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FIXED POINT THEOREMS FOR NONSELF SINGLE-VALUED ALMOST CONTRACTIONS

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Abstract. Let X be a Banach space, K a non-empty closed subset of X and let $T : K \to X$ be a non-self almost contraction. The main result of this paper shows that if T has the so called property (M) and satisfies Rothe's boundary condition, i.e., maps ∂K (the boundary of K) into K, then T has a fixed point in K. This theorem generalizes several fixed point theorems for non-self mappings and also extends several important results in the fixed point theory of self mappings to the case on non-self mappings.
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1. Introduction

Most of the results in metric fixed point theory deal with single-valued self mapping $T : X \to X$ and multi-valued self mappings $T : X \to \mathcal{P}(X)$ satisfying a certain contraction type condition, where X is a set endowed with a certain metric structure (metric space, convex metric space, Banach space etc.), see [49]. These results are mainly generalizations of Banach contraction mapping principle, which can be briefly stated as follows.

Theorem B. *Let (X, d) be a complete metric space and $T : X \to X$ a strict contraction, i.e., a map satisfying*

$$d(Tx, Ty) \le a \cdot d(x, y), \quad \forall\, x, y \in X, \tag{1.1}$$

where $0 < a < 1$ is a constant. Then T is a Picard operator (that is, T has a unique fixed point in X, say x^*, and Picard iteration $\{T^n x_0\}$ converges to x^* for all $x_0 \in X$).

The Banach's fixed point theorem is one of the most useful results in nonlinear analysis, which, together with its local variant, has many applications in solving nonlinear functional equations, optimization problems, variational inequalities etc., by transforming them in an equivalent fixed point problem. Under the present form it however has at least two drawbacks: first, the contraction condition (1.1) forces T to be continuous and, secondly, the condition $T(X) \subset X$ makes it not applicable to most of the nonlinear problems where the associated operator T is actually a **non-self** operator.

This is the reason why, in continuation and completion to the abundant fixed point theory for self-mappings, produced in the last 45 years, it was also an important and challenging research topic to obtain fixed point theorems for non-self mappings.

It 1972 Assad and Kirk [8] extended Banach contraction mapping principle to non-self multi-valued contraction mappings $T : K \to \mathcal{P}(X)$ in the case (X, d) is a convex metric space in the sense of Menger and K is a non-empty closed subset of X such that T maps ∂K (the boundary of K) into K. In 1976, by using an alternative and weaker condition, i.e., T is metrically inward, Caristi [25] has shown that any nonself singlevalued contraction has a fixed point. Next, in 1978, Rhoades [42] proved a fixed point result in Banach spaces for single-valued non-self mapping satisfying the following contraction condition:

$$d(Tx, Ty) \leq \lambda \max\left\{\frac{d(x, y)}{2},\, d(x, Tx),\, d(y, Ty),\, \frac{d(x, Ty) + d(y, Tx)}{1 + 2\lambda}\right\}, \tag{1.2}$$

for all $x, y \in K$, where $0 < \lambda < 1$.

Rhoades' result [42] has been slightly extended by Ćirić in [30]. Note that although the class of mappings satisfying (1.2) is large enough to include some discontinuous mappings, it however does not include contraction mappings satisfying (1.1) for $\frac{1}{2} \leq \lambda < 1$.

A more general result, which also solved a very hard problem that was open for more than 20 years, has been obtained by Ćirić [32], who considered instead of (1.2) the quasi-contraction condition previously introduced and studied by himself in [29]:

$$d(Tx, Ty) \leq \lambda \max\{d(x, y),\, d(x, Tx),\, d(y, Ty),\, d(x, Ty),\, d(y, Tx)\}, \tag{1.3}$$

for all $x, y \in K$, where $0 < \lambda < 1$. More recently, Ćirić, Ume, Khan and Pathak [32] have considered a contraction condition which is more general than (1.2) and (1.3), i.e.,

$$d(Tx, Ty) \leq \max\{\varphi(d(x, y)),\, \varphi(d(x, Tx)),\, \varphi(d(y, Ty)),\, \varphi(d(x, Ty)),\, \varphi(d(y, Tx))\}, \tag{1.4}$$

for all $x, y \in K$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a certain comparison function.

For some other fixed point results for non-self mappings, see also [3]-[7], [22] and Problem 5 in [51].

On the other hand, the first author [13], [14], [17], see also [18], introduced a new class of self mappings (usually called weak contractions, almost contractions or Berinde operators) that satisfy a simple but more general contraction condition that includes most of the conditions in Rhoades' classification [41]. The corresponding fixed point theorems, established mainly in [17], have two important features that differentiate them from similar results in literature: 1) the fixed points set of almost contractions is not a singleton, in general; and 2) the fixed points of almost contractions can be obtained by means of Picard iteration, like in the case of Banach contractions and, moreover, the error estimate is of the same form as in the case of contraction mapping principle (this motivated the term of "almost contractions"). Note also that almost contractions are particular graphic contractions, see [49]. For a graphic contraction T, Picard iteration associated to T always converges but its limit is, in general, not a fixed point of T. The class of almost contractions collects those graphic contractions for which Picard iteration associated to T always converges to a fixed point of T.

As shown in [17] and [18], quasi-contractions and almost contractions are independent classes of mappings as the latter have a unique fixed point, while the former do not.

Starting from these facts, the aim of the present paper is to obtain fixed point theorems for non-self almost contractions. In order to do so, we first present in the next section a few aspects and results related to self almost contractions and then, in section 3, we extend them to non-self almost contractions. Thus, we shall give a solution to Problem 5 in [51] in the case of almost contractions. In order to do so, we first present in the next section a few aspects and results related to self almost contractions and then, in Section 3, we extend them to non-self almost contractions.

2. Single-valued self almost contractions

It is easy to see that any contraction mapping satisfying (1.1) is continuous. Kannan [34] in 1968 has proved a fixed point theorem which extends Theorem B to mappings that need not be continuous on X (but are continuous *at* their fixed point, see [44]), by considering instead of (1.1) the next contractive condition: there exists a constant $b \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \le b\big[d(x, Tx) + d(y, Ty)\big], \quad \text{for all } x, y \in X\,. \tag{2.1}$$

Following the Kannan's theorem, a lot of papers were devoted to obtaining fixed point or common fixed point theorems for various classes of contractive type conditions that do not require the continuity of T, see, for example, [45], [47], [18] and references therein.

One of them, actually a sort of dual of Kannan fixed point theorem, due to Chatterjea [26], is based on a condition similar to (2.1): there exists a constant $c \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \le c\big[d(x, Ty) + d(y, Tx)\big], \quad \text{for all } x, y \in X. \tag{2.2}$$

For the presentation and comparison of such kind of fixed point theorems, see [18], [36], [41], [43] and [45].

On the other hand, in 1972, Zamfirescu [52] obtained a very interesting fixed point theorem which gather together all three contractive conditions mentioned above, i.e., condition (1.1) of Banach, condition (2.1) of Kannan and condition (2.2) of Chatterjea, in a rather unexpected way: if T is such that, for any pair $x, y \in X$, at least one of the conditions (1.1), (2.1) and (2.2) holds, then T is a Picard operator. Note that considering conditions (1.1), (2.1) and (2.2) all together is not trivial since, as shown later by Rhoades [41], see also [33], the contractive conditions (1.1), (2.1) and (2.2), are independent to each other.

Zamfirescu's fixed point theorem [52] is a particular case of the next fixed point theorem [17], see also the papers [13], [14] and [18].

Theorem 2.1. ([17], Theorem 2.1) *Let (X, d) be a complete metric space and $T : X \to X$ an almost contraction, that is, a mapping for which there exist a constant $\delta \in [0, 1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx)\,, \quad \text{for all } x, y \in X. \tag{2.3}$$

Then

1) $Fix\,(T) = \{x \in X : Tx = x\} \neq \emptyset$*;*

2) *For any $x_0 \in X$, Picard iteration $\{x_n\}_{n=0}^{\infty}$, $x_n = T^n x_0$, converges to some $x^* \in Fix\,(T)$;*

3) *The following estimate holds*

$$d(x_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta}\, d(x_n, x_{n-1})\,, \quad n = 0, 1, 2, \ldots ;\, i = 1, 2, \ldots \tag{2.4}$$

Let us recall, see [47], that a mapping T possessing properties 1) and 2) above is called a *weakly Picard operator*.

Notice also that while any quasi-contraction is a Picard operator (that is, it has a *unique* fixed point), an almost contraction is a weakly Picard operator, i.e., it does not have a unique fixed point, in general, as shown by the next Example.

Example 2.2. Let $X = [0, 1]$ be the unit interval with the usual norm and let $T : [0, 1] \to [0, 1]$ be given by $Tx = \dfrac{1}{2}$ for $x \in [0, 2/3)$ and $Tx = 1$, for $x \in [2/3, 1]$.

As T has two fixed points, that is, $Fix\,(T) = \left\{\dfrac{1}{2}, 1\right\}$, it does not satisfy neither Ćirić's condition (1.3), nor Banach, Kannan, Chatterjea, Zamfirescu or Ćirić [27] contractive conditions, but T satisfies the contraction condition (2.3).

Indeed, for $x, y \in [0, 2/3)$ or $x, y \in [2/3, 1]$, (2.3) is obvious. For $x \in [0, 2/3)$ and $y \in [2/3, 1]$ or $y \in [0, 2/3)$ and $x \in [2/3, 1]$ we have $d(Tx, Ty) = 1/2$ and $d(y, Tx) = |y - 1/2| \in [1/6, 1/2]$, in the first case, and $d(y, Tx) = |y - 1| \in [1/3, 1]$, in the second case, which show that it suffices to take $L = 3$ in order to ensure that (2.3) holds for $0 < \delta < 1$ arbitrary and all $x, y \in X$.

These facts motivate us to try to extend Theorem 2.1 to the case of non-self almost contractions, and thus to extend some of the most important fixed point theorems of this kind for self mappings, amongst which we mention the results due to Banach [11], Kannan [34], Chatterjea [26], Zamfirescu [52], and Ćirić [27], to the more general case of non-self mappings. In particular, we also generalize several fixed point theorems for non-self mappings, see [3]-[8], [30], [32] etc.

3. Fixed point theorems for non-self almost contractions

Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a non-self mapping. If $x \in K$ is such that $Tx \notin K$, then we can always choose an

$y \in \partial K$ (the boundary of K) such that $y = (1-\lambda)x + \lambda Tx$ $(0 < \lambda < 1)$, which actually expresses the fact that

$$d(x, Tx) = d(x, y) + d(y, Tx),\ y \in \partial K, \tag{3.1}$$

where we denoted $d(x, y) = \|x - y\|$.

In general, the set Y of points y satisfying condition (3.1) above may contain more than one element.

In this context we shall need the following concept.

Definition 3.1. Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a non-self mapping. Let $x \in K$ with $Tx \notin K$ and let $y \in \partial K$ be the corresponding elements given by (3.1). If, for any such elements x, we have

$$d(y, Ty) \leq d(x, Tx), \tag{3.2}$$

for at least on of the corresponding $y \in Y$, then we say that T has property (M).

Note that the non-self mapping T in the next example has property (M).

Example 3.2. Let X be the set of real numbers with the usual metric, $K = [0, 1]$ and let $T : K \to X$ be defined (see the example in Remark 1.3, [32]) by $Tx = -0.1$, if $x = 0.9$ and $Tx = \dfrac{x}{x+1}$, if $x \neq 0.9$.

Then T satisfies condition (1.4), T is discontinuous, 0 is the unique fixed point of T and T is continuous at 0, T has property (M) but T does not satisfy the almost contraction condition (3.3) below. Indeed, the only $x \in K$ with $Tx \notin K$ is $x = 0.9$ and the corresponding $y \in \partial K$ is $y = 0$. It is now easy to check that (3.2) holds. To prove the last claim take $x \neq 0.9$ and $y = \dfrac{x}{1+x}$ in (3.3) to get, for any $x > 0$,

$$\frac{1+x}{1+2x} \leq \delta < 1,\ x > 0.$$

If we let now $x \to 0$ in the previous double inequality, we get the contradiction

$$1 \leq \delta < 1.$$

We now state and prove our main result in this paper.

Theorem 3.3. *Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a non-self almost contraction, that is, a mapping for which there exist two constants $\delta \in [0, 1)$ and $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx)\,, \quad \text{for all } x, y \in K. \tag{3.3}$$

If T has property (M) and satisfies Rothe's boundary condition

$$T(\partial K) \subset K, \tag{3.4}$$

then T has a fixed point in K.

Proof. If $T(K) \subset K$, then T is actually a self mapping on the closed set K and the conclusion follows by Theorem 2.1 for $X = K$. Therefore, we consider the case $T(K) \not\subset K$. Let $x_0 \in \partial K$. By (3.4) we know that $Tx_0 \in K$. Denote $x_1 = Tx_0$. Now, if $Tx_1 \in K$, set $x_2 = Tx_1$. If $Tx_1 \notin K$, we can choose an element x_2 on the segment $[x_1, Tx_1]$ which also belong to ∂K, that is,

$$x_2 = (1-\lambda)x_1 + \lambda Tx_1 \, (0 < \lambda < 1).$$

Continuing in this way we obtain a sequence $\{x_n\}$ whose terms are satisfying one of the following properties:

i) $x_n = Tx_{n-1}$, if $Tx_{n-1} \in K$;

ii) $x_n = (1-\lambda)x_{n-1} + \lambda Tx_{n-1} \in \partial K \, (0 < \lambda < 1)$, if $Tx_{n-1} \notin K$.

To simplify the argumentation in the proof, let us denote

$$P = \{x_k \in \{x_n\} : x_k = Tx_{k-1}\}$$

and

$$Q = \{x_k \in \{x_n\} : x_k \neq Tx_{k-1}\}.$$

Note that $\{x_n\} \subset K$ and that, if $x_k \in Q$, then both x_{k-1} and x_{k+1} belong to the set P. Moreover, by virtue of (3.4), we cannot have two consecutive terms of $\{x_n\}$ in the set Q (but we can have two consecutive terms of $\{x_n\}$ in the set P) .

We claim that $\{x_n\}$ is a Cauchy sequence. To prove this, we must discuss three different cases: Case I. $x_n, \, x_{n+1} \in P$; Case II. $x_n \in P, \; x_{n+1} \in Q$; Case III. $x_n \in Q$, $x_{n+1} \in P$;

Case I. $x_n, \, x_{n+1} \in P$.

In this case we have $x_n = Tx_{n-1}$, $x_{n+1} = Tx_n$ and by (3.3) we get

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \delta d(x_n, x_{n-1}) + Ld(x_n, Tx_{n-1})$$

that is,

$$d(x_{n+1}, x_n) \leq \delta d(x_n, x_{n-1}), \tag{3.5}$$

since $x_n = Tx_{n-1}$.

Case II. $x_n \in P, \; x_{n+1} \in Q$.

In this case we have $x_n = Tx_{n-1}$ but $x_{n+1} \neq Tx_n$ and

$$d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, Tx_n).$$

Hence

$$d(x_n, x_{n+1}) \leq d(x_n, Tx_n) = d(Tx_{n-1}, Tx_n)$$

and so by using (3.3) we get

$$d(x_n, x_{n+1}) \leq \delta d(x_n, x_{n-1}) + Ld(x_n, Tx_{n-1}) = \delta d(x_n, x_{n-1}),$$

which yields again inequality (3.5).

Case III. $x_n \in Q, \; x_{n+1} \in P$.

In this situation, we have $x_{n-1} \in P$. Having in view that T has property (M), it follows that

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) \leq d(x_{n-1}, Tx_{n-1})).$$

Since $x_{n-1} \in P$ we have $x_{n-1} = Tx_{n-2}$ and by (3.3) we get

$$d(Tx_{n-2}, Tx_{n-1}) \leq \delta d(x_{n-2}, x_{n-1}) + Ld(x_{n-1}, Tx_{n-2}) = \delta d(x_{n-2}, x_{n-1}).$$

which shows that

$$d(x_n, x_{n+1}) \leq \delta d(x_{n-2}, x_{n-1}). \tag{3.6}$$

Therefore, by summarizing all three cases and using (3.5) and (3.6), it follows that the sequence $\{x_n\}$ satisfies the inequality

$$d(x_n, x_{n+1}) \le \delta \max\{d(x_{n-2}, x_{n-1}),\, d(x_{n-1}, x_n)\}, \tag{3.7}$$

for all $n \ge 2$. Now, by induction for $n \ge 2$, from (3.7) one obtains

$$d(x_n, x_{n+1}) \le \delta^{[n/2]} \max\{d(x_0, x_1),\, d(x_1, x_2)\},$$

where $[n/2]$ denotes the greatest integer not exceeding $n/2$.

Further, for $m > n > N$,

$$d(x_n, x_m) \le \sum_{i=N}^{\infty} d(x_i, x_{i-1}) \le 2\frac{\delta^{[N/2]}}{1-\delta} \max\{d(x_0, x_1),\, d(x_1, x_2)\},$$

which shows that $\{x_n\}$ is a Cauchy sequence.

Since $\{x_n\} \subset K$ and K is closed, $\{x_n\}$ converges to some point in K.

Denote

$$x^* = \lim_{n\to\infty} x_n\,, \tag{3.8}$$

and let $\{x_{n_k}\} \subset P$ be an infinite subsequence of $\{x_n\}$ (such a subsequence always exists) that we denote in the following for simplicity by $\{x_n\}$, too.

Then

$$d(x^*, Tx^*) \le d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) = d(x_{n+1}, x^*) + d(Tx_n, Tx^*)\,.$$

By (3.3) we have

$$d(Tx_n, Tx^*) \le \delta\, d(x_n, x^*) + L\, d(x^*, Tx_n)$$

and hence

$$d(x^*, Tx^*) \le (1+L) d(x^*, x_{n+1}) + \delta \cdot d(x_n, x^*)\,, \tag{3.9}$$

for all $n \ge 0$. Letting $n \to \infty$ in (3.9) we obtain

$$d(x^*, Tx^*) = 0,$$

which shows that x^* is a fixed point of T. □

Remark 3.4. Note that although T satisfying (3.3) may be discontinuous (see Example 1), however T is continuous at the fixed point. Indeed, if $\{y_n\}$ is a sequence in K convergent to $x^* = Tx^*$, then by (3.3) we have

$$d(Ty_n, x^*) = d(Tx^*, Ty_n) \le \delta d(x^*, y_n) + L d(y_n, Tx^*),$$

and letting $n \to \infty$ in the previous inequality, we get exactly the continuity of T at the fixed point x^*:

$$d(Ty_n, x^*) \to 0 \text{ as } n \to \infty, \text{ that is, } Ty_n \to x^*.$$

Example 3.5. Let X be the set of real numbers with the usual norm, $K = [0, 1]$ be the unit interval and let $T : [0, 1] \to \mathbb{R}$ be given by $Tx = \dfrac{2}{3}x$ for $x \in [0, 1/2)$, $T\left(\frac{1}{2}\right) = -1$, and $Tx = \dfrac{2}{3}x + \dfrac{1}{3}$, for $x \in (1/2, 1]$.

As T has two fixed points, that is, $Fix\,(T) = \{0\,, 1\}$, it does not satisfy neither Ćirić's conditions (1.3) and (1.4), nor Banach, Kannan, Chatterjea, Zamfirescu or

Ćirić [27] contractive conditions in the corresponding non-self form, but T satisfies the contraction condition (3.3).

Indeed, for the cases 1) $x \in [0,1/2)$, $y \in (1/2,1]$; 2) $y \in [0,1/2)$, $x \in (1/2,1]$; 3) $x,y \in [0,1/2)$ and 4) $x,y \in (1/2,1]$, we have by Example 1.3.10 in [39], pp. 28-29, that (3.3) is satisfied with $\delta = 2/3$ and $L \geq 6$.

We have to cover the remaining four cases: 5) $x = 1/2$, $y \in [0,1/2)$; 6) $x \in [0,1/2)$, $y = 1/2$; 7) $x = 1/2$, $y \in (1/2,1]$; and 8) $x \in (1/2,1]$, $y = 1/2$.

Case 5) $x = 1/2$, $y \in [0,1/2)$. In this case, (3.3) reduces to

$$\left|-1-\frac{2}{3}y\right| \leq \delta\left|\frac{1}{2}-y\right| + L|y+1|,\ y \in [0,1/2).$$

Since $\left|-1-\frac{2}{3}y\right| \leq \frac{4}{3}$ and $1 \leq |y+1|$, in order to have the previous inequality satisfied, we simply need to take $L \geq \frac{4}{3}$.

Case 6) $x \in [0,1/2)$, $y = 1/2$. In this case, (3.3) reduces to

$$\left|\frac{2}{3}x+1\right| \leq \delta\left|x-\frac{1}{2}\right| + L|\frac{1}{2}-\frac{2}{3}x|,\ x \in [0,1/2).$$

Since $\left|\frac{2}{3}x+1\right| \leq \frac{4}{3}$ and $|\frac{1}{2}-\frac{2}{3}x| \geq \frac{1}{6}$, to have the previous inequality satisfied, it is enough to take $L \geq 8$.

Case 7) $x = 1/2$, $y \in (1/2,1]$. In this case, (3.3) reduces to

$$\left|-1-\frac{2}{3}y-\frac{1}{3}\right| \leq \delta\left|\frac{1}{2}-y\right| + L|y+1|,\ y \in (1/2,1].$$

Since $\left|1+\frac{2}{3}y+\frac{1}{3}\right| \leq 2$ and $|y+1| > \frac{3}{2}$, to have the previous inequality satisfied, it is enough to take $L \geq \frac{4}{3}$.

Case 8) $x \in (1/2,1]$, $y = 1/2$. Similarly, we find that (3.3) holds with $L \geq 8$ and $0 < \delta < 1$ arbitrary.

By summarizing all possible cases, we conclude that T satisfies (3.3) with $\delta = 2/3$ and $L = 8$.

As we have shown in [17], it is possible to force the uniqueness of the fixed point of an almost contraction, by imposing an additional contractive condition, quite similar to (3.3), as shown by the next theorem. For other conditions that ensure the uniqueness of the fixed point of almost contractions we refer to [10], [37], [39].

Theorem 3.6. *Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a non-self almost contraction for which there exist $\theta \in (0,1)$ and some $L_1 \geq 0$ such that*

$$d(Tx,Ty) \leq \theta \cdot d(x,y) + L_1 \cdot d(x,Tx)\,, \quad \text{for all } x,y \in K\,. \tag{3.10}$$

If T has property (M) and satisfies Rothe's boundary condition $T(\partial K) \subset K$, then T has a unique fixed point in K.

Remark 3.7. By the considerations in Section 2 of this paper we immediately obtain various fixed point results as corollaries of Theorem 3.3, for T satisfying one of the conditions (2.1), (2.2) and so on.

4. Conclusions and further study

1. As illustrated by Examples 2.2-3.5, our main result in this paper (Theorem 3.3) is very general in comparison to other related results existing in literature. Many other fixed point theorems for non-self mappings may be obtained as particular cases of Theorem 3.3, see [3]-[8], [30], [32] etc.

2. The proof of Theorem 3.3 is essentially based on the assumption that the mapping T possesses property (M). Theorem 3.3 and Examples 3.2-3.5 naturally raise the following

Open Problem. Does any non-self almost contraction possess property (M)?

3. Note also that all results established here in the setting of a Banach space could be transposed without difficulty for the case of a convex metric space.

It is our aim to try to answer this question in a future work and also to extend Theorem 3.3 to the case of multi-valued almost contractions, in view of the work [12], which extended the fixed point theorems for self single-valued almost contractions established in [17] to the case of self multi-valued almost contractions.

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