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APPROXIMATING FIXED POINTS OF WEAK φ-CONTRACTIONS USING THE PICARD ITERATION

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Abstract. Two fixed point theorems for weak contractions, established in [4], are extended to the more general class of weak φ-contractions. These results also extend and improve several results in literature.

1. Introduction

In [4], the author introduced and studied the class of the so called *weak contractions.*

Let (X,d) be a metric space and $T:X\longrightarrow X$ a self operator. T is said to be a **weak contraction** if there exist a constant $\delta\in(0,1)$ and some $L\geq 0$ such that

$$d(Tx,Ty)\leq \delta\, d(x,y)+L\, d(y,Tx)\,,\quad \text{for all } \ x,y\in X\,. \tag{1.1}$$

Note that, due to the symmetry of the distance, the weak contraction condition (1.1) implicitly includes the following dual inequality

$$d(Tx,Ty)\leq \delta\cdot d(x,y)+L\cdot d(x,Ty)\,,\quad \text{for all } \ x,y\in X\,, \tag{1.2}$$

obtained from (1.1) by formally replacing $d(Tx,Ty)$ and $d(x,y)$ by $d(Ty,Tx)$ and $d(y,x)$, respectively, and then interchanging x and y.

Therefore, in order to check the weak contractiveness of a given operator, it is necessary to check both conditions (1.1) and (1.2).

The main results in [4] are the following two theorems.

Theorem 1. *Let (X,d) be a complete metric space and $T:X\longrightarrow X$ a (δ,L)-weak contraction, i.e., a mapping satisfying (1.1) with $\delta\in(0,1)$ and some $L\geq 0$. Then*

1) $F(T)=\{x\in X: Tx=x\}\neq\phi$;

2) *For any $x_0\in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by*

$$x_{n+1}=Tx_n\,,\quad n=0,1,2,\dots \tag{1.3}$$

converges to some $x^*\in F(T)$;

3) *The following estimates*

$$d(x_n,x^*)\leq\frac{\delta^n}{1-\delta}\,d(x_0,x_1)\,,\quad n=0,1,2,\dots$$

$$d(x_n,x^*)\leq\frac{\delta}{1-\delta}\,d(x_{n-1},x_n)\,,\quad n=1,2,\dots$$

hold, where δ is the constant appearing in (1.1).

Theorem 2. *Let (X,d) be a complete metric space and $T : X \longrightarrow X$ a weak contraction for which there exist a constant $\theta \in (0,1)$ and some $L_1 \geq 0$ such that*

$$d(Tx,Ty) \leq \theta \cdot d(x,y) + L_1 \cdot d(x,Tx)\,, \quad \text{for all } x,y \in X\,. \tag{1.4}$$

Then

1) *T has a unique fixed point, i.e. $F(T) = \{x^*\}$;*

2) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.3) converges to x^*, for any $x_0 \in X$;*

3) *The a priori and a posteriori error estimates*

$$\begin{aligned} d(x_n,x^*) &\leq \frac{\delta^n}{1-\delta}\, d(x_0,x_1)\,, \quad n = 0,1,2,\ldots \\ d(x_n,x^*) &\leq \frac{\delta}{1-\delta}\, d(x_{n-1},x_n)\,, \quad n = 1,2,\ldots \end{aligned}$$

hold.

4) *The rate of convergence of the Picard iteration is given by*

$$d(x_n,x^*) \leq \theta\, d(x_{n-1},x^*)\,, \quad n = 1,2,\ldots$$

It was shown in [4] that any strict contraction, the Kannan [15] and Zamfirescu [35] operators, as well as a large class of quasi-contractions [9], are all weak contractions.

A weak contraction has always at least one fixed point and there exist weak contractions that have infinitely many fixed points, see Example 4.

Note also that the weak contraction condition (1.1) implies the so called Banach orbital condition

$$d(Tx,T^2x) \leq \delta\, d(x,Tx), \quad \text{for all } x \in X\,,$$

studied by various authors in the context of fixed point theorems, see for example Hicks and Rhoades [13], Ivanov [14], Rus [26], [27], [29] and Taskovic [34].

Moreover, the class of weak contractions offers a large class of weakly Picard operators. Recall, see Rus [31], [32], that in a metric space setting, an operator $T : X \longrightarrow X$ is said to be a *weakly Picard operator* if the sequence $\{T^n x_0\}_{n=0}^{\infty}$ converges for all $x_0 \in X$ and the limits are fixed points of T.

Theorem 1 shows, in particular, that any weak contraction is a weakly Picard operator.

Starting from the fact that φ-contractions are natural generalizations of strict contractions, it is the aim of this paper to extend the results in [4] from weak contractions to the more general class of weak φ-contractions. To this end, let us first remind some concepts from Rus [30], [32] and Berinde [2].

A map $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called *comparison function* if it satisfies:

(i_φ) φ is monotone increasing, i.e., $t_1 < t_2 \Rightarrow \varphi(t_1) \leq \varphi(t_2)$;

(ii_φ) the sequence $\{\varphi^n(t)\}_{n=0}^{\infty}$ converges to zero, for all $t \in \mathbb{R}_+$, where φ^n stands for the n^{th} iterate of φ.
If φ satisfies (i_φ) and

(iii_φ) $\sum\limits_{k=0}^{\infty} \varphi^k(t)$ converges for all $t \in \mathbb{R}_+$,

then φ is said to be a *(***c***) - comparison function* [2].
It was shown in [2] that φ satisfies (iii_φ) if and only if there exist $0 < c < 1$ and a convergent series of positive terms, $\sum\limits_{n=0}^{\infty} u_n$, such that

$$\varphi^{k+1}(t) \leq c\varphi^k(t) + u_k\,, \quad \text{for all } t \in \mathbb{R}_+ \quad \text{and } k \geq k_0 \text{ (fixed)}.$$

It is also known that if φ is a (c) - comparison function, then the sum of the comparison series, i.e.,

$$s(t) = \sum_{k=0}^{\infty} \varphi^k(t)\,, \quad t \in \mathbb{R}_+, \tag{1.5}$$

is monotone increasing and continuous at zero, and that any (c) - comparison function is a comparison function.

A prototype for comparison functions is

$$\varphi(t) = a\,t\,, \quad t \in \mathbb{R}_+ \quad (0 \leq a < 1)$$

but, as shown by Example 1, the comparison functions need not be neither linear, nor continuous.

Note however that any comparison function is continuous at zero.

Example 1. Let $\varphi_1(t) = \dfrac{t}{t+1}$, $t \in \mathbb{R}_+$ and $\varphi_2(t) = \dfrac{1}{2}t$, if $0 \leq t < 1$ and $\varphi_2(t) = t - \dfrac{1}{3}$, if $t \geq 1$.
Then φ_1 is a nonlinear comparison function, which is not a (c) - comparison function, while φ_2 is a discontinuous (c) - comparison function.

By replacing the well known strict contractiveness condition appearing in Banach's fixed point theorem, i.e.

$$d(Tx, Ty) \leq a\,d(x, y)\,, \quad \text{for all } x, y \in X\,,$$

by a more general one

$$d(Tx, Ty) \leq \varphi\big(d(x, y)\big)\,, \quad \text{for all } x, y \in X\,, \tag{1.6}$$

where φ is a certain comparison function, several fixed point theorems have been obtained, see for example Taskovic [34], Rus [32] and Berinde [2], and references therein. One of the first fixed point theorems of this type is due to Browder [5].

Recall that an operator T which satisfy a condition of the form (1.6) is commonly named φ **- contraction**.

Following the way in which the strict contractions were extended to φ - contractions, it is the aim of this paper to extend Theorems 1 and 2 to weak φ - contractions.

Their merit is that, as in the case of weak contractions, they provide a constructive method for approximating fixed points, i.e. the method of successive approximations. Moreover, both a priori and a posteriori error estimates are available for this method, also known as the Picard iteration.

Our results extend, unify and improve numerous fixed points theorems in literature, see [1], [2], [6], [14], [15], [29], [30], [35].

2. Weak φ - contractions

Definition 1. Let (X, d) be a metric space. A self operator $T : X \longrightarrow X$ is said to be a *weak φ-contraction* or (φ, L)*-weak contraction*, provided that there exist a comparison function φ and some $L \geq 0$, such that

$$d(Tx, Ty) \leq \varphi\big(d(x, y)\big) + L\, d(y, Tx)\,, \quad \text{for all } \ x, y \in X. \tag{2.1}$$

Remark 1. Clearly, any weak contraction is a weak φ - contraction, with $\varphi(t) = \delta t$, $t \in \mathbb{R}_+$ and $0 < \delta < 1$.
There exist weak φ - contractions which are not weak contractions with respect to the same metric, see Example 1.
Also, all φ - contractions are weak φ - contractions with $L \equiv 0$ in (2.1).

Remark 2. Similar to the case of weak contractions, the fact that T satisfies (2.1), for all $x, y \in X$, does imply that the following dual inequality

$$d(Tx, Ty) \leq \varphi\big(d(x, y)\big) + L\, d(x, Ty)\,, \tag{2.2}$$

obtained from (2.1) by formally replacing $d(Tx, Ty)$ and $d(x, y)$ by $d(Ty, Tx)$ and $d(y, x)$, respectively and then interchanging x and y, is also satisfied.

Consequently, in order to prove that a certain operator T is a weak φ - contraction, we must check the both inequalities (2.1) and (2.2).

Remark 3. The class of weak φ - contractions includes not only contractive type operators which have a unique fixed point, but also operators with more than one fixed point, see Example 4 below.

To illustrate de diversity of weak (φ) - contractions we give a few examples.

Example 2. Any strict contraction, any operator satisfying the conditions in either Chatterjea [6], Kannan [15] or Zamfirescu [34] fixed point theorems, are weak contractions and hence weak φ - contractions. See also Rhoades [22], [24] and Meszaros [19] for other contractive type conditions that imply weak contractiveness.

Example 3. ([4]) Any quasi contraction, i.e. any operator for which there exists $0 < h < 1$ such that

$$d(Tx, Ty) \leq h \cdot \max\big\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big\} \tag{2.3}$$

for all $x, y \in X$, is a weak contraction if $h < \dfrac{1}{2}$.

All operators mentioned in Examples 2 and 3 have a unique fixed point. The next example shows that a weak contraction may have infinitely many fixed points.

Example 4. ([4]) Let $[0,1]$ be the unit interval with the usual norm and $T : [0,1] \longrightarrow [0,1]$ the identity map, i.e. $Tx = x$, for all $x \in [0,1]$. Then, taking $\varphi(t) = a \cdot t$, $t \in \mathbb{R}$, $0 < a < 1$; $\delta = a$ and $L \geq 1 - a$, condition (2.1) leads to

$$|x - y| \leq a \cdot |x - y| + L \cdot |y - x| \,,$$

which is valid for all $x, y \in [0,1]$.
Note that $F(T) = \big\{x \in [0,1] : Tx = x\big\} = [0,1]$.

3. Main results

Theorem 3. *Let (X,d) be a complete metric space and $T : X \longrightarrow X$ a weak φ - contraction with φ a (c) - comparison function. Then*

1) *$F(T) = \{x \in X : Tx = x\} \neq \phi$;*

2) *For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_0 \in X$ and*

$$x_{n+1} = Tx_n \,, \quad n = 0, 1, 2, \ldots \tag{3.1}$$

converges to a fixed point x^* of T;

3) *The following estimate*

$$d(x_n, x^*) \leq s\big(d(x_n, x_{n+1})\big) \,, \quad n = 0, 1, 2, \ldots \tag{3.2}$$

holds, where $s(t)$ is given by (1.5).

Proof. We shall prove that T has at least one fixed point in X. To this end, let $x_0 \in X$ be arbitrary and $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration defined by (3.1).

Since T is a weak φ - contraction, there exist a (c) - comparison function φ and some $L \geq 0$, such that

$$d(Tx, Ty) \leq \varphi\big(d(x,y)\big) + L \cdot d(y, Tx) \,, \tag{3.3}$$

holds, for all $x, y \in X$.

Take $x := x_{n-1}$, $y := x_n$ in (3.3). We get

$$d(x_n, x_{n+1}) \leq \varphi\big(d(x_{n-1}, x_n)\big) \,, \quad \text{for all } \; n = 1, 2, \ldots \tag{3.4}$$

Since φ is not decreasing, by (3.4) we have

$$d(x_{n+1}, x_{n+2}) \leq \varphi\big(d(x_n, x_{n+1})\big)$$

which inductively yields

$$d(x_{n+k}, x_{n+k+1}) \leq \varphi^k\big(d(x_n, x_{n+1})\big)\,, \quad k = 0, 1, 2, \ldots \tag{3.5}$$

By triangle rule we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\leq r + \varphi(r) + \cdots + \varphi^{n+p-1}(r)\,, \end{aligned} \tag{3.6}$$

where we denoted $r = d(x_n, x_{n+1})$.
Again by (3.4) we find that

$$d(x_n, x_{n+1}) \leq \varphi^n\big(d(x_0, x_1)\big)\,, \quad n = 0, 1, 2, \ldots \tag{3.7}$$

which, by property (ii_φ) of a comparison function implies

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.8}$$

As φ is positive, it is obvious that

$$r + \varphi(r) + \cdots + \varphi^{n+p-1}(r) < s(r)\,, \tag{3.9}$$

where $s(t)$ is the sum of the series $\sum\limits_{k=0}^{\infty} \varphi^k(r)$.
Then by (3.6) and (3.9) we get

$$d(x_n, x_{n+p}) \leq s\big(d(x_n, x_{n+1})\big)\,, \quad n \in \mathbb{N},\ p \in \mathbb{N} \tag{3.10}$$

Since s is continuous at zero, (3.8) and (3.9) implies that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence.
As X is complete, $\{x_n\}_{n=0}^{\infty}$ is convergent.

Let $x^* = \lim\limits_{n\to\infty} x_n$.
We shall prove that x^* is a fixed point of T. Indeed,

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) = \\ &= d(x_{n+1}, x^*) + d(Tx_n, Tx^*)\,. \end{aligned}$$

By (3.3) we have

$$d(Tx_n, Tx^*) \leq \varphi(d(x_n, x^*)) + L\, d(x^*, Tx_n)$$

and hence

$$d(x^*, Tx^*) \leq (1 + L)\, d(x_{n+1}, x^*) + \varphi(\, d(x_n, x^*))\,, \tag{3.11}$$

valid for all $n \geq 0$.

Now letting $n \to \infty$ in (3.11) and using the continuity of φ at zero, it results

$$d(x^*, Tx^*) = 0\,,$$

i.e., x^* is a fixed point of T.
The estimate (3.2) is obtained by (3.6) letting $p \to \infty$.

The proof is complete. □

Remark 4. 1) Using the a posteriori error estimate (3.2) and (3.7) we easily obtain

$$d(x_n, x^*) \leq s\Big(\varphi^n\big(d(x_0, x_1)\big)\Big), \quad n = 0, 1, 2, \ldots$$

which is the *a priori* estimate for the Picard iteration $\{x_n\}_{n=0}^{\infty}$.

2) Note that a weak φ - contraction is not generally continuous, as shown by Example 5.

3) If we take $\varphi(t) = \delta \cdot t$, $t \in \mathbb{R}_+$, $0 < \delta < 1$, by Theorem 3 obtain the corresponding result for weak contractions in [4], i.e. Theorem 1.

Example 5. ([4]). Let $T : [0, 1] \longrightarrow [0, 1]$ be given by $Tx = \dfrac{1}{2}$, for $x \in [0, 1)$ and $T1 = 0$.

Then: 1) T is not a strict contraction;

2) T is a quasi contraction, i.e. satisfies (2.3) with $h = \dfrac{1}{2}$.

3) T is a weak contraction, hence a weak φ - contraction with $\varphi(t) = \dfrac{1}{2} \cdot t$, and $L \geq 1$.

4) T has a unique fixed point.

Remark 5. As shown by Example 4, a weak φ - contraction generally possesses more than one fixed point. The fixed point x^* determined by the Picard iteration $\{x_n\}_{n=0}^{\infty}$ in Theorem 3 generally depends on the initial guess x_0.

As in the case of weak contractions, in order to guarantee the uniqueness of the fixed point of T, we have to consider an additional weak contractive type condition, as in the next theorem.

Theorem 4. *Let X and T as in* Theorem 1. *Suppose T also satisfies the following condition: there exist a comparison function ψ and some $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \psi\big(d(x, y)\big) + L_1 d(x, Tx)\,, \tag{3.12}$$

holds, for all $x, y \in X$.

Then

1) *T has a unique fixed point, i.e. $F(T) = \{x^*\}$;*

2) *The estimate* (3.2) *holds;*

3) *The rate of convergence of the Picard iteration is given by*

$$d(x_n, x^*) \leq \varphi\big(d(x_{n-1}, x^*)\big)\,, \quad n = 1, 2, \ldots\,. \tag{3.13}$$

Proof. Assume there are two distinct fixed points $x^*, y^* \in X$. Then by (3.12) with $x := x^*$ and $y := y^*$, it results

$$d(x^*, y^*) \leq \psi\big(d(x^*, y^*)\big)$$

which by induction yields

$$d(x^*, y^*) \leq \psi^n\big(d(x^*, y^*)\big)\,, \quad n = 1, 2, \ldots \tag{3.14}$$

Letting $n \longrightarrow \infty$ in (3.14) we get

$$d(x^*, y^*) = 0$$

i.e. $x^* = y^*$, a contradiction.
Therefore, T has a unique fixed point.
To obtain (3.13), we let $x := x^*$, $y := x_n$ in (3.12).
The proof is complete. □

Remark 6. 1) Similarly to the case of the pairs of dual conditions (1.1) and (1.2), (2.1) and (2.2), condition (3.8) holds for all $x, y \in X$ if and only if its dual

$$d(Tx, Ty) \leq \psi\big(d(x, y)\big) + L_1 d(y, Ty)\,,$$

is also satisfied, for all $x, y \in X$.

2) Condition (3.12) is not necessary for the fixed point to be unique, as shown by the function T in Example 5, which has a unique fixed point $x^* = \dfrac{1}{2}$ and does not satisfy (3.12).
Indeed, if we take $x = \dfrac{1}{2}$, $y = 1$ in (3.12) we get

$$\frac{1}{2} \leq \psi\left(\frac{1}{2}\right)$$

which is not true, since any comparison function satisfies

$$\varphi(t) < t\,, \quad \text{for } \ t > 0\,.$$

3) However, if T has a unique fixed point x^* and the Picard iteration $\{T^n x_0\}_{n=0}^{\infty}$ converges to x^*, for all $x_0 \in X$, then by Bessaga theorem, see [30], for any $a \in (0, 1)$, there exist a metric ρ on X such that (X, ρ) is complete and T is an a-contraction with respect to the metric ρ.
Therefore, condition (3.12) can be reformulated in terms of an other metric, thus obtaining the following more general result.

Theorem 5. *Let X be a nonempty set and d, ρ two metrics on X, such that (X, d) is complete.*
Let $T : X \longrightarrow X$ be a self operator satisfying

(i) *There exists a (c) - comparison function φ and $L \geq 0$ such that*

$$d(Tx, Ty) \leq \varphi\big(d(x, y)\big) + L\, d(y, Tx)\,, \quad \text{for all } \ x, y \in X\,.$$

(ii) *There exists a comparison function ψ and $L_1 \geq 0$ such that*

$$\rho(Tx, Ty) \leq \psi\big(\rho(x, y)\big) + L_1 \rho(x, Tx)\,, \quad \text{for all } \ x, y \in X\,.$$

Then

1) *T has a unique fixed point x^* ;*

2) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$, $x_{n+1} = Tx_n$, $n \geq 0$, converges to x^*, for all $x_0 \in X$;*

3) *The a posteriori error estimate*

$$d(x_n, x^*) \leq s\big(d(x_n, x_{n+1})\big)\,, \quad n = 0, 1, 2, \dots$$

holds, where $s(t) = \sum\limits_{k=0}^{\infty} \varphi^k(t)$ *;*

4) *The rate of convergence of the Picard iteration is given by*

$$\rho(x_n, x^*) \leq \psi\big(\rho(x_{n-1}, x^*)\big)\,, \quad n \geq 1\,.$$

Particular case. If we set $d \equiv \rho$, by Theorem5 we obtain Theorem 4.

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