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A FIXED POINT PROOF OF THE CONVERGENCE OF A NEWTON-TYPE METHOD

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Abstract. By applying an appropriate fixed point technique, it is shown that a certain Newton-type iterative method converges to the unique solution of the scalar nonlinear equation $f(x) = 0$, under weak smoothness conditions, involving only the function f and its first derivative f'. For this Newton-like method, an error estimate, better than the one known in the case of the classical Newton method, is also established.

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1. Introduction

Newton's method or Newton-Raphson method, as it is generally called in the case of scalar equations $f(x) = 0$, is one of the most used iterative procedures for solving nonlinear equations. It is defined by the iterative sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n \geq 0, \tag{1}$$

under appropriate assumptions on f and its first derivatives. Notice that there is a close connection between Newton type iterative methods and fixed point theory, in the sense that (1) can be also viewed as the sequence of successive approximations of the Newton iteration function

$$G(x) = x - \frac{f(x)}{f'(x)},$$

and moreover, under appropriate conditions, α is a solution of the equation $f(x) = 0$ if and only if α is a fixed point of the iteration function G.

There exist several convergence theorems in literature for the Newton's method, see for example [12], [13], [15], which, in order to ensure a quadratic convergence for the iterative process (1), are requiring strong smoothness assumptions, that involve f, f' and f''. These theorems usually also provide appropriate error estimates.

Theorem 1. ([12]) *Let $f : [a, b] \to \mathbb{R}$, $a < b$, be a function such that the following conditions are satisfied*
1) $f(a)\, f(b) < 0$; 2) $f \in C^2[a, b]$ and $f'(x)\, f''(x) \neq 0$, $x \in [a, b]$;

Then the sequence $\{x_n\}$, defined by (1) and $x_0 \in [a, b]$, converges to α, the unique solution of $f(x) = 0$ in $[a, b]$, and the following estimation

$$|x_n - \alpha| \leq \frac{M_2}{2m_1} |x_n - x_{n-1}|, \, n \geq 1, \tag{2}$$

holds, where

$$m_1 = \min_{x \in [a,b]} \left| f'(x) \right| \text{ and } M_2 = \max_{x \in [a,b]} \left| f''(x) \right|.$$

For concrete applications, Theorem 1 is widely used but there exist more general results, based on weaker smoothness conditions. We state here such a result, due to Ostrowski ([14], Theorem 7.2, pp. 60), based on weaker conditions on f but still involving the second derivative f''.

Theorem 2. ([14]) *Let $f(x)$ be a real function of the real variable x, $f(x_0)f'(x_0) \neq 0$, and put $h_0 = -f(x_0)/f'(x_0)$, $x_1 = x_0 + h_0$.*

Consider the interval $I_0 = [x_0, x_0 + 2h_0]$ and assume that $f''(x)$ exists in I_0, that $\max_{x \in I_0} |f''(x)| = M_2$ and

$$2\,|h_0|\, M_2 \leq \left| f'(x_0) \right|.$$

Then the sequence $\{x_n\}$ given by (1) lie in I_0 and $x_n \to \alpha$ $(n \to \infty)$, where α is the unique zero of f in I_0.

The smoothness assumptions in Theorem 2 are still very sharp, as shown by the next Example.

Example 1. ([2]) Let $f : [-1,1] \to \mathbb{R}$ be given by $f(x) = -x^2 + 2x$, if $x \in [-1,0)$, and $f(x) = x^2 + 2x$, if $x \in [0,1]$. The Newton iteration (1) converges to the unique solution of $f(x) = 0$ in $[-1,1]$ but Theorem 2 cannot be applied, because f'' does not exist at $0 \in I_0 = [-1,1]$.

In a series of papers [1] - [11], the first author obtained more general convergence theorems for what was called there the *extended Newton's method*, for both scalar equations ([1] - [8], [10] - [11]) and n-dimensional equations [9], theorems that can be applied to weakly smooth functions, including the function in the previous example. The term *extended Newton method* was adopted in view of the fact that the iterative process (1) has been extended from $[a,b]$ to the whole real axis $\mathbb{R}$, in order to cover possible overflowing of $[a,b]$ at a certain step. A sample scalar variant of these results is contained in the following theorem.

Theorem 3. ([4]-[5]) *Let $f : [a,b] \to \mathbb{R}$, $a < b$, be a function such that the following conditions are satisfied*
(f_1) $f(a)\,f(b) < 0$;
(f_2) $f \in C^1[a,b]$ *and* $f'(x) \neq 0$, $x \in [a,b]$;
(f_3) $2m > M$, *where*

$$m = \min_{x\in[a,b]} \left|f'(x)\right| \text{ and } M = \max_{x\in[a,b]} \left|f'(x)\right|. \tag{3}$$

Then the Newton iteration $\{x_n\}$, defined by (1) and $x_0 \in [a,b]$, converges to α, the unique solution of $f(x) = 0$ in $[a,b]$, and the following estimation

$$|x_n - \alpha| \leq \frac{M}{m}\left|x_n - x_{n+1}\right|, \; n \geq 0, \tag{4}$$

holds.

A slightly more general variant of Theorem 3 has been obtained in ([4], Theorem 5), by replacing condition (f_3) by the next one

$$(f_3') \quad 2m \geq M.$$

All proofs in [2], [4] - [7] are based on a rather classical technique, which focuses on the behavior of the Newton sequence (1). In an other paper [3], without large circulation, the first author succeeded to prove Theorem 3 using an elegant fixed point argument.

Very recently, Sen at all [16] have extended Theorem 3 to the case of a Newton-like iteration of the form

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + M_1 f(x)},\ n \geq 0, \tag{5}$$

where $M_1 f(x) = \text{sgn } f'(x) \cdot M$, with M defined by (3).

This result was then extended by Sen et all [17] to the n-dimensional case. In both cases an *extended* Newton-like algorithm was used.

It is the main aim of this paper to obtain a convergence theorem for the process (5), by means of a fixed point argument, under the same general assumptions in Theorem 3, that involve only f and its first derivative f'. Finally, a numerical example is also discussed.

2. Quasi-contractive operators

We shall use the results in [3] to construct the main tool to be used in Section 3 of this paper for proving a convergence theorem for the iterative method (5). Let (X, d) be a metric space.

An operator $F : X \to X$ is called a *quasi-contractive operator* if there exist $\overline{y} \in X$ and a constant $a \in [0, 1)$ such that

$$d(F(x), F(\overline{y})) \leq a \cdot d(x, \overline{y}),\ \forall x \in X. \tag{6}$$

If $\overline{y}$ is a fixed point of F and F satisfies (6) with $a = 1$, then F is called a *quasi-nonexpansive operator*, see for example [13]. We shall need the following Lemma.

Lemma 1. *Let (X, d) be a complete metric space and $F : X \to X$ be a quasi-contractive operator with $\overline{y}$ a fixed point of F. Then $\overline{y}$ is the unique fixed point of F and the sequence of successive approximations $\{F^n(x_0)\}$ converges to $\overline{y}$, for each $x_0 \in X$.*

Proof. Since $\overline{y} = F(\overline{y})$, by (6) we get

$$d(F(x), \overline{y}) \leq a \cdot d(x, \overline{y}),\ \forall x \in X. \tag{7}$$

Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be the sequence of successive approximations starting from x_0, that is, $x_{n+1} = F(x_n)$, $n \geq 0$.

Using (7) we get, by induction, that

$$d(x_n, \overline{y}) \leq a^n \cdot d(x_0, \overline{y}),\ n \geq 0$$

which, based on the fact that $0 \leq a < 1$, yields

$$d(x_n, \overline{y}) \to 0 \text{ as } n \to \infty,$$

that is, $\{x_n\}$ converges to $\overline{y}$.

To prove that the fixed point is unique, assume that there is an other fixed point of F, say $\overline{x}$. If $\overline{x} \neq \overline{y}$, then $d(\overline{x}, \overline{y}) \neq 0$ and then by (7) we get to the contradiction

$$d(\overline{x}, \overline{y}) \leq a \cdot d(\overline{x}, \overline{y}).$$

□

Remark 1. If X is a *compact* metric space, as in the case of $X = [a, b]$, then Lemma 1 remains true if we replace the quasi-contractiveness assumption by the quasi-nonexpansiveness assumption. Note that a quasi-contractive (quasi-nonexpansive) operator is not generally a contraction, that is, an operator which, instead of (6), satisfies the classical contraction condition

$$d(F(x), F(y)) \leq a \cdot d(x, y),\ \forall x, y \in X,$$

with $0 \leq a < 1$.

3. The convergence theorem

Theorem 4. *Let $f : [a, b] \to \mathbb{R}$ $(a, b \in \mathbb{R}, a < b)$, be a function such that the following conditions are satisfied*
(f_1) $f(a)\, f(b) < 0$;
(f_2) $f \in C^1[a, b]$ *and* $f'(x) \neq 0$, $x \in [a, b]$;
(f_3) $2m > M$, *where*

$$m = \min_{x \in [a,b]} \left|f'(x)\right| \text{ and } M = \max_{x \in [a,b]} \left|f'(x)\right|.$$

Then the Newton-like iteration $\{x_n\}$ defined by (5) and $x_0 \in [a, b]$, converges to α, the unique solution of $f(x) = 0$ in $[a, b]$, and the estimation

$$|x_n - \alpha| \leq \frac{2M}{m + M} \left|x_n - x_{n+1}\right|,\ n \geq 0, \tag{4'}$$

holds.

Proof. By (f_1) and (f_2) it follows that the equation $f(x) = 0$ has a unique solution α in (a, b).

Let $F : [a, b] \to \mathbb{R}$ be the Newton-like iteration function associated to f, that is,

$$F(x) = x - \frac{2f(x)}{f'(x) + M_1 f(x)},\ x \in [a, b],$$

where $M_1(x) = \text{sgn } f'(x) \cdot M$ and M is given by (3). Since $f(\alpha) = 0$, we get

$$F(\alpha) = \alpha$$

and hence

$$F(x) - \alpha = x - \frac{2f(x)}{f'(x) + M_1 f(x)} - \alpha = x - \alpha - \frac{2f(x)}{f'(x) + M_1 f(x)}.$$

As

$$f(x) = f(x) - 0 = f(x) - f(\alpha),$$

by (f_2) and the mean value theorem, we get

$$f(x) = f'(\overline{y}) \cdot (x - \alpha),$$

where $\overline{y} = \alpha + \lambda(x - \alpha)$, $0 < \lambda < 1$. Then

$$F(x) - \alpha = (x - \alpha) \cdot \left(1 - \frac{2f'(\overline{y})}{f'(x) + M_1 f(x)}\right),\ \forall x \in [a, b]. \tag{8}$$

Using (f_2), it results that f' preserves sign on $[a, b]$. Hence

$$\frac{2f'(\overline{y})}{f'(x) + M_1 f(x)} > 0$$

which leads to the conclusion that

$$1 - \frac{2f'(\overline{y})}{f'(x) + M_1 f(x)} < 1,\ \forall x \in [a, b] \tag{9}$$

and for any $\overline{y}$ between α and x. On the other hand, by (f_3) we obtain that, $\forall x \in [a, b]$,

$$\frac{2f'(\overline{y})}{f'(x) + M_1 f(x)} = \left|\frac{2f'(\overline{y})}{f'(x) + M_1 f(x)}\right| = \frac{2\,|f'(\overline{y})|}{|f'(x) + M_1 f(x)|} \le \frac{2M}{m + M} < 2,$$

which shows that

$$1 - \frac{2f'(\overline{y})}{f'(x) + M_1 f(x)} > -1,\ \forall x \in [a, b] \tag{10}$$

and for any $\overline{y}$ between α and x. Now, by (9), (10) and the continuity of f' one obtains that there exists

$$k = \max_{x,\overline{y}\in[a,b]} \left| 1 - \frac{2f'(\overline{y})}{f'(x) + M_1 f(x)} \right| < 1,$$

which, together with (8), yields the quasi-contractive condition

$$|F(x) - \alpha| \leq k \cdot |x - \alpha| \,, \forall x, y \in [a,b]. \tag{11}$$

Note that we cannot apply Lemma 1 to F directly, because, under the weak differentiability assumptions in Theorem 4, $[a,b]$ is generally not an invariant set under F. We shall prove, however, that there exists a certain iteration of F, say F^N, which satisfies

$$F^N([a,b]) \subset [a,b].$$

To this end, let us first note that, by (11) and the proof of Lemma 1, we have that

$$F^n(x_0) \to \alpha, \text{ as } n \to \infty. \tag{12}$$

for any $x_0 \in [a,b]$, providing that all terms of $\{F^n(x_0)\}$ lie in $[a,b]$.

Because $[a,b]$ is not an invariant set under F, then, by starting with the Newton-like iteration $\{x_n\}$ from a given initial guess x_0, it is possible to obtain at a certain step, say p, $x_p = F^p(x_0) \notin [a,b]$. In order to send back the Newton-like iteration into $[a,b]$, we shall consider an *extended* algorithm, defined by means of the prolongation by continuity and first order differentiability of f to the whole real axis. Denote this function by $\overline{f}$: $\overline{f}(x) = f'(a)\cdot(x-a) + f(a)$, if $x < a$ and $\overline{f}(x) = f'(b)\cdot(x-b) + f(b)$, if $x > b$.

Then, we will correspondingly define $M_1\overline{f}(x)$ on a set $A \subset \mathbb{R} \setminus [a,b]$ by

$$M_1\overline{f}(x) = \text{sgn}\overline{f}\,'(x) \cdot \max_{x\in A} \left|\overline{f}\,'(x)\right|, \; x \in A.$$

Notice that, if $x \in (-\infty, a]$, then

$$M_1\overline{f}(x) = \text{sgn}\overline{f}\,'(x) \cdot \max_{x\in(-\infty,a]} \left|\overline{f}\,'(x)\right| = \text{sgn}f'(a) \left|f'(a)\right| = f'(a)$$

and similarly, if $x \in [b, +\infty)$, then

$$M_1\overline{f}(x) = \text{sgn}\overline{f}\,'(x) \cdot \max_{x\in[b,+\infty)} \left|\overline{f}\,'(x)\right| = \text{sgn}f'(b) \left|f'(b)\right| = f'(b).$$

Now, if some iterate x_p of the Newton-like process (5) does not lie in $[a,b]$, we can have either $x_p < a$ or $x_p > b$. In the first case,

$$x_{p+1} = x_p - \frac{2\overline{f}(x_p)}{\overline{f}\,'(x_p) + M_1(\overline{f}(x_p))} = x_p - \frac{2\left(f'(a)(x-a) + f(a)\right)}{f'(a) + f'(a)} =$$

$$x_p - \frac{2\left(f'(a)(x-a) + f(a)\right)}{2f'(a)} = x_p - x_p + a - \frac{f(a)}{f'(a)} = a - \frac{f(a)}{f'(a)} > a,$$

because, from $(f_1) - (f_2)$, we get $f(a)f'(a) < 0$.
If the second alternative occurs, that is, $x_p > b$, then

$$x_{p+1} = x_p - \frac{2\overline{f}(x_p)}{\overline{f}\,'(x_p) + M_1(\overline{f}(x_p))} = x_p - \frac{2\left(f'(b)(x-b) + f(b)\right)}{f'(b) + f'(b)} =$$

$$x_p - \frac{2\left(f'(b)(x-b) + f(b)\right)}{2f'(b)} = x_p - x_p + b - \frac{f(b)}{f'(b)} = b - \frac{f(b)}{f'(b)} < b,$$

because, from $(f_1) - (f_2)$, we get $f(b)f'(b) > 0$.

Therefore, in both cases, any iteration that goes outside $[a,b]$ will be sent back to $[a,b]$ at the next step and, since $\{x_n\}$ converges to α, with $a < \alpha < b$, it follows that, starting from a certain rank $N > 0$, we shall necessarily have

$$x_n = F^n(x_0) \in [a,b],$$

which means that $F^N([a,b]) \subset [a,b]$, as claimed.

By an other hand, using (11) we obtain

$$|F^n(x) - \alpha| \leq k^n \; |x - \alpha| \,, \; \forall x \in [a,b], \; n \geq 0,$$

and thus F^N satisfies all requirements of Lemma 1.
This leads to the conclusion that α is the unique fixed point of F^N, therefore α is a fixed point of F, too.

To obtain the estimation (4'), we use (5) and the mean value theorem to get

$$x_{n+1} - x_n = -2\frac{f'(c_n)}{f'(x_n) + M_1 f(x_n)}(x_n - \alpha),$$

where

$$c_n = \alpha + \mu(x_n - \alpha), \; 0 < \mu < 1,$$

which immediately yields the desired estimation. □

Remark 2. Having in view the obvious inequality $m \leq M$, it is clear that the error estimate (4') in Theorem 4 is better than the corresponding error estimate (4) in Theorem 3.

4. Numerical examples and Conclusions

Note that, similarly to the case of Theorem 3, condition (f_3) in Theorem 4 can be replaced by (f_3') to obtain a convergence theorem that includes functions like f in Example 1. In that case the equation $f(x) = 0$ has a unique solution $\alpha = 0$ situated in the interval $[-1, 1]$ and $m = 2$, $M = 4$. Starting from $x_0 = 0.5$, the classical Newton's method, that is, the iterative process

$$x_{n+1} = \frac{x_n^2}{2x_n + 2}, \ n \geq 0,$$

yields the decreasing sequence $x_1 = 0.83333$, $x_2 = 0.0032051$, $x_3 = 0000129$, $x_4 = 0.0000001$ and $x_5 = 0$, while the Newton-like method considered on the interval $[0, 1]$, defined by

$$x_{n+1} = \frac{x_n}{x_n + 6}, \ n \geq 0,$$

produces the decreasing sequence $x_1 = 0.076923$, $x_2 = 0.012658$, $x_3 = 0.002105$, $x_4 = 0.000351$ and $x_5 = 0.000058$ which is slightly slower than the Newton's iteration. The Newton-like method studied in this paper could be an alternative to the classical Newton's method, if a better expression to replace $M_1 f(x)$ in (5) could be determined.

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