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Iterative Methods for the Class of Quasi-Contractive Type Operators and Comparsion of their Rate of Convergence in Convex Metric Spaces

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Abstract. We introduce modified Noor iterative method in a convex metric space and apply it to approximate fixed points of quasi-contractive operators introduced by Berinde [6]. Our results generalize and improve upon, among others, the corresponding results of Berinde [6], Bosede [9] and Phuengrattana and Suantai [20]. We also compare the rate of convergence of proposed iterative method to the iterative methods due to Noor [26], Ishikawa [14] and Mann [18]. It has been observed that the proposed method is faster than the other three methods. Incidently the results obtained herein provide analogues of the corresponding results of normed spaces and holds in $CAT(0)$ spaces, simultaneously.

1. Introduction

Let C be a nonempty convex subset of a normed space E and $T : C \to C$ be an operator. In 2005, Suantai [23] introduced the following modified Noor iterative method with $x_1 = x \in C$ and

$$\begin{aligned} z_n &= a_n T x_n + (1 - a_n) x_n \\ y_n &= b_n T z_n + c_n T x_n + (1 - b_n - c_n) x_n \\ x_{n+1} &= \alpha_n T y_n + \beta_n T z_n + (1 - \alpha_n - \beta_n) x_n, \; n \geq 1 \end{aligned} \tag{1}$$

where $0 \leq \alpha_n, \beta_n, a_n, b_n, c_n \leq 1$. He used (1) (Replacing T by T^n) to approximate fixed points of asymptotically nonexpansive operators in a uniformly convex Banach space. Moreover, (1) can be viewed as a generalization of the iterative methods by Noor [19], Glowinski and Le Tallec [13], Xu and Noor [26], Ishikawa [14] and Mann [18], simultaneously.

An extension of a linear version (usually in normed spaces) of a known result to metric fixed point theory has its own importance. As (1) involves general convex combinations, we need some convex structure in

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a metric space to translate it on a nonlinear domain. A mapping $W : X^2 \times [0,1] \to X$ is called a convex structure on a metric space (X, d) [24] if

$$d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1-\alpha) d(u, y) \tag{2}$$

for all $x, y, u \in X$ and $\alpha \in [0,1]$. The triplet (X, d, W) is known as a convex metric space and will be denoted simply by X. A nonempty subset C of a convex metric space X is convex if $W(x, y, \alpha) \in C$ for all $x, y \in C$ and $\alpha \in [0,1]$. An important and interesting example of convex metric spaces is a $CAT(0)$ space . To get details for $CAT(0)$ spaces, we refer the reader to [10, 12, 16].

From the definition of convex structure W on X, it is obvious that

$$d(u, W(x, y, \alpha)) \geq (1-\alpha) d(u, y) - \alpha d(u, x) \tag{3}$$

for all $x, y, u \in X$ and $\alpha \in [0,1]$.

Let C be a nonempty convex subset of a convex metric space X and $T : C \to C$ be an operator. Let $\{\alpha_n\}$, $\{\beta_n\}, \{a_n\}, \{b_n\}$ and $\{c_n\}$ be appropriately chosen sequences in $[0,1]$.

Then for a given $x_1 \in C$, we compute the sequence $\{x_n\}$ by

$$\begin{aligned} z_n &= W(Tx_n, x_n, a_n), \\ y_n &= W\left(Tz_n, W\left(Tx_n, x_n, \tfrac{c_n}{1-b_n}\right), b_n\right), \\ x_{n+1} &= W\left(Ty_n, W\left(Tz_n, x_n, \tfrac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), \quad n \geq 1. \end{aligned} \tag{4}$$

Uusing " $W(x, y, 0) = y$ for any x, y in X ([25], Proposition 1.2(a))", the iterative method (4) provides, in convex metric spaces, analogues of:

Noor iterative sequence $\left\{x_n^{(3)}\right\}$[26] (with $c_n = \beta_n = 0$)

$$\begin{aligned} z_n^{(3)} &= W\left(Tx_n^{(3)}, x_n^{(3)}, a_n\right), \\ y_n^{(3)} &= W\left(Tz_n^{(3)}, x_n^{(3)}, b_n\right), \\ x_{n+1}^{(3)} &= W\left(Ty_n^{(3)}, x_n^{(3)}, \alpha_n\right), \quad n \geq 1 \end{aligned} \tag{5}$$

Ishikawa iterative sequence $\left\{x_n^{(2)}\right\}$ [14] (with $a_n = c_n = \beta_n = 0$)

$$\begin{aligned} y_n^{(2)} &= W\left(Tx_n^{(2)}, x_n^{(2)}, b_n\right) \\ x_{n+1}^{(2)} &= W\left(Ty_n^{(2)}, x_n^{(2)}, \alpha_n\right), \quad n \geq 1 \end{aligned} \tag{6}$$

and Mann iterative sequence $\left\{x_n^{(1)}\right\}$ [18] (with $a_n = b_n = c_n = \beta_n = 0$)

$$x_{n+1}^{(1)} = W\left(Tx_n^{(1)}, x_n^{(1)}, \alpha_n\right), \quad n \geq 1. \tag{7}$$

If X is a normed space and $W(x, y, \alpha) = \alpha x + (1-\alpha) y$, then (4) coincides with (1).

Berinde [6] introduced a new class of quasi-contractive type operators and used Ishikawa iterative method [14] to approximate fixed points of this new class of operators in a normed space. To appreciate this class of operators, we recall some definitions.

An operator $T : X \to X$ is (i) an $a-$ contraction if

$$d(Tx, Ty) \leq a d(x, y) \text{ for all } x, y \in X, \tag{8}$$

where $0 < a < 1$,

(ii) Kannan operator [15] if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X, \tag{9}$$

(iii) Chatterjea operator [11] if there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \text{ for all } x, y \in X. \tag{10}$$

Combining the definitions (8)-(10), Zamfirescu [27] proved the following important result.

Theorem 1.1. *Let X be a complete metric space and $T : X \to X$ be an operator for which there exist real numbers a, b and c satisfying $0 < a < 1$, $b, c \in (0, \frac{1}{2})$ such that for $x, y \in X$, at least one of the follwing conditions holds:*
(z_1) $d(Tx, Ty) \leq ad(x, y)$
(z_2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$
(z_3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.
Then T has a unique fixed point p and the Picard iteration $\{x_n\}$ defined by: $x_1 \in X$, $x_{n+1} = Tx_n, n \geq 1$, converges to p.

An operator T satisfying the contractive conditions $(z_1) - (z_3)$ is called Zamfirescu operator. The class of Zamfirescu operators is one of the most studied class of contractive type operators. For this class of operators, Mann iterative method [18] and Ishikawa iterative method [14] are known to converge to a unique fixed point of T in a normed space.

An operator T on a metric space X is a quasi-contractive type [6] if

$$d(Tx, Ty) \leq \delta d(x, y) + \varepsilon d(Tx, x) \tag{11}$$

for any $x, y \in X$, $0 < \delta < 1$ and $\varepsilon \geq 0$.

Note that the contractive condition (8) includes T as a continuous operator on X while this is not the case with the contractive conditions (9)-(11).

Berinde [6] proved that the class of operators satisfying the condition (11) is wider than the class of Zamfirescu operators. He used Ishikawa iterative method [14] to approximate fixed points of this class of operators in a normed space. Actually, he proved:

Theorem 1.2. *Let C be a nonempty, bounded, closed and convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying (11). Let $\{x_n\}$ be defined by: $x_1 \in C, x_{n+1} = \alpha_n T[\beta_n Tx_n + (1 - \beta_n) x_n] + (1 - \alpha_n) x_n$ where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T)$(the set of fixed points of T) is nonempty, then $\{x_n\}$ converges strongly to a fixed point of T.*

In this paper, we prove a convergence therorem in a convex metric space for modified Noor iterative method (4) of quasi-contractive type operator (11). As a consequence, our results generalize, improve and unify a number of results including the results of Berinde [3, 6], Bosede [9] and Phuengrattana and Suantai [20] in Banach spaces. We also compare the rate of convergence of our new iterative method (4) with classical iterative methods in convex metric spaces.

2. Main Results

The main theorem of this paper goes as follows.

Theorem 2.1. *Let C be a nonempty, closed and convex subset of a convex metric space X. Let $T : C \to C$ be an operator satisfying (11). If $F(T) \neq \emptyset$ and $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{b_n + c_n\}$ and $\{\alpha_n + \beta_n\}$ are sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} (\alpha_n + \beta_n) = \infty$, then $\{x_n\}$ in (4), converges to a unique fixed point p of T.*

Proof. Applying inequality (2) to the sequences $\{z_n\}$ and $\{y_n\}$ in (4) with $p \in F(T)$, we estimate

$$d(z_n, p) = d(W(Tx_n, x_n, a_n), p) \leq a_n d(Tx_n, p) + (1 - a_n) d(x_n, p) \leq a_n \delta d(x_n, p) + (1 - a_n) d(x_n, p)$$

$$\leq (1 - a_n(1 - \delta)) d(x_n, p) \tag{12}$$

and

$$d(y_n,p) = d\left(W\left(Tz_n, W\left(Tx_n, x_n, \frac{c_n}{1-b_n}\right), b_n\right), p\right) \le b_n d(Tz_n,p) + (1-b_n)\, d\left(W\left(Tx_n, x_n, \frac{c_n}{1-b_n}\right), p\right)$$

$$\le b_n d(Tz_n,p) + (1-b_n)\left(\frac{c_n}{1-b_n} d(Tx_n,p) + \left(1-\frac{c_n}{1-b_n}\right) d(x_n,p)\right)$$

$$= b_n d(Tz_n,p) + c_n d(Tx_n,p) + (1-b_n-c_n) d(x_n,p) \le b_n \delta d(z_n,p) + c_n \delta d(x_n,p) + (1-b_n-c_n) d(x_n,p). \quad (13)$$

Inserting (12) into (13), we get that

$$d(y_n,p) \le (b_n\delta(1-a_n(1-\delta)) + c_n\delta + (1-b_n-c_n))\, d(x_n,p). \quad (14)$$

Again applying inequality (2) to the sequence $\{x_n\}$ in (4), we obtain

$$\begin{aligned} d(x_{n+1},p) &= d\left(W\left(Ty_n, W\left(Tz_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) \\ &\le \alpha_n d(Ty_n,p) + (1-\alpha_n)\, d\left(W\left(Tz_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\ &\le \alpha_n d(Ty_n,p) \\ &+ (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n} d(Tz_n,p) + \left(1-\frac{\beta_n}{1-\alpha_n}\right) d(x_n,p)\right) \\ &= \alpha_n d(Ty_n,p) + \beta_n d(Tz_n,p) + (1-\alpha_n-\beta_n) d(x_n,p) \\ &\le \alpha_n \delta d(y_n,p) + \beta_n \delta d(z_n,p) + (1-\alpha_n-\beta_n) d(x_n,p). \end{aligned} \quad (15)$$

Finally substituting (12) and (14) into (15) and simplifying, we get

$$\begin{aligned} d(x_{n+1},p) &\le \alpha_n d(Ty_n,p) + \beta_n d(Tz_n,p) + (1-\alpha_n-\beta_n) d(x_n,p) \\ &\le \alpha_n \delta d(y_n,w) + \beta_n \delta d(z_n,p) + (1-\alpha_n-\beta_n) d(x_n,p) \\ &\le \left\{ \begin{array}{c} \alpha_n\delta\,[b_n\delta(1-a_n(1-\delta)) + c_n\delta + (1-b_n-c_n)] \\ +\beta_n\delta(1-a_n(1-\delta)) + (1-\alpha_n-\beta_n) \end{array} \right\} d(x_n,p) \\ &\le \left\{ \begin{array}{c} 1-(1-\delta)\,\alpha_n\,[1+a_nb_n+(b_n+c_n)\,\delta] \\ -(1-\delta)\,\beta_n\,[1+a_n\delta] \end{array} \right\} d(x_n,p) \\ &\le (1-(1-\delta)\,\alpha_n - (1-\delta)\,\beta_n)\, d(x_n,p) \\ &= (1-(1-\delta)(\alpha_n+\beta_n))\, d(x_n,p) \\ &\quad . \\ &\quad . \\ &\quad . \\ &\le \prod_{k=1}^{n} [1-(1-\delta)(\alpha_k+\beta_k)]\, d(x_1,p). \end{aligned}$$

That is,

$$d(x_{n+1},p) \le \prod_{k=1}^{n} [1-(1-\delta)(\alpha_k+\beta_k)]\, d(x_1,p). \quad (16)$$

In view of the fact that $1+x \le e^x$ for $x \ge 0$, the inequality (16) reduces to

$$d(x_{n+1},p) \le \prod_{k=1}^{n} e^{-(1-\delta)(\alpha_k+\beta_k)} d(x_1,p)$$

$$= e^{-(1-\delta)\sum_{k=1}^{n}(\alpha_k+\beta_k)} d(x_1, p) = \frac{1}{e^{(1-\delta)\sum_{k=1}^{n}(\alpha_k+\beta_k)}} d(x_1, p).$$

Since $\sum_{k=1}^{\infty}(\alpha_k + \beta_k) = \infty$, therefore

$$d(x_{n+1}, p) \to 0 \text{ as } n \to \infty.$$

Consequently, $x_n \to p \in F(T)$.

Next we show that $F(T) = \{p\}$. To prove this, we take $p_1, p_2 \in F(T)$ and $p_1 \neq p_2$.

Using quasi-contractive condition (11), we have

$$d(p_1, p_2) = d(Tp_1, p_2) \leq \delta d(p_1, p_2) < d(p_1, p_2),$$

a contradiction. Therefore $F(T) = \{p\}$. □

The following results are the immediate consequences of Theorem 2.1.

Theorem 2.2. *Let C be a nonempty, closed and convex subset of a convex metric space X. Let $T : C \to C$ be an operator satisfying (11). If $F(T) \neq \emptyset$ and $\{\alpha_n\}, \{b_n\}$ and $\{a_n\}$ are sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\{x_n^{(3)}\}$ in (5) with initial point x_1, converges to a unique fixed point p of T.*

Proof. Choose $c_n = 0 = \beta_n$ in Theorem 2.1. □

Theorem 2.3. *Let C be a nonempty, closed and convex subset of a convex metric space X. Let $T : C \to C$ be an operator satisfying (11). If $F(T) \neq \emptyset$ and $\{\alpha_n\}, \{b_n\}$ are sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\{x_n^{(2)}\}$ in (6) with initial point x_1, converges to a unique fixed point p of T.*

Proof. Take $a_n = c_n = \beta_n = 0$ in Theorem 2.1. □

Theorem 2.4. *Let C be a nonempty, closed and convex subset of a convex metric space X. Let $T : C \to C$ be an operator satisfying (11). If $F(T) \neq \emptyset$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\{x_n^{(1)}\}$ in (7) with initial point x_1, converges to a unique fixed point p of T.*

Proof. Set $a_n = b_n = c_n = \beta_n = 0$ in Theorem 2.1. □

In 2004, Berinde [4] gave the concept to compare the rate of convergence of different iteration methods, see also [1, 2, 5, 7, 8, 17, 21, 22].

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive numbers that converge to a, b, respectively and

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = l.$$

If $l = 0$, then it is said that $\{a_n\}$ converges to a faster than $\{b_n\}$ to b. If $0 < l < \infty$, then we say that $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.

Suppose that for two fixed point iterative sequences $\{x_n\}$ and $\{y_n\}$ converging to the same fixed point z of T, the error estimates $d(x_n, z) \leq a_n$ and $d(y_n, z) \leq b_n$ for all $n \geq 1$, are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive real numbers (converging to zero). Then, in view of above definition, the following concept appears to be very natural.

If $\{a_n\}$ converges faster than $\{b_n\}$, then we say that the fixed point iterative sequence $\{x_n\}$ converges faster than the fixed point iterative sequence $\{y_n\}$ to z.

It has been observed that the comparison of the rate of convergence in the above definition depends on the choice of sequences $\{a_n\}$ and $\{b_n\}$ which are error bounds of $\{x_n\}$ and $\{y_n\}$, respectively. This method of comparing the rate of convergence of two fixed point iterative sequences seems to be unclear. In 2013, Phuengrattana and Suantai [20] modified his concept as follows:

If $\{x_n\}$ and $\{y_n\}$ are two iterative sequences converging to the same fixed point z of T, then we say that $\{x_n\}$ converges faster than $\{y_n\}$ to z if

$$\lim_{n\to\infty} \frac{d(x_n, z)}{d(y_n, z)} = 0.$$

The following result compares the rate of convergence of the proposed iterative method (4) with the classical iterative methods due to Mann[18] , Ishikawa [14] and Noor [19] for quasi-contractive operators.

Theorem 2.5. *Let C be a nonempty, closed and convex subset of a convex metric space X. Let $T : C \to C$ be an operator satisfying (11). If $F(T) \neq \emptyset$ and $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{b_n + c_n\}$ and $\{\alpha_n + \beta_n\}$ are sequences in $[0, 1]$ which satisfy the conditions:*
(C1): $0 < \eta < \alpha_n + \beta_n < 1$
(C2): $0 < \alpha_n < \frac{1}{1+\delta}, \sum_{n=1}^{\infty} (\alpha_n + \beta_n) = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$, then $\{x_n\}$ in (4), converges faster than Noor iterative method (5), Ishikawa iterative method (6) and Mann iterative method (7), to a unique fixed point of T provided that all the methods have the same initial guess x_1.

Proof. By Theorems 2.1-2.4, the sequences $\{x_n\}$, $\left\{x_n^{(3)}\right\}$, $\left\{x_n^{(2)}\right\}$ and $\left\{x_n^{(1)}\right\}$generated by (4-7), respectively, converges to a unique fixed point p of T.

Applying the inequality (3) to Noor iterations (5), we have

$$\begin{aligned}
d\left(x_{n+1}^{(3)}, p\right) &\geq (1 - \alpha_n) d\left(x_n^{(3)}, p\right) - \alpha_n d\left(T y_n^{(3)}, p\right) \\
&\geq (1 - \alpha_n) d\left(x_n^{(3)}, p\right) - \alpha_n d\left(T y_n^{(3)}, p\right) \\
&\geq (1 - \alpha_n) d\left(x_n^{(3)}, p\right) - \alpha_n \delta d\left(y_n^{(3)}, p\right) \\
&= (1 - \alpha_n) d\left(x_n^{(3)}, p\right) - \alpha_n \delta \left(W\left(T z_n^{(3)}, x_n^{(3)}, b_n\right), p\right) \\
&\geq (1 - \alpha_n - \alpha_n \delta (1 - b_n)) d\left(x_n^{(3)}, p\right) \\
&\quad -\alpha_n b_n \delta^2 (1 - b_n (1 - \delta)) d\left(x_n^{(3)}, p\right) \\
&\geq (1 - \alpha_n (1 + \delta)) d\left(x_n^{(2)}, p\right) \\
&\quad . \\
&\quad . \\
&\quad . \\
&\geq \prod_{k=1}^{n} (1 - \alpha_k (1 + \delta)) d(x, p)
\end{aligned}$$

Similarly for Mann iterative sequence $\left\{x_n^{(1)}\right\}$ and Ishikawa iterative sequence $\left\{x_n^{(2)}\right\}$, we can calculate that

$$d\left(x_{n+1}^{(i)}, p\right) \geq \prod_{k=1}^{n} (1 - \alpha_n (1 + \delta)) d\left(x_n^{(i)}, p\right) \text{ where } i = 1, 2 \tag{17}$$

Combining (17) with the similar inequality obtained for Noor iterations, we have

$$d\left(x_{n+1}^{(i)}, p\right) \geq (1 - \alpha_n (1 + \delta)) d\left(x_n^{(i)}, p\right) \text{ where } i = 1, 2, 3. \tag{18}$$

It follows by (C1), (16) and (18) that

$$\frac{d(x_{n+1}, p)}{d\left(x_{n+1}^{(i)}, p\right)} \leq \frac{(1 - \eta (1 - \delta))^n}{\prod_{k=1}^{n} (1 - \alpha_k (1 + \delta))}.$$

Let $\theta_n = \frac{(1-\eta(1-\delta))^n}{\prod\limits_{k=1}^{n}(1-\alpha_k(1+\delta))}$. By the assumption in (C2), we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{\theta_{n+1}}{\theta_n} &= \lim_{n\to\infty}\frac{(1-\eta(1-\delta))^{n+1}}{\prod\limits_{k=1}^{n+1}(1-\alpha_k(1+\delta))\,d\left(x_1^{(2)},p\right)}\cdot\frac{\prod\limits_{k=1}^{n}(1-\alpha_k(1+\delta))\,d\left(x_1^{(3)},p\right)}{(1-\eta(1-\delta))^n}\\ &= \lim_{n\to\infty}\frac{(1-\eta(1-\delta))}{(1-\alpha_{n+1}(1+\delta))} = (1-\eta(1-\delta)) < 1.\end{aligned}$$

By the ratio test of real sequences, we conclude that $\sum_{n=1}^{\infty}\theta_n < \infty$. That is, $\lim_{n\to\infty}\theta_n = 0$.

Hence the sequence $\{x_n\}$ generated by (4) converges faster than all the three classical iterative methods, namely, Noor iterations, Ishikawa iterations and Mann iterations. □

Now approximation of unique fixed point of Zamfirescu operators by iterative method (4) in convex metric spaces, can be formulated as follows.

Theorem 2.6. *Let C be a nonempty, closed and convex subset of a complete convex metric space X. Let $T : C \to C$ be a Zamfirescu operator. If $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$, $\{b_n + c_n\}$ and $\{\alpha_n + \beta_n\}$ are sequences in $[0,1]$ such that $\sum_{n=1}^{\infty}(\alpha_n + \beta_n) = \infty$, then $\{x_n\}$ in (4), converges to a unique fixed point of T.*

Proof. The Zamfirescu operator T has a unique fixed point by Theorem 1.1 and satisfies the inequality (11). Therefore the conclusion follows from Theorem 2.1. □

The following result compares the rate of convergence of the proposed iterative method (4) and the classical iterative methods of Mann[18], Ishikawa [14] and Noor [19] for Zamfirescu operator.

Theorem 2.7. *Let C be a nonempty, closed and convex subset of a convex metric space X. Let $T : C \to C$ be a Zamfirescu operator. If $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$, $\{b_n + c_n\}$ and $\{\alpha_n + \beta_n\}$ are sequences in $[0,1]$ which satisfy the conditions:*
(C1): $0 < \eta < \alpha_n + \beta_n < 1$
(C2): $0 < \alpha_n < \frac{1}{1+\delta}$, $\sum_{n=1}^{\infty}(\alpha_n + \beta_n) = \infty$ and $\lim_{n\to\infty}\alpha_n = 0$, then $\{x_n\}$ in (4), converges faster than Noor iterative method (5), Ishikawa iterative method (6) and Mann iterative method (7) to a unique fixed point of T provided that all the methods have the same initial guess x_1.

Proof. The Zamfirescu operator T has a unique fixed point by Theorem 1.1 and satisfies the inequality (11). Therefore the conclusion follows from Theorem 2.1. □

The following results in normed spaces, are immediate consequences of our theorems.

Theorem 2.8. *([20],Theorem 2.1). Let C be a nonempty, closed and convex subset of a Banach space. Let $T : C \to C$ be an operator satisfying (11). If $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$, $\{b_n + c_n\}$ and $\{\alpha_n + \beta_n\}$ are sequences in $[0,1]$ such that $\sum_{n=1}^{\infty}(\alpha_n + \beta_n) = \infty$, then $\{x_n\}$ in (1), converges to a unique fixed point p of T.*

Theorem 2.9. *([20],Theorem 2.2). Let C be a nonempty, closed and convex subset of a Banach space. Let $T : C \to C$ be an operator satisfying (11). If $\{a_n\}, \{b_n\}, \{\alpha_n\}$ are sequences in $[0,1]$ such that $\sum_{n=1}^{\infty}\alpha_n = \infty$, then Mann iterative sequences $\{x_n^{(1)}\}$, Ishikawa iterative sequence $\{x_n^{(2)}\}$ and Noor iterative sequence $\{x_n^{(3)}\}$ converge to a unique fixed point p of T.*

Theorem 2.10. *([20],Theorem 2.4). Let C be a nonempty, closed and convex subset of a Banach space. Let $T : C \to C$ be an operator satisfying (11). If $\{a_n\}, \{b_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $0 \le b_n + c_n, \alpha_n + \beta_n \le 1$ which satisfy the conditions:*
(C1): $0 < \eta < \alpha_n + \beta_n < 1$
(C2): $0 < \alpha_n < \frac{1}{1+\delta}$, $\sum_{n=1}^{\infty}(\alpha_n + \beta_n) = \infty$ and $\lim_{n\to\infty}\alpha_n = 0$, then $\{x_n\}$ in (4) with $c_n = 0$, converges faster than Noor iterative method (5), Ishikawa iterative method (6 and Mann iterative method (7) to a unique fixed point of T provided that all the methods have the same initial guess x_1.

Theorem 2.11. *([9],Theorem 3.1). Let* $(E, \|.\|)$ *be a Banach space,* $T : E \to E$ *a self mapping of E with a fixed point p, satisfying the contractive condition:* $\left\|Tx - p\right\| \leq a \left\|x - p\right\|$ *for each* $x \in E$ *and* $0 \leq a < 1$*. For* $x_1^{(3)} \in E$*, let* $\{x_n^{(3)}\}$ *be the Noor iteration method in (5) where* $\{\alpha_n\}, \{b_n\}$ *and* $\{a_n\}$ *are real sequences in* $[0, 1]$ *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$*. Then* $\{x_n^{(3)}\}$ *converges to p.*

Theorem 2.12. *([3],Theorem 2).Let C be a nonempty closed convex subset of an arbitrary Banach space E and* $T : C \to C$ *be a Zamfirescu operator. If* $\{\alpha_n\}$ *and* $\{b_n\}$ *are sequences in* $[0, 1]$ *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *, then* $\{x_n^{(2)}\}$ *in (6), converges to a unique fixed point p of T.*

Remark 2.13. $CAT(0)$ *space is a convex metric space by taking* $W(x, y, \lambda) = \lambda x \oplus (1 - \lambda)y$*, the geodesic path between the points x and y in a* $CAT\,(0)$ *space X, therefore our results also hold in* $CAT(0)$ *spaces, simultaneously.*

References

[1] G.V. Babu, K.N. Prasad, Mann iteration converges faster than Ishikawa iteration for the class of Zamfirescu operators, Fixed Point Theory and Applications, 2006 (2006), Article ID 49615, 6 pages.
[2] V. Berinde, Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators, Fixed Point Theory and Applications, 1 (2004) 1–9.
[3] V. Berinde, On the convergence of Ishikawa iteration in the class of quasi contractive operators, Acta Mathematica Universitatis Comenianae, 73 (2004) 119–126.
[4] V. Berinde, Comparing Krasnoselskij and Mann iterative methods for Lipschitzian generalized pseudo-contractions, International Conference on Fixed Point Theory and Applications, 15–26, Yokohama Publ., Yokohama, 2004.
[5] V. Berinde, M. Berinde, The fastest Krasnoselskij iteration for approximating fixed points of strictly pseudo-contractive mappings, Carpathian Journal of Mathematics 21 (2005) 13–20.
[6] V. Berinde, A convergence theorem for some mean value fixed point iterations procedures, Demonstratio Mathematica, 38 (2005) 177–184.
[7] V. Berinde, Iterative approximation of fixed points, (2nd edition), Lectures Notes in Mathematics 1912, Springer, Berlin, 2007.
[8] V. Berinde, On a notion of rapidity of convergence used in the study of fixed point iterative methods, Creat. Math. Inform. 25 (2016) 29–40.
[9] A. O. Bosede, Strong convergence of Noor iteration for a general class of functions, Bulletin of Mathematical Analysis and Applications 1 (2011) 140-145.
[10] M. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Springer-Verlag, Berlin, Heidelberg, New York, 1999.
[11] S. K. Chatterjea, Fixed point theorems, Comptes Rendus de l'Academie Bulgare des Sciences 25 (1972) 727–730.
[12] H. Fukhar-ud-din, Strong convergence of an Ishikawa type algorithm in $CAT\,(0)$ spaces, Fixed Point Theory and Applications, 2013, 2013:207.
[13] R. Glowinski, P. Le Tallec, Augmented Lagrangian and operator-splitting methods in nonlinear mechanics, SIAM, Philadelphia,1989.
[14] S. Ishikawa, Fixed points by a new iteration method, Proceedings of the American Mathematical Society 44 (1974) 147–150.
[15] R. Kannan, Some results on fixed points, Bulletin of Calcutta Mathematical Society 10 (1968) 71–76.
[16] A. R. Khan, M. A. Khamsi and H. Fukhar-ud-din, Strong convergence of a general iteration scheme in CAT(0)-spaces, Nonlinear Analysis 74 (2011) 783–791.
[17] A. R. Khan, V. Kumar and N. Hussain, Analytical and numerical treatment of Jungck-type iterative schemes, Applied Mathematics and Computation 231 (2014) 521–535.
[18] W. R. Mann, Mean value methods in iterations, Proceedings of the American Mathematical Society 4 (1953) 506–510.
[19] M. A. Noor, New approximation schemes for general variational ineqaulities, Journal of Mathematical Analysis and Applications 251 (2000) 217–229.
[20] W. Phuengrattana and S. Suantai, Comparison of the rate of convergence of various iterative methods for the class of weak contractions in Banach spaces, Thai Journal of Mathematics 11 (2013) 217–226.
[21] B. E. Rhoades, Fixed point iteration using infinite matrices, Transactions of the American Mathematical Society 196 (1974) 161–176.
[22] B. E. Rhoades and Z. Xue, Comparison of the rate of convergence among Picard, Mann, Ishikawa, and Noor iterations applied to quasi-contractive maps, Fixed Point Theory and Applications, 2010 (2010), Article ID 169062, 12 pages.
[23] S. Suantai, Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings, Journal of Mathematical Analysis and Applications 311 (2005) 506–517.
[24] W. Takahashi, A convexity in metric spaces and nonexpansive mappings, Kodai Mathematical Seminar Reports 22 (1970) 142–149.
[25] L. A. Talman, Fixed points for condensing multifunctions in metric spaces with convex structure, Kodai Mathematical Seminar Reports 29 (1977) 62–70.
[26] B. Xu and M. A. Noor, Ishikawa and Mann iteration process with errors for nonlinear strongly accretive operator equations, Journal of Mathematical Analysis and Applications 224 (1998) 91–101.
[27] T. Zamfirescu, Fix point theorems in metric spaces, Archiv der Mathematik (Basel) 23 (1972) 292–298.