008;24:8–12.
48. Babu GVR, Subhashini P, Coupled coincidence points of ϕ almost generalized contractive mappings in partially ordered metric spaces. J. Adv. Res. Pure Math. 2013;5:1–16.
49. Banach S, Sur les opérations dans les ensembles abstraits et leur applications aux equations intégrales. Fund. Math. 1922;3:133–181.
50. Banach S, Théorie des Operations Linéaires. Monografie Matematyczne. Warszawa-Lwow;1932.
51. Bekeshie T, Naidu GA, Recent fixed point theorems for T-contractive mappings and T-weak (almost) contractions in metric and cone metric spaces are not real generalizations. J. Nonlinear Anal. Optim. 2013;4:219–225.
52. Berinde V, Contracţii generalizate şi aplicaţii. Baia Mare:Editura Cub Press 22;1997.
53. Berinde V, On the approximation of fixed points of weak contractive mappings. Carpathian J. Math. 2003;19:7–22.
54. Berinde V, Approximating fixed points of weak ϕ-contractions using the Picard iteration. Fixed Point Theory 2003;4:131–142.
55. Berinde V, Approximating fixed points of weak contractions using the Picard iteration. Nonlinear Anal. Forum 2004;9:43–53.
56. Berinde V, A common fixed point theorem for nonself mappings. Miskolc Math. Notes. 2004;5:137–144.
57. Berinde V, Approximation of fixed points of some nonself generalized ϕ-contractions. Math. Balkanica (N.S.) 2004;18:85–93.
58. Berinde V, Berinde M, On Zamfirescu's fixed point theorem. Rev. Roumaine Math. Pures Appl. 2005;50:443–453.
59. Berinde V, A convergence theorem for some mean value fixed point iteration procedures. Demonstratio Math. 2005;38:177–184.
60. Berinde V, Iterative Approximation of Fixed Points. Berlin:Springer;2007.
61. Berinde V, A convergence theorem for Mann iteration in the class of Zamfirescu operators. An. Univ. Vest Timiş. Ser. Mat.-Inform. 2007;45:33–41.
62. Berinde V, General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces. Carpathian J. Math. 2008;24:10–19.
63. Berinde V, Approximating common fixed points of noncommuting discontinuous weakly contractive mappings in metric spaces. Carpathian J. Math. 2009;25:13–22.
64. Berinde V, Some remarks on a fixed point theorem for Ćirić-type almost contractions. Carpathian J. Math. 2009;25:157–162.
65. Berinde V, Approximating common fixed points of noncommuting almost contractions in metric spaces. Fixed Point Theory. 2010;11:179–188.
66. Berinde V, Common fixed points of noncommuting discontinuous weakly contractive mappings in cone metric spaces. Taiwanese J. Math. 2010;14:1763–1776.
67. Berinde V, Common fixed points of noncommuting almost contractions in cone metric spaces. Math. Commun. 2010;15:229–241.
68. Berinde V, Stability of Picard iteration for contractive mappings satisfying an implicit relation. Carpathian J. Math. 2011;27:13–23.
69. Berinde V, Approximating fixed points of implicit almost contractions. Hacet. J. Math. Stat. 2012;41:93–102.
70. Berinde M, Berinde V, On a general class of multi-valued weakly Picard mappings. J. Math. Anal. Appl. 2007;326:772–782.

71. Berinde V, Măruşter Ş, Rus IA, An abstract point of view on iterative approximation of fixed points of nonself operators. J. Nonlinear Convex Anal. 2014;15:851–865.
72. Berinde V, Păcurar M, Fixed points and continuity of almost contractions. Fixed Point Theory 2008;9:23–34.
73. Berinde V, Păcurar M, A note on the paper "Remarks on fixed point theorems of Berinde". Nonlinear Anal. Forum. 2009;14:119–124.
74. Berinde V, Păcurar M, Fixed point theorems for nonself single-valued almost contractions. Fixed Point Theory 2013;14:301–311.
75. Berinde V, Păcurar M, Stability of k-step fixed point iterative methods for some Prešić type contractive mappings. J. Inequal. Appl. 2014;2014:Article ID 149.
76. Berinde V, Păcurar M, A constructive approach to coupled fixed point theorems in metric spaces. Carpathian J. Math. 2015;31:277–287.
77. Berinde V, Păcurar M, Rus IA, From a Dieudonné theorem concerning the Cauchy problem to an open problem in the theory of weakly Picard operators. Carpathian J. Math. 2014;30:283–292.
78. Berinde V, Vetro F, Common fixed points of mappings satisfying implicit contractive conditions. Fixed Point Theory Appl. 2012;2012:Article ID 105.
79. Bose RK, Some Suzuki type fixed point theorems for generalized contractive multifunctions. Int. J. Pure Appl. Math. 2013;84:13–27.
80. Bota MF, Karapinar E, A note on "Some results on multi-valued weakly Jungck mappings in b-metric space". Cent. Eur. J. Math. 2013;11:1711–1712.
81. Camouzis E, Chatterjee E, Ladas G, On the dynamics of $x_{n+1} = (\delta x_{n-2} + x_{n-3})/(A + x_{n-3})$. J. Math. Anal. Appl. 2007;331:230–239.
82. Caristi J, Fixed point theorems for mappings satisfying inwardness conditions. Trans. Am. Math. Soc. 1976;215:241–251.
83. Caristi J, Fixed point theory and inwardness conditions. In: Applied nonlinear analysis (Proc. Third Int. Conf., Univ. Texas, Arlington, Tex., 1978), New York:Academic Press;1979; 479–483.
84. Chandok S, Choudhury BS, Metiya N, Fixed point results in ordered metric spaces for rational type expressions with auxiliary functions. J. Egyptian Math. Soc. 2015;23:95–101.
85. Chatterjea SK, Fixed-point theorems. C.R. Acad. Bulgare Sci. 1972;25:727–730.
86. Chen YZ, A Presić type contractive condition and its applications. Nonlinear Anal. 2009;71:2012–2017.
87. Chifu C, Petruşel G, Generalized contractions in metric spaces endowed with a graph. Fixed Point Theory Appl. 2012;2012:Article ID 161.
88. Cho SH, Fixed point theorems for weak $\alpha_* - (\Phi, L)$-contractive set-valued maps in cone metric spaces. Int. J. Math. Anal. 2013; 7:2967–2979.
89. Cho SH, A fixed point theorem for a Ćirić-Berinde type mapping in orbitally complete metric spaces. Carpathian J. Math. 2014;30:63–70.
90. Cho SH, Fixed point theorems for Ćirić-Berinde type contractive multivalued mappings. Abstr. Appl. Anal. 2015;2015:Article ID 768238.
91. Choudhury BS, Metiya N, Fixed point theorems for almost contractions in partially ordered metric spaces. Ann. Univ. Ferrara Sez. VII Sci. Mat. 2012;58:21–36.
92. Choudhury BS, Metiya N, Coincidence point theorems for a family of multivalued mappings in partially ordered metric spaces. Acta Univ. M. Belii Ser. Math. 2013;10–23.
93. Choudhury BS, Metiya N, Som T, Bandyopadhyay C, Multivalued fixed point results and stability of fixed point sets in metric spaces. Facta Univ. Ser. Math. Inform. 2015;30:501–512.
94. Ćirić LB, Generalized contractions and fixed-point theorems. Publ. l'Inst. Math. (Beograd) 1971;12:19–26.
95. Ćirić LB, On contraction type mappings. Math. Balkanica. 1971;1:52–57.
96. Ćirić LB, A generalization of Banach's contraction principle. Proc. Am. Math. Soc. 1974;45:267–273.
97. Ćirić LB, Convergence theorems for a sequence of Ishikawa iteration for nonlinear quasi-contractive mappings. Indian J. Pure Appl. Appl. Math. 1999;30:425–433.
98. Ćirić LB, A remark on Rhoades' fixed point theorem for non-self mappings. Int. J. Math. Math. Sci. 1993;16:397–400.

99. Ćirić LB, Quasi contraction non-self mappings on Banach spaces. Bull. Cl. Sci. Math. Nat. Sci. Math. 1998;23:25–31.
100. Ćirić LB, Ume JS, Khan MS, Pathak HK, On some nonself mappings. Math. Nachr. 2003;251:28–33.
101. Ćirić LB, Prešić SB, On Prešić type generalization of the Banach contraction mapping principle. Acta Math. Univ. Comenianae 2007;76:143–147.
102. Ćirić LB, Abbas M, Damjanović B, Saadati R, Common fuzzy fixed point theorems in ordered metric spaces. Math. Comput. Modelling 2011;53:1737–1741.
103. Ćirić LB, Abbas M, Saadati R, Hussain N, Common fixed points of almost generalized contractive mappings in ordered metric spaces. Appl. Math. Comput. 2011;217:5784–5789.
104. Cvetkovic M, Rakocevic V, Extensions of Perov theorem. Carpathian J. Math. 2015;31:181–188.
105. Das KM, Naik KV, Common fixed point theorems for commuting maps on metric spaces. Proc. Am. Math. Soc. 1979;77:369–373.
106. De la Sen M, Agarwal RP, Ibeas A, Results on proximal and generalized weak proximal contractions including the case of iteration-dependent range sets. Fixed Point Theory Appl. 2014;2014:Article ID 169.
107. De la Sen M, Some further results on weak proximal contractions including the case of iteration-dependent image sets. In: Mathematical Methods in Engineering and Economics, Proceedings of the 2014 International Conference on Applied Mathematics and Computational Methods in Engineering II (AMCME '14). Proceedings of the 2014 International Conference on Economics and Business Administration II (EBA '14), Prague, Czech Republic April 2–4, 2014, pp. 15–20.
108. Devault R, Dial G, Kocic VL, Ladas G, Global behavior of solutions of $x_{n+1} = ax_n + f(x_n, x_{n-1})$. J. Difference Eq. Appl. 1998;3:311–330.
109. Di Bari C, Vetro P, Common fixed points for three or four mappings via common fixed point for two mappings. arXiv:1302.3816.
110. Du WS, Fixed point theorems for generalized multivalued weak contractions. Int. J. Math. Anal. (Ruse) 2008;2:181–186.
111. Du WS, Some new results and generalizations in metric fixed point theory. Nonlinear Anal. 2010;73:1439–1446.
112. Du WS, New cone fixed point theorems for nonlinear multivalued maps with their applications. Appl. Math. Lett. 2011;24:172–178.
113. Du WS, On coincidence point and fixed point theorems for nonlinear multivalued maps. Topology Appl. 2012;159:49–56.
114. Du WS, On approximate coincidence point properties and their applications to fixed point theory. J. Appl. Math. 2012; 2012:Article ID 302830.
115. Du WS, On generalized weakly directional contractions and approximate fixed point property with applications. Fixed Point Theory Appl. 2012;2012:Article ID 6.
116. Du WS, New existence results and generalizations for coincidence points and fixed points without global completeness. Abstr. Appl. Anal. 2013;2012:Article ID 214230.
117. Du WS, He ZH, Chen YL, New existence theorems for approximate coincidence point property and approximate fixed point property with applications to metric fixed point theory. J. Nonlinear Convex Anal. 2012;13:459–474.
118. Du WS, Karapinar E, Shahzad N, The study of fixed point theory for various multivalued non-self-maps. Abstr. Appl. Anal. 2013;2013:Article ID 938724.
119. Du WS, Khojasteh F, New results and generalizations for approximate fixed point property and their applications. Abstr. Appl. Anal. 2014;2014:Article ID 581267.
120. Du WS, Khojasteh F, Chiu YN, Some generalizations of Mizoguchi-Takahashi's fixed point theorem with new local constraints. Fixed Point Theory Appl. 2014;2014:Article ID 31.
121. Du WS, Zheng SX, Nonlinear conditions for coincidence point and fixed point theorems. Taiwanese J. Math. 2012;16:857–868.
122. Du WS, Zheng SX, New nonlinear conditions and inequalities for the existence of coincidence points and fixed points. J. Appl. Math. 2012;2012:Article ID 196759.
123. Dubey AK, Tiwari SK, Dubey RP, Common fixed point theorems for T-weak contraction mapping in a cone metric space. Mathematica Aeterna 2013;3:121–131.

124. Dugundji J, Granas A, Weakly contractive maps and elementary domain invariance theorem. Bull. Greek Math. Soc. 1978;19:141–151.
125. Durmaz G, Minak G, Altun I, Fixed point results for $\alpha - \Psi$-contractive mappings including almost contractions and applications. Abstr. Appl. Anal. 2014;2014:Article ID 869123.
126. Eisenfeld J, Lakshmikantham V, Fixed point theorems on closed sets through abstract cones. Appl. Math. Comput. 1977;3:155–167.
127. El-Metwally H, Grove EA, Ladas G, Levins R, Radin M, On the difference equation $x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}$. Nonlinear Anal. 2001;47:4623–4634.
128. Erduran A, Kadelburg Z, Nashine HK, Vetro C, A fixed point theorem for (ϕ, L)-weak contraction mappings on a partial metric space. J. Nonlinear Sci. Appl. 2014;7:196–204.
129. Filip AD, Petru PT, Fixed point theorems for multivalued weak contractions. Stud. Univ. Babeş-Bolyai Math. 2009;54:33–40.
130. Filip AD, Petruşel A, Fixed point theorems on spaces endowed with vector-valued metrics. Fixed Point Theory Appl. 2010;2010:Article ID 28138.
131. Gabeleh M, Best proximity point theorems for single- and set-valued non-self mappings. Acta Math. Sci. Ser. B Engl. Ed. 2014;34:1661–1669.
132. Gabeleh M, Best proximity point theorems via proximal non-self mappings. J. Optim. Theory Appl. 2015;164:565–576.
133. George R, Reshma KP, Padmavati A, Fixed point theorems for cyclic contractions in b-metric spaces. J. Nonlinear Funct. Anal. 2015;2015:Article ID 5.
134. Gül, Karapinar E, On almost contractions in partially ordered metric spaces via implicit relations. J. Inequal. Appl. 2012;2012:Article ID 217.
135. He ZH; Du WS, Lin IJ, The existence of fixed points for new nonlinear multivalued maps and their applications. Fixed Point Theory Appl. 2011;2011:Article ID 84.
136. Hussain N, Amini-Harandi A, Cho YJ, Approximate endpoints for set-valued contractions in metric spaces. Fixed Point Theory Appl. 2010;2010:Article ID 614867.
137. Hussain N, Cho YJ, Weak contractions, common fixed points, and invariant approximations. J. Inequal. Appl. 2009;2009:Article ID 390634.
138. Hussain N, Parvaneh V, Roshan JR, Kadelburg Z, Fixed points of cyclic weakly (ψ, φ, L, A, B)-contractive mappings in ordered b-metric spaces with applications. Fixed Point Theory Appl. 2013;2013:Article ID 256.
139. Javahernia M, Razani A, Khojasteh F, Common fixed point of the generalized Mizoguchi-Takahashi's type contractions. Fixed Point Theory Appl. 2014;2014:Article ID 195.
140. Jleli M, Karapinar E, Samet B, Fixed point results for almost generalized cyclic (ψ, ϕ)-weak contractive type mappings with applications. Abstr. Appl. Anal. 2012;2012:Article ID 917831.
141. Jleli M, Karapinar E, Samet B, Further generalizations of the Banach contraction principle. J. Inequal. Appl. 2014;2014:Article ID 439.
142. Jleli M, Samet B, A new generalization of the Banach contraction principle. J. Ineq. Appl. 2014;2014:Article ID 38.
143. Jleli M, Samet B, Vetro C, Fixed point theory in partial metric spaces via ϕ-fixed point's concept in metric spaces. J. Inequal. Appl. 2014;2014:Article ID 426.
144. Jungck G, Commuting maps and fixed points. Am. Math. Monthly 1976;83:261–263.
145. Jungck G, Common fixed points for noncontinuous nonself maps on non-metric spaces. Far East J. Math. Sci. 1996;4:199–215.
146. Kalinde AK, Mishra S. N., Pathak HK, Some results on common fixed points with applications. Fixed Point Theory 2005;6:285–301.
147. Kamran T, Multivalued f-weakly Picard mappings. Nonlinear Anal. 2007;67:2289–2296.
148. Kamran T, Cakić N, Hybrid tangential property and coincidence point theorems. Fixed Point Theory 2008; 9:487–496.
149. Kannan R, Some results on fixed points. Bull. Calcutta Math. Soc. 1968;10:71–76.
150. Kannan R, Some results on fixed points. III. Fund. Math. 1971;70:169–177.
151. Kannan R, Construction of fixed points of a class of nonlinear mappings. J. Math. Anal. Appl. 1973;41:430–438.

152. Karapinar E, Sadarangani K, Berinde mappings in ordered metric spaces. Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM 2015;109:353–366.
153. Karapinar E, Sintunavarat W, The existence of optimal approximate solution theorems for generalized α-proximal contraction non-self mappings and applications. Fixed Point Theory Appl. 2013;2013:Article ID 323.
154. Karuppiah U, Dharsini AMP, Some fixed point theorems satisfying (s,t)-contractive condition in partially ordered partial metric space. Far East J. Math. Sci. 2014;95:19–50.
155. Khandani H, Vaezpour SM, Sims B, Fixed point and common fixed point theorems of contractive multivalued mappings on complete metric spaces. J. Comput. Anal. Appl. 2011;13:1025–1039.
156. Khan AR, Abbas M, Nazir T, Ionescu C, Fixed points of multivalued contractive mappings in partial metric spaces. Abstr. Appl. Anal. 2014;2014:Article ID 230708.
157. Khan MS, Swaleh M, Sessa S, Fixed point theorems by altering distances between the points. Bull. Aust. Math. Soc. 1984;30:1–9.
158. Khan MS, Berzig M, Samet B, Some convergence results for iterative sequences of Prešić type and applications. Adv. Difference Equ. 2012;2012:Article ID 38.
159. Khan MS, Jhade PK, On a fixed point theorem with PPF dependence in the Razumikhin class. Gazi Univ. J. Sci . 2015;28:211–219.
160. Khan SH, Common fixed points of two quasi-contractive operators in normed spaces by iteration. Int. J. Math. Anal. (Ruse) 2009;3:145–151.
161. Kikina L, Kikina K, Vardhami I, Fixed point theorems for almost contractions in generalized metric spaces. Creat. Math. Inform. 2014;23:65–72.
162. Kiran Q, Kamran T, Nadler's type principle with high order of convergence. Nonlinear Anal. 2008;69:4106–4120.
163. Klanarong C, Suantai S, Coincidence point theorems for some multi-valued mappings in complete metric spaces endowed with a graph. Fixed Point Theory Appl. 2015;2015:Article ID 129.
164. Kocic VL, A note on the non-autonomous Beverton-Holt model. J. Difference Equ. Appl. 2005;11:415–422.
165. Kocic VL, Ladas G, Global Asymptotic Behavior of Nonlinear Difference Equations of Higher Order with Applications. Dordrecht:Kluwer Academic Publishers; 1993.
166. Kumar A, Rathee S, Fixed point and common fixed point results in cone metric space and application to invariant approximation. Fixed Point Theory Appl. 2015;2015:Article ID 1.
167. Kuruklis SA, The asymptotic stability of $x_{n+1} - ax_n + bx_{n-k} = 0$. J. Math. Anal. Appl. 1994;188:719–731.
168. Kutbi MA, Sintunavarat W, On sufficient conditions for the existence of past-present-future dependent fixed point in the Razumikhin class and application. Abstr. Appl. Anal. 2014;2014:Article ID 342687.
169. Latif A, Mongkolkeha C, Sintunavarat W, Fixed point theorems for generalized $\alpha - \beta$-weakly contraction mappings in metric spaces and applications. Scientific World J. 2014;2014:Article ID 784207.
170. Lin IJ, Chen TH, New existence theorems of coincidence points approach to generalizations of Mizoguchi-Takahashi's fixed point theorem. Fixed Point Theory Appl. 2012;2012:Article ID 156.
171. Lin IJ, Jian KR, New fixed point theorems for nonlinear multivalued maps and mt-functions in complete metric spaces. Nonlinear Anal. Diff. Eq. 2013;1:29–41.
172. Lin IJ, Wang TY, New fixed point theorems for generalized distances. Int. J. Math. Anal. 2013;7:1843–1855.
173. Măruşter L, Măruşter S, On the error estimation and T-stability of the Mann iteration. J. Comput. Appl. Math. 2015;276:110–116.
174. Mehmood N, Azam A, Aleksić S, Topological vector-space valued cone Banach spaces. Int. J. Anal. Appl. 2014;6:205–219.
175. Minak G, Acar Ö, Altun I, Multivalued pseudo-Picard operators and fixed point results. J. Funct. Spaces Appl. 2013;2013:Article ID 827458.
176. Minak G, Altun I, Some new generalizations of Mizoguchi-Takahashi type fixed point theorem. J. Inequal. Appl. 2013;2013:Article ID 493.

177. Minak G, Altun I, Multivalued weakly Picard operators on partial metric spaces. Nonlinear Funct. Anal. Appl. 2014;19:45–59.
178. Minak G, Altun I, Romaguera S, Recent developments about multivalued weakly Picard operators. Bull. Belg. Math. Soc. Simon Stevin 2015;22:411–422.
179. Minak G, Helvaci A, Altun I, Ćirić type generalized F-contractions on complete metric spaces and fixed point results. Filomat 2014;28:1143–1151.
180. Mohsenalhosseini SAM, Approximate best proximity pairs in metric space for contraction maps. Adv. Fixed Point Theory 2014;4:310–324.
181. Mongkolkeha C, Kongban C, Kumam P, Existence and uniqueness of best proximity points for generalized almost contractions. Abstr. Appl. Anal. 2014;2014:Article ID 813614.
182. Morales JR, Rojas EM, Coincidence points for multivalued mappings. An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. 2011;19:37–150.
183. Mustafa Z, Karapinar E, Aydi H, A discussion on generalized almost contractions via rational expressions in partially ordered metric spaces. J. Inequal. Appl. 2014;2014:Article ID 219.
184. Mustafa Z, Parvaneh V, Roshan JR, Kadelburg Z, b_2-metric spaces and some fixed point theorems. Fixed Point Theory Appl. 2014;2014:Article ID 144.
185. Nashine HK, Kadelburg Z, Common fixed point theorems for a pair of multivalued mappings under weak contractive conditions in ordered metric spaces. Bull. Belg. Math. Soc. Simon Stevin 2012;19:577–596.
186. Olatinwo MO, Some results on multi-valued weakly Jungck mappings in *b*-metric space. Cent. Eur. J. Math. 2008;6:610–621.
187. Olatinwo MO, An extension of some fixed point theorems for single-valued and multi-valued Picard operators in b-metric spaces. Nonlinear Anal. Forum. 2009;14:103–111.
188. Olatinwo MO, Imoru CO, A generalization of some results on multi-valued weakly Picard mappings in b-metric space. Fasc. Math. 2008;40:45–56.
189. Olatinwo MO, Postolache M, Stability results for Jungck-type iterative processes in convex metric spaces. Appl. Math. Comput. 2012;218:6727–6732.
190. Osilike MO, Stability results for fixed point iteration procedures. J. Nigerian Math. Soc. 1995/96;14/15:17–29.
191. Osilike MO, Stability of the Ishikawa iteration method for quasi-contractive maps. Indian J. Pure Appl. Math. 1997;28(9):1251–1265.
192. Osilike MO, Short proofs of stability results for fixed point iteration procedures for a class of contractive-type mappings. Indian J. Pure Appl. Math. 1999;30(12):1229–1234.
193. Ortega JM, Rheinboldt WC, Iterative Solution of Nonlinear Equations in Several Variables. New York:Academic Press;1970.
194. Pathak HK, Agarwal RP, Cho YJ, Coincidence and fixed points for multi-valued mappings and its application to nonconvex integral inclusions. J. Comput. Appl. Math. 2015;283:201–217.
195. Pathak HK, George R, Nabwey HA, El-Paoumy MS, Reshma KP, Some generalized fixed point results in a b-metric space and application to matrix equations. Fixed Point Theory Appl. 2015;2015:Article ID 101.
196. Păcurar M, Approximating common fixed points of Prešić-Kannan type operators by a multi-step iterative method. An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. 2009;17:153–168.
197. Păcurar M, Iterative Methods for Fixed Point Approximation. Cluj-Napoca:Editura Risoprint;2009.
198. Păcurar M, Sequences of almost contractions and fixed points in *b*-metric spaces. An. Univ. Vest Timiş. Ser. Mat.-Inform. 2010;48:125–137.
199. Păcurar M, Remark regarding two classes of almost contractions with unique fixed point. Creat. Math. Inform. 2010;19:178–183.
200. Păcurar M, A multi-step iterative method for approximating common fixed points of Prešić-Rus type operators on metric spaces. Studia Univ. "Babeş-Bolyai". Math. 2010;55:149–162.
201. Păcurar M, A multi-step iterative method for approximating fixed points of Prešić-Kannan operators. Acta Math. Univ. Comen. New Ser. 2010;79:77–88.
202. Păcurar M, Fixed point theory for cyclic Berinde operators. Fixed Point Theory 2011;12:419–428.
203. Păcurar M, Fixed points of almost Prešić operators by a *k*-step iterative method. An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) 2011;57:199–210.

204. Păcurar M, Common fixed points for almost Prešić type operators. Carpathian J. Math. 2012;28:117–126.
205. Păcurar M, Păcurar RV, Approximate fixed point theorems for weak contractions on metric spaces. Carpathian J. Math. 2007;23:149–155.
206. Petruşel A, Petruşel G, Multivalued contractions of Feng-Liu type in complete gauge spaces. Carpathian J. Math. 2008;24:392–396.
207. Petruşel A, Petruşel G, Urs C, Vector-valued metrics, fixed points and coupled fixed points for nonlinear operators. Fixed Point Theory Appl. 2013;2013:Article ID 218.
208. Phuengrattana W, Suantai S, Comparison of the rate of convergence of various iterative methods for the class of weak contractions in Banach spaces. Thai J. Math. 2013;11:217–226.
209. Phon-on A, Sama-Ae A, Makaje N, Riyapan P, Busaman S, Coincidence point theorems for weak graph preserving multi-valued mapping. Fixed Point Theory Appl. 2014;2014:248.
210. Picard E, Memoire sur la théorie des equations aux derivées partielles et la methode des approximations successives. J. Math. Pures et Appl. 1890;6:145–210.
211. Popa V, Fixed point theorems for implicit contractive mappings. Stud. Cerc. St. Ser. Mat. Univ. Bacău. 1997;7:127–133.
212. Popa V, Some fixed point theorems for compatible mappings satisfying an implicit relation. Demonstratio Math. 1999;32:157–163.
213. Popa V, On some fixed point theorems for implicit almost contractive mappings. Carpathian J. Math. 2013;29:223–229.
214. Popa V, Common (E.A)-property and altering distance in metric spaces. Sci. Stud. Res. Ser. Math. Inform. 2014;24:115–133.
215. Popa V, Patriciu AM, A general fixed point theorem for a pair of self mappings with common limit range property in G-metric spaces. Facta Univ. Ser. Math. Inform. 2014;29:351–370.
216. Popescu O, Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators. Math. Commun. 2007;12:195–202.
217. Popescu O, Comparison of fastness of the convergence among Krasnoselskij iterations. Bull. Transilv. Univ. Braşov Ser. B (N.S.) 2007;14(49):27–32.
218. Prešić SB, Sur une classe d’ inéquations aux différences finite et sur la convergence de certaines suites. Publ. Inst. Math. (Beograd) (N.S.) 1965;5(19):75–78.
219. Rao KPR, Bindu HS, Ali MM, Coupled fixed point theorems in d-complete topological spaces. J. Nonlinear Sci. Appl. 2012;5:186–194.
220. Rao KPR, Rao KRK, Karapinar E, Common coupled fixed point theorems in d-complete topological spaces. Ann. Funct. Anal. 2012;3:107–114.
221. Raphael P, Pulickakunnel S, Approximate fixed point theorems for generalized T-contractions in metric spaces. Stud. Univ. Babeş-Bolyai Math. 2012;57:551–559.
222. Rashwan RA, Hammad HA, Common fixed point theorems for weak contraction mapping of integral type in modular spaces. Universal J. Comput. Math. 2014;2:69–86.
223. Rathee S, Kumar A, Some common fixed-point and invariant approximation results with generalized almost contractions. Fixed Point Theory Appl. 2014;2014:Article ID 23.
224. Rathee S, Kumar A, Tas K, Invariant approximation results via common fixed point theorems for generalized weak contraction maps. Abstr. Appl. Anal. 2014;2014:Article ID 752107.
225. Redjel N, On some extensions of Banach’s contraction principle and applications to the convergence and stability of some iterative processes. Adv. Fixed Point Theory 2014;4:555–570.
226. Rhoades BE, A comparison of various definitions of contractive mappings. Trans. Am. Math. Soc. 1977;226:257–290.
227. Rhoades BE, Extensions of some fixed theorems of Ciric, Maiti and Pal. Math. Seminar Notes 1978;6:41–46.
228. Rhoades BE, A fixed point theorem for some non-self-mappings. Math. Japon. 1978/79;23:457–459.
229. Rhoades BE, Contractive definitions revisited. Contemporary Math. 1983;21:189–205.
230. Rhoades BE, Contractive definitions and continuity. Contemporary Math. 1988;72:233–245.
231. Romaguera S, On Nadler’s fixed point theorem for partial metric spaces. Math. Sci. Appl. E-Notes 2013;1:1–8.

232. Roshan JR, Parvaneh V, Sedghi S, Shobkolaei N, Shatanawi W, Common fixed points of almost generalized (Ψ,Φ) s-contractive mappings in ordered b-metric spaces. Fixed Point Theory Appl. 2013;2013:Article ID 159.
233. Rus IA, An iterative method for the solution of the equation $x = f(x,\ldots,x)$. Anal. Numér. Théor. Approx. 1981;10:95–100.
234. Rus IA, An abstract point of view in the nonlinear difference equations. In: Conf. on An., Functional Equations, App. and Convexity, Cluj-Napoca, October 15–16, 1999, 272–276.
235. Rus IA, Generalized Contractions and Applications. Cluj-Napoca: Cluj Univ. Press;2001.
236. Rus IA, Picard operators and applications. Sci. Math. Jpn. 2003;58:191–219.
237. Rus IA, Heuristic introduction to weakly Picard operator theory. Creat. Math. Inform. 2014;23:243–252.
238. Rus IA, Five open problems in fixed point theory in terms of fixed point structures (I): Singlevalued operators. In: Espinola R, Petruşel A, Prus, S, editors, Fixed Point Theory and Its Applications. Cluj-Napoca:Casa Cărţii de Ştiinţă;2013;39–60.
239. Rus IA, Petruşel A, Petruşel G, Fixed Point Theory. Cluj-Napoca: Cluj Univ. Press;2008.
240. Rus IA, Şerban MA, Basic problems of the metric fixed point theory and the relevance of a metric fixed point theorem. Carpathian J. Math. 2013;29:239–258.
241. Sadeghi-Hafshejani A, Amini-Harandi A, A fixed point result for a new class of set-valued contractions. Int. J. Nonlinear Anal. Appl. 2014;5:64–70.
242. Saha M, Dey D, Some random fixed point theorems for (θ,L)-weak contractions. Hacet. J. Math. Stat. 2012;41:795–812.
243. Samet B, Fixed points for α-ψ contractive mappings with an application to quadratic integral equations. Electron. J. Differential Equations 2014;2915:Article ID 152.
244. Samet B, The class of (α,ψ)-type contractions in b-metric spaces and fixed point theorems. Fixed Point Theory Appl. 2015;2015:Article ID 92.
245. Samet B, Vetro C, Berinde mappings in orbitally complete metric spaces. Chaos Solitons Fractals 2011;44:1075–1079.
246. Sarma KKM, Kumari VA, Fixed points of almost generalized (α,ψ) contractive maps in G-metric spaces. Thai J. Math. (in press).
247. Sarwar M, Rahman MU, Fixed point theorems for Ciric's and generalized contractions in b-metric spaces. Int. J. Anal. Appl. 2015;7:70–78.
248. Sastry KPR, Babu GVR, Sandhya ML, Weak contractions in Menger spaces. J. Adv. Res. Pure Math. 2010;2:26-35.
249. Şerban MA, Teoria punctului fix pentru operatori definiţi pe produs cartezian. Cluj-Napoca:Presa Universitară Clujeană;2002.
250. Shaddad F, Noorani MSM, Alsulami SM, Fixed point results in cone metric spaces for multivalued maps. Malays. J. Math. Sci. 2014;8:83–102.
251. Shaddad F, Noorani MSM, Alsulami SM, Common fixed-point results for generalized Berinde-type contractions which involve altering distance functions. Fixed Point Theory Appl. 2014;2014:Article ID 24.
252. Shatanawi W, Some fixed point results for generalized ψ-weak contraction mappings in orbitally metric spaces. Chaos Solitons Fractals. 2012;45:520–526.
253. Shatanawi W, Al-Rawashdeh A, Common fixed points of almost generalized (ψ,φ)-contractive mappings in ordered metric spaces. Fixed Point Theory Appl. 2012;2012:Article ID 80.
254. Shatanawi W, Postolache M, Some fixed-point results for a G-weak contraction in G-metric spaces. Abstr. Appl. Anal. 2012;2012:Article ID 815870.
255. Shatanawi W, Postolache M, Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces. Fixed Point Theory Appl. 2013;2013:Article ID 54.
256. Shatanawi W, Postolache M, Common fixed point theorems for dominating and weak annihilator mappings in ordered metric spaces. Fixed Point Theory Appl. 2013;2013:Article ID 271.
257. Shatanawi W, Saadati R, Park C, Almost contractive coupled mapping in ordered complete metric spaces. J. Inequal. Appl. 2013;2013:Article ID 565.
258. Shobkolaei N, Sedghi S, Roshan JR, Altun I, Common fixed point of mappings satisfying almost generalized (S,T)-contractive condition in partially ordered partial metric spaces. Appl. Math. Comput. 2012;219:443–452.

259. Shukla S, Sen R, Set-valued Prešić-Reich type mappings in metric spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM 2014;108:431–440.
260. Singh A, Prajapati DJ, Dimri RC, Some fixed point results of almost generalized contractive mappings in ordered metric spaces. Int. J. Pure Appl. Math. 2013;86:779–789.
261. Singh SL, Pant R, Remarks on fixed point theorems of V. Berinde: "Approximating fixed points of weak contractions using the Picard iteration" [Nonlinear Anal. Forum 9 (2004), no. 1, 43–53]. Nonlinear Anal. Forum. 2007;12:231–234.
262. Sintunavarat W, Kim JK, Kumam P, Fixed point theorems for a generalized almost (Φ, φ)-contraction with respect to S in ordered metric spaces. J. Inequal. Appl. 2012;2012:Article ID 263.
263. Sintunavarat W, Kumam P, Common fixed point theorem for hybrid generalized multi-valued contraction mappings. Appl. Math. Lett. 2012;25:52–57.
264. Stević S, Asymptotic behavior of a class of nonlinear difference equations. Discrete Dynamics in Nature and Society 2006;2006:Article ID 47156.
265. Stević S, On the recursive sequence $x_{n+1} = A + (x_n^p)/(x_{n-1}^p)$. Discrete Dynamics in Nature and Society 2007;2007:Article ID 34517.
266. Suzuki T, Fixed point theorems for Berinde mappings. Bull. Kyushu Inst. Technol. Pure Appl. Math. 2011;58:13–19.
267. Sz.-Nagy B, Foiaş C, Analyse harmonique des opérateurs de l'espace de Hilbert. Paris:Masson et Cie;Budapest:Akadémiai Kiadó;1967.
268. Tahat N, Aydi H, Karapinar E, Shatanawi W, Common fixed points for single-valued and multi-valued maps satisfying a generalized contraction in G-metric spaces. Fixed Point Theory Appl. 2012;2012:Article ID 48.
269. Tiammee J, Suantai S, Coincidence point theorems for graph-preserving multi-valued mappings. Fixed Point Theory Appl. 2014;2014:Article ID 70.
270. Timiş I, Stability of the Picard iterative procedure for mappings which satisfy implicit relations. Comm. Appl. Nonlinear Anal. 2012;19:37–44.
271. Tiwari R, Prajapati PB, Bhardwaj R, Fixed point theorems for generalized almost contractive mappings in ordered metric spaces for integral type. Math. Theory Model. 2015;5:112–120.
272. Turinici M, Fixed points in complete metric spaces. Proceedings of the Institute of Mathematics Iaşi (1974), pp. 179–182. Bucharest:Editura Acad. R. S. R.;1976.
273. Turinici M, Weakly contractive maps in altering metric spaces. ROMAI J. 2013;9:175-183.
274. Turinici M, Linear contractions in product ordered metric spaces. Ann. Univ. Ferrara Sez. VII Sci. Mat. 2013;59:187–198.
275. Turinici M, Contractive operators in relational metric spaces. In: Handbook of Functional Equations, New York:Springer;2014;419–458.
276. Turkoglu AD, Özturk V, Common fixed point results for four mappings on partial metric spaces. Abstr. Appl. Anal. 2012;2012:Article ID 190862.
277. Udo-utun X, On inclusion of F-contractions in (δ, k)-weak contractions. Fixed Point Theory Appl. 2014;2014:Article ID 65.
278. Walter W, Remarks on a paper by F. Browder about contraction. Nonlinear Anal. TMA 1981;5:21–25.
279. Weinitschke H, Über eine Klasse von Iterationsverfahren. Numer. Math. 1964;6:395–404.
280. Zamfirescu T, Fix point theorems in metric spaces. Arch. Math. (Basel) 1972;23:292–298.