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Fixed point for cyclic weak (ψ, C)-contractions in 0-complete partial metric spaces

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Abstract. In this paper, following [W.A. Kirk, P.S. Srinivasan, P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theory, **4** (2003), 79-89], we give a fixed point result for cyclic weak (ψ, C)-contractions on partial metric space. A Maia type fixed point theorem for cyclic weak (ψ, C)-contractions is also given.

1. Introduction

Matthews [17] introduced the notion of a partial metric space as a part of the study of denotational semantics of data for networks, showing that the contraction mapping principle can be generalized to the partial metric context for applications in program verification. In [2, 3, 9, 11–13, 18, 20, 25, 28, 30–32] we have some generalizations of the result of Matthews.

A very interesting work on cyclic contractions is the Ph.D. thesis of Petric [22]. In this thesis, the reader can find a background introduction to the study of fixed point theory for cyclical contractive operators and its applications. Some other fixed point results for cyclic mappings are obtained in the papers [4, 11, 15, 16, 19, 21, 23, 24]. Useful applications of cyclic contractions are devoted to obtain the existence and uniqueness of best proximity points of mappings. In fact, there is a fruitful research branch on this topic that is concretized in various papers published in the last years [1, 5, 10, 14, 26, 27, 29].

In this paper, we give fixed point results for cyclic weak (ψ, C)-contractions on partial metric space. A Maia type fixed point theorem for cyclic weak (ψ, C)-contractions is also given. Our results generalize some interesting results of [15, 23].

2. Preliminaries

First, we recall some definitions and some properties of partial metric spaces that can be found in [9, 11, 17, 18, 20, 25, 28]. A partial metric on a nonempty set X is a function $p : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$:

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(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;

(p_2) $p(x,x) \leq p(x,y)$;

(p_3) $p(x,y) = p(y,x)$;

(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x,y) = 0$, then from (p_1) and (p_2) it follows that $x = y$. But if $x = y$, $p(x,y)$ may not be 0. A basic example of a partial metric space is the pair $([0,+\infty), p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in [0,+\infty)$.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where

$$B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$.

Definition 2.1. *Let (X, p) be a partial metric space. Then*
(i) a sequence $\{x_n\}$ in (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$;
(ii) a sequence $\{x_n\}$ in (X,p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n,x_m)$;
(iii) a partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m)$;
(iv) a sequence $\{x_n\}$ in (X,p) is called 0-Cauchy if $\lim_{n,m\to+\infty} p(x_n,x_m) = 0$. We say that (X,p) is 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = 0$.

On the other hand, the partial metric space $(\mathbb{Q} \cap [0,+\infty), p)$, where $\mathbb{Q}$ denotes the set of rational numbers and the partial metric p is given by $p(x,y) = \max\{x,y\}$, provides an example of a 0-complete partial metric space which is not complete.

It is easy to see that every closed subset of a complete partial metric space is complete. We have the following useful lemma.

Lemma 2.2. *Let (X,p) be a partial metric space and $\{x_n\} \subset X$. If $x_n \to x \in X$ and $p(x,x) = 0$, then $\lim_{n\to+\infty} p(x_n,z) = p(x,z)$ for all $z \in X$.*

Proof. By the triangle inequality

$$p(x,z) - p(x_n,x) \leq p(x_n,z) \leq p(x,z) + p(x_n,x).$$

Letting $n \to +\infty$, we obtain that $p(x_n,z) \to p(x,z)$. □

3. Main results

Our results are inspired from the definition given in [7, Definition 4]. First we introduce the notion of cyclic weak (ψ, C)-contraction in partial metric space.

In the sequel, we denote with:

1. Ψ the class of functions $\psi : [0,+\infty) \to [0,+\infty)$ strictly increasing and continuous such that $\psi(t) \leq \frac{1}{2}t$ for all $t > 0$ and $\psi(0) = 0$;
2. Φ the class of functions $\phi : [0,+\infty)^2 \to [0,+\infty)$ nondecreasing in each coordinate such that $\phi(x,y) = 0$ if and only if $x = y = 0$, $\phi(x,y) \leq x + y$ for all $x, y \in [0,+\infty)$ and $\phi(\cdot, 0)$ continuous.

Let X be a nonempty set, m a positive integer and $T : X \to X$ a mapping. By definition, a finite family $A_1, \dots, A_m$ of nonempty subsets of X is a cyclic representation of X with respect to T if

(i) $\bigcup_{j=1}^{m} A_j = X$;

(ii) $T(A_1) \subset A_2,\ T(A_2) \subset A_3,\ \dots,\ T(A_m) \subset A_1$.

Let (X, p) be a partial metric space, m a positive integer, $A_1, \dots, A_m$ nonempty subsets of X and $Y = \bigcup_{j=1}^{m} A_j$. A mapping $T : Y \to Y$ is a cyclic weak (ψ, C)-contraction if

(i) $A_1, \dots, A_m$ is a cyclic representation of Y with respect to T;

(ii) $p(Tx, Ty) \le \psi(p(x, Ty) + p(y, Tx) - \phi(p(x, Ty), p(y, Tx)))$, for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \dots, m$ where $A_{m+1} = A_1$, $\psi \in \Psi$ and $\phi \in \Phi$.

Obiviously, condition (ii) is a generalization of the condion of Chatterjea [6]

Lemma 3.1. *Let (X, p) be a partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \dots, A_m$ nonempty subsets of X and $X = \bigcup_{i=1}^{m} A_i$. Suppose that $T : X \to X$ is a cyclic weak (ψ, C)-contraction, then*

(i) $p(Tx, Ty) \le \psi(p(x, Ty) + p(y, Tx))$ for all $x \in A_i$ and $y \in A_{i+1}$;
(ii) $p(Tx, Ty) < \psi(p(x, Ty) + p(y, Tx))$ for all $x \in A_i$ and $y \in A_{i+1}$ such that $p(x, Ty) + p(y, Tx) > 0$;
(iii) $p(T^n x_0, T^{n+1} x_0) \to 0$ as $n \to +\infty$ for all $x_0 \in X$.

Proof. (i) and (ii) hold as ψ is strictly increasing and $\phi(u, v) > 0$ if $u + v > 0$. We prove (iii). Take $x_0 \in X$ and consider the sequence given by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \dots$. As $X = \bigcup_{i=1}^{m} A_i$, for any $n > 0$ there exists $i_n \in \{1, 2, \dots, m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_n+1}$. Since ψ is strictly increasing and

$$p(x_{n-1}, x_{n+1}) + p(x_n, x_n) \le p(x_{n-1}, x_n) + p(x_n, x_{n+1}),$$

using (i), we deduce that

$$\begin{aligned} p(x_n, x_{n+1}) &= p(Tx_{n-1}, Tx_n) \\ &\le \psi(p(x_{n-1}, x_{n+1}) + p(x_n, x_n)) \\ &\le \frac{1}{2}[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)] \\ &\le \frac{1}{2}[p(x_{n-1}, x_n) + p(x_n, x_{n+1})]. \end{aligned} \tag{1}$$

Therefore,

$$p(x_n, x_{n+1}) \le p(x_{n-1}, x_n) \text{ for any } n = 1, 2, \dots.$$

Thus $\{p(x_n, x_{n+1})\}$ is a nonincreasing sequence of non negative real numbers. Consequently, there exists $\alpha \ge 0$ such that

$$\lim_{n \to +\infty} p(x_n, x_{n+1}) = \alpha. \tag{2}$$

Passing to the limit as $n \to +\infty$ in (1) we deduce

$$\alpha \le \lim_{n \to +\infty} \frac{1}{2}[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)] \le \alpha$$

and hence

$$\lim_{n \to +\infty} [p(x_{n-1}, x_{n+1}) + p(x_n, x_n)] = 2\alpha. \tag{3}$$

As T is a cyclic weak (ψ, C)-contraction and ϕ is nondecreasing with respect to the second component, we obtain

$$\begin{aligned} p(x_n, x_{n+1}) &= p(Tx_{n-1}, Tx_n) \\ &\leq \psi(p(x_{n-1}, Tx_n) + p(x_n, Tx_{n-1}) - \phi(p(x_{n-1}, Tx_n), p(x_n, Tx_{n-1}))) \\ &= \psi(p(x_{n-1}, x_{n+1}) + p(x_n, x_n) - \phi(p(x_{n-1}, x_{n+1}), p(x_n, x_n))) \\ &\leq \psi(p(x_{n-1}, x_{n+1}) + p(x_n, x_n) - \phi(p(x_{n-1}, x_{n+1}), 0)). \end{aligned} \tag{4}$$

As $\limsup_{n\to+\infty} p(x_n, x_n) \leq \limsup_{n\to+\infty} p(x_n, x_{n+1}) = \alpha$, by (3) we get

$$\begin{aligned} 2\alpha &= \liminf_{n\to+\infty}[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)] \\ &\leq \liminf_{n\to+\infty} p(x_{n-1}, x_{n+1}) + \limsup p(x_n, x_n) \\ &\leq \liminf_{n\to+\infty} p(x_{n-1}, x_{n+1}) + \alpha. \end{aligned}$$

Therefore

$$\liminf_{n\to+\infty} p(x_{n-1}, x_{n+1}) \geq \alpha. \tag{5}$$

By (3), (5) and the continuity of $\phi(\cdot, 0)$, we obtain

$$\begin{aligned} &\limsup_{n\to+\infty}[p(x_{n-1}, x_{n+1}) + p(x_n, x_n) - \phi(p(x_{n-1}, x_{n+1}), 0)] \\ = \; & 2\alpha - \liminf_{n\to+\infty} \phi(p(x_{n-1}, x_{n+1}), 0) \\ \leq \; & 2\alpha - \phi(\alpha, 0). \end{aligned} \tag{6}$$

Taking the upper limit as $n \to +\infty$ in (4), using (6) and the continuity of ψ, we get

$$\alpha \leq \psi(2\alpha - \phi(\alpha, 0)) \leq \alpha - \frac{1}{2}\phi(\alpha, 0) \leq \alpha.$$

Hence, we have $\phi(\alpha, 0) = 0$, that is, $\alpha = 0$, and so

$$\lim_{n\to+\infty} p(x_n, x_{n+1}) = \lim_{n\to+\infty} p(T^n x_0, T^{n+1} x_0) = 0. \tag{7}$$

□

Lemma 3.2. *Let (X, p) be a partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ nonempty subsets of X and $X = \bigcup_{i=1}^m A_i$. Suppose that $T : X \to X$ is a cyclic weak (ψ, C)-contraction. For every $x_0 \in X$, let $x_n = T^n x_0$, then $\{x_n\}$ is a 0-Cauchy sequence.*

Proof. First, we prove the following claim.

Claim: For every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that if $r > q \geq n$ with $r - q \equiv 1(m)$, then $p(x_r, x_q) < \epsilon$.

In fact, suppose the contrary case. This means that there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$ we can find $r_n > q_n \geq n$ with $r_n - q_n \equiv 1(m)$ satisfying

$$p(x_{q_n}, x_{r_n}) \geq \epsilon. \tag{8}$$

Now, we take $n > 2m$. Then, corresponding to $q_n \geq n$, one can choose r_n in such a way that it is the smallest integer with $r_n > q_n$ satisfying $r_n - q_n \equiv 1(m)$ and $p(x_{q_n}, x_{r_n}) \geq \epsilon$. Therefore, $p(x_{q_n}, x_{r_n - m}) < \epsilon$. By the property (p_4) of a partial metric, we have

$$\begin{aligned} \epsilon &\leq p(x_{q_n}, x_{r_n}) \\ &\leq p(x_{q_n}, x_{r_n - m}) + \sum_{i=1}^m p(x_{r_n - i}, x_{r_n - i + 1}) - \sum_{i=1}^m p(x_{r_n - i}, x_{r_n - i}) \\ &< \epsilon + \sum_{i=1}^m p(x_{r_n - i}, x_{r_n - i + 1}). \end{aligned}$$

Passing to the limit as $n \to +\infty$ in the last inequality and taking into account that $\lim_{n\to+\infty} p(x_n, x_{n+1}) = 0$, we obtain

$$\lim_{n\to+\infty} p(x_{q_n}, x_{r_n}) = \epsilon. \tag{9}$$

Again, by the property (p_4), we have

$$\begin{aligned}\epsilon &\leq p(x_{q_n}, x_{r_n}) \\ &\leq p(x_{q_n}, x_{q_n+1}) + p(x_{q_n+1}, x_{r_n+1}) + p(x_{r_n+1}, x_{r_n}) \\ &\leq p(x_{q_n}, x_{q_n+1}) + p(x_{q_n+1}, x_{q_n}) + p(x_{q_n}, x_{r_n}) + p(x_{r_n}, x_{r_n+1}) + p(x_{r_n+1}, x_{r_n}) \\ &= 2p(x_{q_n}, x_{q_n+1}) + p(x_{q_n}, x_{r_n}) + 2p(x_{r_n}, x_{r_n+1}).\end{aligned} \tag{10}$$

Passing to the limit as $n \to +\infty$ in (10), using $\lim_{n\to+\infty} p(x_n, x_{n+1}) = 0$ and (9), we get

$$\lim_{n\to+\infty} p(x_{q_n+1}, x_{r_n+1}) = \epsilon. \tag{11}$$

In the same way, we deduce that

$$\lim_{n\to+\infty} p(x_{q_n+1}, x_{r_n}) = \lim_{n\to+\infty} p(x_{q_n}, x_{r_n+1}) = \epsilon. \tag{12}$$

Since x_{q_n} and x_{r_n} lie in different adjacently labelled sets A_i and A_{i+1} for certain $1 \leq i \leq m$, using the fact that T is a cyclic weak (ψ, C)-contraction, we have

$$\begin{aligned}p(x_{q_n+1}, x_{r_n+1}) &= p(Tx_{q_n}, Tx_{r_n}) \\ &\leq \psi(p(x_{q_n}, Tx_{r_n}) + p(x_{r_n}, Tx_{q_n}) - \phi(p(x_{q_n}, Tx_{r_n}), p(x_{r_n}, Tx_{q_n}))) \\ &\leq \psi(p(x_{q_n}, x_{r_n+1}) + p(x_{r_n}, x_{q_n+1}) - \phi(p(x_{q_n}, x_{r_n+1}), 0)).\end{aligned}$$

Taking into account (11) and (12) and the continuity of ψ and $\phi(\cdot, 0)$, passing to the limit as $n \to +\infty$ in the last inequality, we obtain

$$\epsilon \leq \psi(2\epsilon - \phi(\epsilon, 0)) \leq \epsilon - \frac{1}{2}\phi(\epsilon, 0) \leq \epsilon$$

and from the last inequality, $\phi(\epsilon, 0) = 0$. From the fact that $\phi(x, y) = 0 \Leftrightarrow x = y = 0$, we have $\epsilon = 0$, which is a contradiction. Therefore, our claim is proved.

Now, we prove that $\{x_n\}$ is a 0-Cauchy sequence. Fix $\epsilon > 0$. By the claim, we find $n_0 \in \mathbb{N}$ such that if $r > q \geq n_0$ with $r - q \equiv 1(m)$

$$p(x_r, x_q) \leq \frac{\epsilon}{2}. \tag{13}$$

Since $\lim_{n\to+\infty} p(x_n, x_{n+1}) = 0$ we also find $n_1 \in \mathbb{N}$ such that

$$p(x_n, x_{n+1}) \leq \frac{\epsilon}{2m} \tag{14}$$

for any $n \geq n_1$. Suppose that $r, s \geq \max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, \ldots, m\}$ such that $s - r \equiv k(m)$. Therefore, $s + j - r \equiv 1(m)$ for $j = m - k + 1$. So, we have

$$\begin{aligned}p(x_r, x_s) &\leq p(x_r, x_{s+j}) + \sum_{i=1}^{j} p(x_{s+i}, x_{s+i-1}) - \sum_{i=1}^{j} p(x_{s+i}, x_{s+i}) \\ &\leq p(x_r, x_{s+j}) + \sum_{i=1}^{j} p(x_{s+i}, x_{s+i-1}).\end{aligned}$$

By (13) and (14) and from the last inequality, we get

$$p(x_r, x_s) \leq \frac{\epsilon}{2} + j\,\frac{\epsilon}{2m} \leq \frac{\epsilon}{2} + m\,\frac{\epsilon}{2m} = \epsilon.$$

This proves that $\{x_n\}$ is a 0-Cauchy sequence. □

The main result of the paper is the following theorem.

Theorem 3.3. *Let (X, p) be a 0-complete partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ closed nonempty subsets of X and $X = \bigcup_{i=1}^m A_i$. Suppose that $T : X \to X$ is a cyclic weak (ψ, C)-contraction. Then, T has a unique fixed point $z \in \bigcap_{i=1}^n A_i$.*

Proof. Take $x_0 \in X$ and consider the sequence $\{x_n\}$ given by $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$. By Lemma 3.2, $\{x_n\}$ is a 0-Cauchy sequence. Since X is a 0-complete partial metric space, there exists $x \in X$ such that

$$\lim_{n\to+\infty} p(x_n, x) = \lim_{n\to+\infty} p(x_n, x_m) = p(x, x) = 0.$$

In what follows, we prove that x is a fixed point of T. In fact, since $\lim_{n\to+\infty} x_n = x$ and, as the family $A_1, A_2, ..., A_m$ is a cyclic representation of X with respect to T, the sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1, 2, \ldots, m\}$. As A_i is closed for all $i \in \{1, 2, \ldots, m\}$, we deduce that $x \in \bigcap_{i=1}^m A_i$. Using the contractive condition, we can obtain

$$\begin{aligned} p(x_{n+1}, Tx) &= p(Tx_n, Tx) \\ &\leq \psi(p(x_n, Tx) + p(x, Tx_n) - \phi(p(x_n, Tx), p(x, Tx_n))) \\ &\leq \psi(p(x_n, Tx) + p(x, x_{n+1}) - \phi(p(x_n, Tx), 0)). \end{aligned}$$

Passing to the limit as $n \to +\infty$ and using $x_n \to x$, Lemma 2.2, continuity of ψ and $\phi(\cdot, 0)$, we have

$$\begin{aligned} p(x, Tx) &\leq \psi(p(x, Tx) - \phi(p(x, Tx), 0)) \\ &\leq \frac{1}{2} p(x, Tx) - \frac{1}{2}\phi(p(x, Tx), 0) \\ &\leq \frac{1}{2} p(x, Tx) \end{aligned}$$

which is a contradiction unless $p(x, Tx) = 0$ and, therefore, x is a fixed point of T. Finally, to prove the uniqueness of the fixed point, we assume that $y, z \in X$ are fixed points of T. The cyclic character of T and the fact that $y, z \in X$ are fixed points of T, imply that $y, z \in \bigcap_{i=1}^m A_i$. If $p(y, z) > 0$, using Lemma 3.1(ii), we obtain

$$p(y, z) = p(Ty, Tx) < \psi(p(y, Tz) + p(z, Ty)) = \psi(2\, p(y, z)) \leq p(y, z).$$

This gives us $p(y, z) = 0$, that is, $y = z$. □

A Maia type result regarding cyclic weak (ψ, C)-contractions is given in the following theorem.

Theorem 3.4. *Let X be a nonempty set, p and ρ two partial metrics on X, $m \in \mathbb{N}$, $A_1, \ldots, A_m$ closed nonempty subsets of (X, p), $X = \bigcup_{i=1}^m A_i$ and $T : X \to X$. Assuming that*

(i) $A_1, \ldots, A_m$ *is a cyclic representation of X with respect to T;*

(ii) $p(x, y) \leq \rho(x, y)$, *for any $x, y \in Y$;*

(iii) (X, p) *is a 0-complete partial metric space;*

(iv) $T : (X, p) \to (X, p)$ *is continuous;*

(v) $T : (X, \rho) \to (X, \rho)$ *is a cyclic weak (ψ, C)-contraction.*

Then T has a unique fixed point.

Proof. Take $x_0 \in X$ and consider the sequence $\{x_n\}$ given by $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$. By Lemma 3.2, $\{x_n\}$ is a 0-Cauchy sequence in (X, ρ). By condition (ii) the sequence $\{x_n\}$ is 0-Cauchy in (X, p). As (X, p) is a 0-complete partial metric space, then there exists $x \in X$ such that $p(x_n, x) \to p(x, x) = 0$. Now, the condition (iv) ensures that

$$x = \lim_{n\to+\infty} x_n = T(\lim_{n\to+\infty} x_{n-1}) = Tx$$

and hence x is a fixed point of T. The uniqueness of the fixed point follows by condition (v). □

Corollary 3.5. *Let (X, p) be a 0-complete partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ closed nonempty subsets of X and $X = \bigcup_{i=1}^m A_i$. Suppose that $T : X \to X$ is a mapping such that*
(1) $A_1, A_2, \ldots, A_m$ is a cyclic representation of X with respect to T;
(ii) there exists $\phi \in \Phi$ such that

$$p(Tx, Ty) \le \frac{1}{2}[p(x, Ty) + p(y, Tx)] - \phi(p(x, Ty), p(y, Tx)) \tag{15}$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$ where $A_{m+1} = A_1$. Then, T has a unique fixed point $z \in \bigcap_{i=1}^n A_i$.

Proof. Let $\psi : [0, +\infty) \to [0, +\infty)$ defined by $\psi(t) = \frac{1}{2}t$ for all $t \ge 0$. Since

$$p(Tx, Ty) \le \frac{1}{2}[p(x, Ty) + p(y, Tx)] - 2\phi(p(x, Ty), p(y, Tx))],$$

$2\phi \in \Phi$ and $\psi \in \Psi$ applying Theorems 3.3, we deduce that T has a unique fixed point $z \in \bigcap_{i=1}^n A_i$. □

In the setting of metric spaces, Corollary 3.5 is related to Theorem 2.1 of [8]. If in Corollary 3.5, we take $\phi(t_1, t_2) = a(t_1 + t_2)$ with $0 < a < 1/2$, we obtain the following result.

Corollary 3.6. *Let (X, p) be a 0-complete partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ closed nonempty subsets of X and $X = \bigcup_{i=1}^m A_i$. Suppose that $T : X \to X$ is a mapping such that*
(1) $A_1, A_2, \ldots, A_m$ is a cyclic representation of X with respect to T;
(ii) there exists $k \in [0, 1/2)$ such that

$$p(Tx, Ty) \le k[p(x, Ty) + p(y, Tx)] \tag{16}$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$ where $A_{m+1} = A_1$. Then, T has a unique fixed point $z \in \bigcap_{i=1}^n A_i$.

If in Corollary 3.6, (X, p) is a metric space, we obtain Theorem 3 of [23]. The following corollary gives us a fixed point theorem with a contractive condition of integral type for cyclic contractions.

Corollary 3.7. *Let (X, p) be a 0-complete partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ closed nonempty subsets of X and $X = \bigcup_{i=1}^m A_i$. Suppose that $T : X \to X$ is a mapping such that*
(1) $A_1, A_2, \ldots, A_m$ is a cyclic representation of X with respect to T;
(ii) there exists $\phi \in \Phi$ such that

$$p(Tx, Ty) \le \int_0^{p(x,Ty)+p(y,Tx)-\phi(p(x,Ty),p(y,Tx))} \sigma(t)dt$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$ where $A_{m+1} = A_1$, and $\sigma : [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable function satisfying $0 < \int_0^\epsilon \sigma(t)dt \le \frac{1}{2}\epsilon$ for $\epsilon > 0$. Then T has a unique fixed point $z \in \bigcap_{i=1}^m A_i$.

If in Corollary 3.7, we take $A_i = X$ for $i = 1, 2, \ldots, m$, we obtain the following result.

Corollary 3.8. *Let (X, p) be a 0-complete partial metric space and $T : X \to X$ is a mapping such that, for any $x, y \in X$, we have*

$$p(Tx, Ty) \leq \int_0^{p(x,Ty)+p(y,Tx)-\phi(p(x,Ty),p(y,Tx))} \sigma(t)dt$$

where $\sigma : [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable function satisfying $0 < \int_0^{\epsilon} \sigma(t)dt \leq \frac{1}{2}\epsilon$ for $\epsilon > 0$ and $\phi \in \Phi$. Then T has a unique fixed point.

If in Theorem 3.3 we put $A_i = X$ for $i = 1, 2, \ldots, m$ we have the following result (see [8]).

Corollary 3.9. *Let (X, p) be a 0-complete partial metric space and $T : X \to X$ a mapping such that, for any $x, y \in X$, we have*

$$p(Tx, Ty) \leq \psi(p(x, Ty) + d(y, Tx) - \phi(p(x, Ty) + p(y, Tx))),$$

where $\phi \in \Phi$ and $\psi \in \Psi$. Then T has a unique fixed point.

Example 3.10. *Let $X = [0, 1] \cap \mathbb{Q}$ and $p : X \times X \to \mathbb{R}$ defined by $p(x, y) = \max\{x, y\}$, then (X, p) is a 0-complete partial metric space. Let $A_1 = A_2 = \cdots = A_m = X$. Define $T : X \to X$, $\psi : [0, +\infty) \to [0, +\infty)$ and $\phi : [0, +\infty)^2 \to [0, +\infty)$ by $Tx = \frac{x^2}{2(1+x)}$ for all $x \in X$, $\psi(t) = \frac{t}{2}$ for all $t \geq 0$ and*

$$\phi(t_1, t_2) = \frac{t_1^2}{1 + t_1} + \frac{t_2^2}{1 + t_2}, \quad \text{for all } t_1, t_2 \geq 0.$$

We prove that T is a cyclic weak (ψ, C)-contraction.

Take $x, y \in X$ and assume $y \leq x$. Then

$$p(Tx, Ty) = \frac{x^2}{2(1 + x)}, \quad \phi(p(x, Ty), p(y, Tx)) = \frac{x^2}{1 + x} + \frac{[\max\{y, Tx\}]^2}{1 + \max\{y, Tx\}}$$

and hence

$$\begin{aligned} & \psi(p(x, Ty) + p(y, Tx) - \phi(p(x, Ty), p(y, Tx))) \\ = \; & \frac{1}{2}[x + \max\{y, \frac{x^2}{2(1 + x)}\} - \frac{x^2}{1 + x} - \frac{[\max\{y, Tx\}]^2}{1 + \max\{y, Tx\}}] \\ \geq \; & \frac{1}{2}[x - \frac{x^2}{1 + x}] = \frac{1}{2}\frac{x}{1 + x} \geq \frac{x^2}{2(1 + x)} = p(Tx, Ty). \end{aligned}$$

Therefore T is a cyclic weak (ψ, C)-contraction and so T has a unique fixed point by Theorem 3.3.

On the other hand, for the same problem in the standard metric $d(x, y) = |x - y|$ it is not possible to make use of other results for deduce that T has a unique fixed point, since (X, d) is not complete.

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