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FIXED POINT ITERATIVE METHODS DEFINED AS ADMISSIBLE PERTURBATIONS OF GENERALIZED PSEUDOCONTRACTIVE OPERATORS

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Abstract. We establish convergence of a Krasnoselskij type fixed point iterative method constructed as the admissible perturbation of a nonlinear Lipschitzian and generalized pseudocontractive operator defined on a convex closed subset of a Hilbert space. Both *a priori* and *a posteriori* error estimates are obtained for the new algorithm. Our convergence theorem extends and unifies several related results in the current literature.

1. Introduction

The fundamental problem of the iterative approximation of fixed points is to solve the nonlinear fixed point equation

$$x = Tx, \tag{1.1}$$

where X is a space, $T : K \to X$ is a given operator and $K \subset X$, i.e., to find sufficient conditions on K, X and T that ensure that (1.1) has at least one solution in K and also to construct the most suitable method to obtain such a solution (which is usually called a *fixed point* of T).

There exists a vast literature on this topic, see for example the recent monographs [3], [9] and [22] and references therein, where most of the fundamental results on the iterative approximation of fixed points are presented.

The most used iterative method, associated in a natural way to the fixed point problem (1.1), is the well-known *Picard algorithm* or *sequence of successive approximations* $\{x_n\}_{n=0}^{\infty}$, which is defined by $x_0 \in K$ and

$$x_{n+1} = Tx_n,\ n \geq 0. \tag{1.2}$$

But, Picard iteration (1.2) converges to a solution of (1.1) under strong conditions on K, X and T, see the sample convergence theorems in [3], [9] and [22].

In order to solve (1.1) under weaker assumptions, more reliable and elaborative fixed point iterative methods are needed, see [3] and [9].

We recall some of the most used such methods.

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Let E be a real vector space and $T : E \to E$ a given operator. Let $x_0 \in E$ be arbitrary and $\{\alpha_n\} \subset [0, 1]$ a sequence of real numbers.
The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by $x_0 \in E$ and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n\,, \quad n = 0, 1, 2, \dots \tag{1.3}$$

is called *Mann Algorithm* [20]. The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by

$$\begin{cases} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n\,, \quad n = 0, 1, 2, \dots \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n\,, \quad n = 0, 1, 2, \dots\,, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0, 1]$, and $x_0 \in E$ arbitrary, is called *Ishikawa algorithm* [12].

Remark 1.1. For $\beta_n = 0$, (1.4) reduces to (1.3), while, for $\alpha_n = \lambda$ (constant), the Mann algorithm process (1.3) reduces to the so called *Krasnoselskij algorithm* ([13])

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n\,, \quad n = 0, 1, 2, \dots \tag{1.5}$$

Picard algorithm (1.2) is obtained from it with $\lambda = 1$.

In the iterative approximation of fixed points several classes of contractive type mappings have been considered, see for example the recent monographs [3], [9] and [22] and references therein. One of the most important classes of such mappings is that of *pseudocontractive* type mappings. In fact, see for example [3], there exist several concepts of pseudocontractive type mappings: pseudocontractive, strictly pseudocontractive, strongly pseudocontractive, generalized pseudocontractive mappings etc.

In a normed space E, a mapping $T : E \to E$ is said to be a *pseudocontraction* if, for all x, y in E,

$$\| Tx - Ty\|^2 \le \| x - y \|^2 + \| Tx - Ty - (x - y) \|^2 .$$

The generalized pseudocontractions, introduced in [34], are more general than the pseudocontractions, introduced by Browder [8], and are different from other classes of pseudocontractive type mappings, like strictly pseudocontractive or strongly pseudocontractive mappings, see for example Chapter 3 in [3].

There is a continuous research interest on various aspects of generalized pseudocontractions and their applications. Indeed, in a recent paper [30], the authors study existence and approximation of solutions of variational inequalities involving generalized pseudocontractive mappings in Banach spaces.

There are many other papers dealing with: a) variational inequalities that involve generalized pseudocontractive mappings ([15]-[18], [31], [35], [36]) or generalized successively pseudocontractions ([14]); b) variational-like inclusions involving generalized pseudocontractive mappings ([19]); c) approximation of fixed point of generalized pseudocontractive mappings through Mann algorithm ([32], [33]).

On the other hand, in a recent paper, I. A. Rus [28] considered a new approach to fixed point iterative methods, based on the concept of *admissible perturbation* of a self operator. The theory of admissible perturbations for nonself operators has been studied in [6].

Berinde [4] continued the study of fixed point iterative methods by means of the theory of admissible perturbations and obtained very general convergence theorems for Krasnoselskij type fixed point iterative methods defined as admissible perturbations of a nonlinear operator for the class of nonexpansive operators on Hilbert spaces.

As the class of generalized pseudocontractions includes, amongst other important nonlinear operators, the class of nonexpansive operators, our aim in the present paper is to establish a convergence theorem for Krasnoselskij type fixed point iterations obtained as admissible perturbations of generalized pseudocontractive operators on Hilbert spaces; thus obtaining general results that extend the corresponding results in [2], [3] and [34].

2. Admissible perturbations of an operator

Definition 2.1 ([28])**.** Let X be a nonempty set. A mapping $G : X \times X \to X$ is called *admissible* if it satisfies the following two conditions:
(A_1) $G(x,x) = x$, for all $x \in X$;
(A_2) $G(x,y) = x$ implies $y = x$.

Definition 2.2 ([28])**.** Let X be a nonempty set. If $f : X \to X$ is a given mapping and $G : X \times X \to X$ is an admissible mapping, then the operator $f_G : X \to X$, defined by

$$f_G(x) = G(x, f(x)),\ \forall x \in X, \tag{2.1}$$

is called the *admissible perturbation* of f.

Remark 2.3. The following property of admissible perturbations is fundamental in the iterative approximation of fixed points: if $f : X \to X$ is a given mapping and $f_G : X \to X$ denotes its admissible perturbation, then

$$Fix\,(f_G) = Fix\,(f) := \{x \in X | x = f(x)\}, \tag{2.2}$$

that is, the admissible perturbation f_G of f has the same set of fixed points as the mapping f itself.

Note that, in general,

$$Fix\,(f_G^n) \neq Fix\,(f^n),\ n \geq 2. \tag{2.3}$$

Example 2.4 ([28])**.** Let $(V, +, \mathbb{R})$ be a real vector space, $X \subset V$ a convex subset, $\lambda \in (0, 1)$, $f : X \to X$ and $G : X \times X \to X$ be defined by

$$G(x, f(x)) := (1 - \lambda)x + \lambda f(x),\ x \in X.$$

Then f_G is an admissible perturbation of f. We shall denote f_G by f_λ and call it the Krasnoselskij perturbation of f.

Example 2.5 ([28])**.** Let $(V, +, \mathbb{R})$ be a real vector space, $X \subset V$ a convex subset, $\chi : X \times X \to (0, 1)$, $f : X \to X$ and $G(x, y) := (1 - \chi(x, y))x + \chi(x, y)y$.

Then f_G is an admissible perturbation of f which reduces to the Krasnoselskij perturbation in the case $\chi(x, y)$ is a constant function.

For other important examples of admissible mappings and admissible perturbations of nonlinear operators, see [28] (for the case of self mappings) and [6] (for the case of nonself mappings).

Definition 2.6 ([28]). Let $f : X \to X$ be a nonlinear mapping and $G : X \times X \to X$ an admissible mapping. Then the iterative algorithm $\{x_n\}_{n=0}^{\infty}$ given by $x_0 \in X$ and

$$x_{n+1} = G(x_n, f(x_n)), n \geq 0, \tag{2.4}$$

is called the *Krasnoselskij algorithm* corresponding to G which we denote by GK-*algorithm for simplicity.*

Remark 2.7. For the particular case

$$G(x, y) := (1 - \lambda)x + \lambda y,\ x, y \in X, \tag{2.5}$$

the GK-algorithm (2.4) reduces to the classical Krasnoselskij algorithm (1.5).

Definition 2.8. Let $G : E \times E \to E$ be an admissible mapping on a normed space E. We say that G is *affine Lipschitzian* if there exists a constant $\mu \in [0, 1]$ such that

$$\|G(x_1, y_1) - G(x_2, y_2)\| \leq \|\mu(x_1 - x_2) + (1 - \mu)(y_1 - y_2)\|, \tag{2.6}$$

for all x_1, x_2, y_1, $y_2 \in E$.

Note that the admissible mapping $G : E \times E \to E$ given by (2.5) is affine Lipschitzian.

Note also that in the very recent paper [4], Definition 3.7, we used a weaker concept of affine Lipschitzianity. It is easy to show that, if G is affine Lipschitzian in the sense of Definition 2.8 in the present paper, then it is also affine Lipschitzian in the sense of Definition 3.7 in [4], but the converse is not true in general.

3. Generalized pseudocontractive operators

Definition 3.1 ([34]). Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. An operator $T : H \to H$ is said to be *generalized pseudo-contractive* if there exists a constant $r > 0$ such that, for all x, y in H,

$$\| Tx - Ty\|^2 \leq r^2 \| x - y \|^2 + \| Tx - Ty - r(x - y) \|^2. \tag{3.1}$$

Example 3.2. Let $\mathbb{R}$ denote the reals with the usual norm.

1) Let $C = [0, 1]$ and define $T : C \to \mathbb{R}$ by $Tx = \frac{1}{2}x + 1$. Then T is a $\frac{1}{2}$-contraction and hence is generalized pseudocontractive. T has no fixed point in C.

2) Let $C = \{1, 2\}$ and define $T : C \to C$ by $T(1) = 2, T(2) = 1$. Then T is generalized pseudocontractive, but T is not a contraction. T has no fixed point in C.

Remark 3.3. (1) Condition (3.1) is equivalent to

$$\langle Tx - Ty,\ x - y\rangle\ \leq\ r\ \| x - y\|^2, \quad \text{for all} \quad x, y \in H. \tag{3.2}$$

(2) Any contraction mapping with contraction coefficient $r < 1$ (and, in general, any nonexpansive mapping) is a generalized pseudo-contraction but the reverse is not true, as shown by Examples 3.2 and 3.4;

(3) By the Cauchy-Schwarz inequality

$$| \langle Tx - Ty,\ x - y \rangle| \leq \| Tx - Ty\| \cdot \| x - y\|,$$

we obtain that any Lipschitzian operator T, that is, any operator for which there exists $s > 0$ such that

$$\| Tx - Ty\| \le s \cdot \| x - y\| \ , \quad x, y \in H, \tag{3.3}$$

is also a generalized pseudo-contractive operator, with $r = s$.

It is possible that a certain operator T is simultaneously Lipschitzian with constant s, and generalized pseudo-contractive with constant r, and $r < s$, see Example 3.4;

Consequently, in the sequel, we shall assume that the Lipschitzian constant s and the generalized pseudo-contractivity constant r of a Lipschitzian and generalized pseudo-contractive operator T satisfy the conditions

$$0 < r < 1 \ \text{ and } \ r \le s. \tag{3.4}$$

Example 3.4. Let H be the real line $\mathbb{R}$ endowed with the Euclidean inner product and norm, $C = \left[\frac{1}{2}, 2\right]$ and $T : C \to C$ be the function given by $Tx = \frac{1}{x}$, for all x in C.

Then T is Lipschitzian with constant $s = 4$ (so T is not a contraction but is generalized pseudo-contractive with constant $r = 4$).

Actually, T is generalized pseudocontractive with any constant $r > 0$. It is easy to see that T has a unique fixed point, $Fix\,(T) = \{1\}$, and that, for any initial choice $x_0 = a \neq 1$, Picard algorithm yields the oscillatory sequence

$$a, \frac{1}{a}, a, \frac{1}{a}, \dots$$

Hence, in order to approximate fixed points of Lipschitzian and generalized pseudo-contractive mappings, it becomes essential to use a Krasnoselskij type algorithm, see Theorem 3.6 in [3], in place of Picard algorithm which is not convergent in general, .

The next result extends Theorem 3.6 in [3] from the case of Krasnoselskij algorithm to the more general GK-algorithm introduced in Definition 2.6.

Theorem 3.5. *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be a generalized pseudocontractive and Lipschitzian operator with the corresponding constants r and s, respectively, satisfying* (3.4).

Then

(i) *T has an unique fixed point p in C;*

(ii) *If $G : C \times C \to C$ is an affine Lipschitzian admissible mapping with constant $\lambda \in (0, 1)$, then the GK-algorithm $\{x_n\}_{n=0}^{\infty}$ given by x_0 in C and*

$$x_{n+1} = G(x_n, f(x_n)), n \ge 0, \tag{3.5}$$

converges (strongly) to p for all $\lambda \in (0, 1)$ and satisfies

$$0 < \lambda < 2(1 - r)/(1 - 2r + s^2). \tag{3.6}$$

(iii) *The priori*

$$\| x_n - p\| \le \frac{\theta^n}{1 - \theta} \cdot \| x_1 - x_0\| \ , \ n = 1, 2, \dots \tag{3.7}$$

and the posteriori

$$\| x_n - p\| \leq \frac{\theta}{1-\theta} \cdot \| x_n - x_{n-1}\| \ , \quad n = 1, 2, \dots \tag{3.8}$$

estimates hold, with

$$\theta = \left((1-\lambda)^2 + 2\lambda(1-\lambda)\, r + \lambda^2 s^2\right)^{1/2} . \tag{3.9}$$

Proof. We consider the admissible perturbation operator F associated with T,

$$F(x) = G(x, Tx)\,, \quad x \in C. \tag{3.10}$$

By the definition of G, we have that $F(C) \subset C$. As a closed subset of a Hilbert space, C is a complete metric space. We claim that F is a $\theta-$contraction with θ given by (3.9).

Indeed, since G is affine Lipschitzian, we have

$$\|Fx - Fy\|^2 = \|G(x, Tx) - G(y, Ty)\|^2 \leq \| (1-\lambda)(x-y) + \lambda(Tx - Ty)\|^2$$

$$= (1-\lambda)^2 \cdot \| x - y\|^2 + 2\lambda(1-\lambda) \cdot \langle Tx - Ty,\, x - y\rangle + \lambda^2 \cdot \|Tx - Ty\|^2 .$$

Now, since T is generalized pseudo-contractive with constant r and Lipschitzian with constant s, by (3.2) and (3.3) we obtain

$$\|Fx - Fy\|^2 \leq \left((1-\lambda)^2 + 2\lambda(1-\lambda)r + \lambda^2 s^2\right) \cdot \| x - y\|^2 ,$$

which yields

$$\|Fx - Fy\| \leq \theta \cdot \| x - y\| \,, \quad \text{for all} \quad x, y \in C,$$

with θ given by (3.9).

In view of condition (3.6), it follows that $0 < \theta < 1$, so the self mapping F is a $\theta-$contraction. In order to get the conclusion we now apply the contraction mapping principle ([3], Theorem 2.1) to the operator F defined on the complete metric space C. □

Remark 3.6. (1) Note that, under assumption (3.4), the upper bound of λ in (3.6) satisfies

$$2(1-r)/(1-2r+s^2) \leq 1,$$

with equality in the case $s = 1$, that is, when T is a nonexpansive operator.

(2) If the admissible mapping G is given by (2.5), then by Theorem 3.5 we obtain Theorem 3.6 in [3] as well as the main result in [34] (which is given there without the error estimates (3.7) and (3.8)).

Example 3.7.

Let H, C and T be as in Example 3.4. Then $s = 4$ and $r > 0$ arbitrary. Taking, for example, $r = 0.5$ we get

$$2(1-r)/(1-2r+s^2) = 1/16,$$

and so, by Theorem 3.5 with G defined by (2.5), we obtain that the sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} = (1-\lambda) \cdot x_n + \lambda \cdot \frac{1}{x_n}, \quad n = 0, 1, 2, \dots$$

converges strongly to the fixed point $p = 1$ of T, for all values of λ in the interval $\left(0, \dfrac{1}{16}\right)$.

Remark 3.8. Theorem 3.5 actually provides a family of GK-algorithms, depending on the parameter λ. The following question arises: amongst all GK-algorithms $\{x_n\}_{n=0}^{\infty}$ in the family (3.5), obtained when λ ranges over the interval $(0, a)$, with

$$a = \frac{2(1-r)}{(1-2r+s^2)},$$

is there a certain algorithm to be the fastest one ?

We can answer this question, like in the case of classical Krasnoleslskij algorithm ([3]), if we adopt the concept of convergence rate used in ([3], Theorem 3.7).

Let $\{x_n\}$ and $\{y_n\}$ be two sequences that converge to p (as $n \to \infty$), satisfying the estimate (3.7) with $\theta = \theta_1$ and $\theta = \theta_2$, respectively, and such that $\theta_1, \theta_2 \in (0, 1)$. We shall say that $\{x_n\}$ *converges faster than* $\{y_n\}$ if

$$\theta_1 < \theta_2.$$

Our next theorem answers in the affirmative, the above posed question.

Theorem 3.9. *Let all the assumptions in Theorem 3.5 be satisfied. Then the fastest iteration* $\{x_n\}_{n=0}^{\infty}$ *in the family of* GK*-algorithms* (3.5)*, with* $\lambda \in (0, 2(1-r)/(1-2r+s^2))$*, is the one obtained from*

$$\lambda_{\min} = (1-r)\,/\,(1-2r+s^2). \tag{3.11}$$

Proof. The proof is similar to that of Theorem 3.7 in [3]. We have to find the minimum of the quadratic function

$$f(x) = (1-x)^2 + 2x(1-x)\,r + x^2 s^2,$$

with respect to x, that is, to minimize the quadratic function

$$f(x) = (1-2r+s^2)\,x^2 - 2(1-r)\,x + 1\,, \quad x \in (0, a),$$

with a given by

$$a = 2(1-r)/(1-2r+s^2). \tag{3.12}$$

Since by (3.4)

$$1 - 2r + s^2 \geq (1-r)^2 > 0,$$

it follows that f does really admit a minimum, which is attained from

$$x = \lambda_{\min},$$

where $\lambda_{\min}$ is given by (3.11). □

Example 3.10. Let T and C be as in Examples 3.4 and 3.7, and $G(x, y) = (1-\lambda)x + \lambda y$, $x, y \in C$. Fixing a certain $r \in (0, 1)$, we obtain the fastest Krasnoselskij algorithm for

$$\lambda = (1-r)\,/\,(1-2r+16).$$

If we take $r = 0.5$, then (3.5) converges for each $\lambda \in \left(0, \dfrac{1}{16}\right)$. The fastest Krasnoselskij algorithm $\{x_n\}_{n=0}^{\infty}$ in this family is then obtained for $\lambda = \dfrac{1}{32}$, and is given by

$$x_{n+1} = \frac{1}{32}\left(31\, x_n + \frac{1}{x_n}\right), \quad n = 0, 1, 2, \dots .$$

Although T is not a contraction, the perturbed operator F associated with T,

$$F(x) = \frac{1}{32}\left(31\, x + \frac{1}{x}\right),$$

is a contraction with the contraction coefficient

$$\theta_{\min} = \frac{\sqrt{63}}{8} = 0.992.$$

Remark 3.11. Examples 3.5 and 3.6, reveal that study of generalized pseudocontractive operators is notrivial indeed.

4. Concluding remarks and further study

(1) If $s < 1$, that is, T in Theorem 3.9 is $s-$contraction, then, for a given by (3.12), we have $a > 1$ and hence $\lambda = 1 \in (0, a)$. This shows that among all GK-algorithms (3.5) that converge to the fixed point of T, we can find the Picard algorithm associated to T from (3.5) when the admissible map is $G(x, y) = y$;

(2) For the Picard algorithm, we have a similar priori estimation, see Theorem 2.1 in [3], we can compare Picard algorithm to the fastest GK-algorithm in the family (3.5), with $\lambda \in (0, a)$:

(a) If $r = s^2 < 1$, then we have

$$\theta_{\min} = s,$$

which means that the fastest Krasnoselskij algorithm in the family (3.5) coincides with the Picard algorithm itself;

(b) If $r \neq s^2$, then it is easy to check that

$$\theta_{\min} < s,$$

(since $s < 1$), which shows that the Krasnoselskij algorithm (3.5) with $\lambda = \lambda_{\min}$ converges *faster* than the Picard algorithm associated with T.

In this case, the fastest iteration from (3.5) may be regarded as an *accelerating* procedure of the Picard algorithm.

Based on the results obtained in the present paper and in [4], we plan to continue our study of fixed point iterative methods defined as admissible permutations of generalized pseudocontractions in other contexts to solve nonlinear variational inequalities or variational-like inclusions or to solve nonlinear fixed point problems for other classes of contractive or pseudocontractive type mappings like the ones considered in [1], [5], [10], [11], [14], [15]-[19], [24], [29], [31]-[33], [35], [36].

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