Volume 16, Number 9, pp. 1925-1936

Journal of Nonlinear and Convex Analysis
Volume 16, Number 9, 2015, 1–

THE CONTRACTION PRINCIPLE FOR NONSELF MAPPINGS ON BANACH SPACES ENDOWED WITH A GRAPH

VASILE BERINDE AND MĂDĂLINA PĂCURAR

????????????

ABSTRACT. Let K be a non-empty closed subset of a Banach space X endowed with a graph G and let $T : K \to X$ be a G-contraction that satisfies Rothe's boundary condition, i.e., T maps ∂K (the boundary of K) into K. The main results obtained in this paper are fixed point theorems for nonself G-contractions on a Banach space endowed with a graph. Our new results are important extensions of recent fixed point theorems for self mappings on metric spaces endowed with a partial order and also of various fixed point theorems for nonself mappings on Banach spaces or convex metric spaces.

1. INTRODUCTION

Let (X, d) be a metric space. Denote by $Fix\,(T)$ the set of fixed points of a mapping $T : X \to X$, i.e.,

$$Fix\,(T) = \{x \in X : Tx = x\}.$$

The self mapping $T : X \to X$ is called a Picard operator (abbr., PO), see for example [46], if

(i) T has a unique fixed point in X, say x^*;
(ii) Picard iteration $\{T^n x_0\}$ converges to x^*, for all $x_0 \in X$.

The classical contraction mapping principle [14] essentially states that any contraction on a complete metric space is a PO. This fundamental theorem in the metrical fixed point theory is one of the most useful results in nonlinear analysis, which, together with its local variant, has many applications in solving nonlinear functional equations, optimization problems, variational inequalities etc., by appropriately transforming them in equivalent fixed point problems.

However, the restrictive condition $T(X) \subset X$ makes it not applicable to most of the nonlinear problems where the associated operator T is actually a **non-self** operator.

This is the reason why, in continuation and completion to the abundant fixed point theory for self-mappings, produced in the last 45 years, it was also an important and challenging research topic to obtain fixed point theorems for single valued and multi-valued non-self mappings.

It 1972 Assad and Kirk [12] extended Banach contraction mapping principle to non-self multi-valued contraction mappings $T : K \to \mathcal{P}(X)$ in the case (X, d) is a

2010 *Mathematics Subject Classification.* ?????????????
Key words and phrases. ???????????
The authors research was supported by the Grant PN-II-RU-TE-2011-3-239 of the Romanian Ministry of Education and Research.

convex metric space in the sense of Menger and K is a non-empty closed subset of X such that T maps ∂K into K. For other related results see [26], [43], [28], [30], [31], [34], [7]-[11], [15]-[16], [18]-[20], [1], [3], [4].

On the other hand, recently many results appeared in literature giving sufficient conditions for a self mapping T to be a PO if (X, d) is a metric space endowed with a partial ordering $\leq$. They mainly deal with a monotone (either order-preserving or order reversing) mapping satisfying a classical contractive condition but not for all $x, y \in X$ as in the classical Banach's contraction principle, but only for comparable elements with respect to the given partial order.

Those kind of results are actually hybrid fixed point theorems that combine two types of fundamental fixed point theorems: the contraction mapping principle and the Knaster-Tarski theorem, see [47]. The most relevant results in this direction start practically from the following important papers [42], [36], [37], [41] and the unifying paper [32]. See also [21]-[25], [27], [35], for other very recent results on this topic.

Starting from these facts, the aim of the present paper is to obtain general fixed point theorems for non-self contractions on Banach spaces endowed with a graph, which will thus extend and unify all the previous mentioned results. In order to do so, we first present in the next section a few preliminary notions and results regarding fixed point theorems for mappings defined on metric spaces endowed with a graph and next, present some basic results related to fixed point theorems for non self contractions in Banach spaces or convex metric spaces. Our main results are then obtained in Section 3 of the paper.

2. Metric spaces endowed with a graph

Let (X, d) be a metric space and let Δ denote the diagonal of the Cartesian product $X \times X$. Consider now a directed simple graph $G = (V(G), E(G))$ such that the set of its vertices, $V(G)$, coincides with X and $E(G)$, the set of its edges, contains all loops, i.e., $\Delta \subset E(G)$.

By G^{-1} we denote the *converse graph* of G, i.e., the graph obtained by G by reversing its edges, i.e.,

$$E(G^{-1}) = \{(y, x) \in X \times X : (x, y) \in E(G)\}.$$

If x, y are vertices in the graph G, then a *path* from x to y of length N is a sequence $\{x_i\}_{i=1}^{N}$ of $N+1$ vertices of G such that

$$x_0 = x,\ x_N = y \text{ and } (x_{i-1}, x_i) \in E(G),\ i = 1, 2, \dots, N.$$

A graph G is said to be connected if there is at least a path between any two vertices. If $G = (V(G), E(G))$ is a graph and $H \subset V(G)$, then the graph $(H, E(H))$ with $E(H) = E(G) \cap (H \times H)$ is called the *subgraph of G determined by H*. Denote it by G_H.

If $\tilde{G} = (X, E(\tilde{G}))$ is the symmetric graph obtained by putting together the vertices of both G and G^{-1}, i.e.,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}),$$

then G is called *weakly connected* if $\tilde{G}$ is connected.

A mapping $T : X \to X$ is said to be (well) defined on a metric space endowed with a graph G if it has the property

$$\forall x, y \in X,\ (x, y) \in E(G) \text{ implies } (Tx, Ty) \in E(G). \tag{2.1}$$

According to [32], a mapping $T : X \to X$, which is well defined on a metric space endowed with a graph G, is called a *G-contraction* if there exists a constant $\alpha \in (0, 1)$ such that for all $x, y \in X$ with $(x, y) \in E(G)$ we have

$$d(Tx, Ty) \le \alpha \cdot d(x, y). \tag{2.2}$$

In the sequel we present a fixed point theorem for contractions on metric spaces endowed with a graph. It is actually a simplified form of a result (Theorem 3.2) established in a more extended form in [32]. The proof presented here is also significantly simplified, in order to meet the purposes of the present paper.

Theorem 2.1. *Let (X, d, G) be a metric space endowed with a simple directed and weakly connected graph G such that the following property (L) holds: for any sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, there exists a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$ satisfying*

$$(x_{k_n}, x) \in E(G),\ \forall n \in \mathbb{N}. \tag{2.3}$$

Let $T : X \to X$ be a G-contraction. If $X_T := \{x \in X : (x, Tx) \in E(G)\} \neq \emptyset$, then T is a Picard operator, i.e.,

(i) T has a unique fixed point in X, say x^*;

(ii) Picard iteration $\{x_n := T^n x_0\}_{n=1}^{\infty}$ converges to x^*, for all $x_0 \in X_T$, and the following estimates hold

$$d(x_n, x^*) \le \frac{\alpha^n}{1-\alpha}\, d(x_0, x_1)\,, \quad n = 0, 1, 2, \dots \tag{2.4}$$

$$d(x_n, x^*) \le \frac{\alpha}{1-\alpha}\, d(x_{n-1}, x_n)\,, \quad n = 1, 2, \dots \tag{2.5}$$

Proof. Let $x_0 \in X_T$. This means that $(x_0, Tx_0) \in E(G)$ and in view of (2.1), we have

$$(T^n x_0, T^{n+1} x_0) \in E(G),\ \forall n \in \mathbb{N}. \tag{2.6}$$

Denote $x_n := T^n x_0$, for all $n \in \mathbb{N}$. Then by the fact that T is a G-contraction and in view of (2.1), we get

$$d(x_n, x_{n+1}) \le \alpha d(x_{n-1}, x_n),\ \forall n \in \mathbb{N}. \tag{2.7}$$

Using (2.7) we obtain by induction

$$d(x_n, x_{n+1}) \le \alpha^n d(x_0, x_1)\,, \quad n = 0, 1, 2, \dots$$

and then

$$\begin{aligned} d(x_n, x_{n+p}) &\le \alpha^n \big(1 + \alpha + \cdots + \alpha^{p-1}\big) d(x_0, x_1) = \\ &= \frac{\alpha^n}{1-\alpha}\,(1 - \alpha^p) \cdot d(x_0, x_1)\,, \quad n, p \in \mathbb{N},\ p \neq 0\,. \end{aligned} \tag{2.8}$$

Since $0 < \alpha < 1$, (2.8) shows that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and hence is convergent in (X, d, G). Denote

$$x^* = \lim_{n\to\infty} x_n\,. \tag{2.9}$$

By the property (L) of (X, d, G), there exists a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$ satisfying

$$(x_{k_n}, x) \in E(G),\ \forall n \in \mathbb{N}.$$

and hence, by the contraction condition (2.2) and in view of (2.1),

$$d(Tx_{k_n}, Tx^*) \le \alpha d(x_{k_n}, x^*). \tag{2.10}$$

Therefore,

$$d(x^*, Tx^*) \le d(x^*, x_{k_n+1}) + d(x_{k_n+1}, Tx^*) = d(x_{k_n+1}, x^*) + d(Tx_{k_n}, Tx^*)\,.$$

which, by (2.10) yields

$$d(x^*, Tx^*) \le d(x^*, x_{k_n+1}) + \alpha \cdot d(x_{k_n}, x^*)\,, \tag{2.11}$$

valid for all $n \ge 1$. Now, by letting $n \to \infty$ in (2.11), we obtain

$$d(x^*, Tx^*) = 0$$

i.e., x^* is a fixed point of T.

Note that the uniqueness of x^* easily follows by the contraction condition (2.2).

The estimate (2.4) is obtained from (2.8) by letting $p \to \infty$.

In order to obtain (2.5), observe that by (2.7) we inductively obtain

$$d(x_{n+k}, x_{n+k+1}) \le \delta^{k+1} \cdot d(x_{n-1}, x_n)\,, \quad k, n \in \mathbb{N}\,,$$

and hence, similarly to deriving (2.8), one obtains

$$d(x_n, x_{n+p}) \le \frac{\delta(1-\delta^p)}{1-\delta}\, d(x_{n-1}, x_n)\,, \quad n \ge 1,\ p \in \mathbb{N}^*\,. \tag{2.12}$$

Now letting $p \to \infty$ in (2.12), the desired estimate (2.5) follows. □

Remark 2.2. A similar result can be obtained by replacing the property (L) of the triple (X, d, G) in Theorem 2.1 by the orbitally G-continuity of T.

Recall, see [32], Definition 2.4, that a mapping $T : X \to X$ defined on the metric space endowed with a graph (X, d, G) is called *orbitally G-continuous* if for all $x, y \in X$ and any sequence $\{k_n\}_{n=1}^{\infty}$ of positive integers,

$T^{k_n}x \to y$ and $(T^{k_n}x, T^{k_n+1}x) \in E(G)$, for all $n \in \mathbb{N}$, imply $T(T^{k_n}x) \to Ty$.

Theorem 2.3. *Let (X, d, G) be a metric space endowed with a simple directed and weakly connected graph G. Let $T : X \to X$ be a G-contraction.*

If $X_T := \{x \in X : (x, Tx) \in E(G)\} \ne \emptyset$ and T is a orbitally G-continuous mapping, then T is a Picard operator and the estimates (2.4) *and* (2.5) *hold.*

Proof. The proof essentially runs as for the previous theorem. The only difference occurs at the step where we have to prove that x^* is a fixed point of T.

We know that $x_n = T^n x_0$ converges to x^* as $n \to \infty$. Since, by hypothesis, T is orbitally G-continuous, we have that

$$x_{n+1} = T(T^n x_0) \to Tx^*.$$

On the other hand, x_{n+1} converges to x^* as well. So, $x^* = Tx^*$, as required. □

We end this section by presenting some examples of graphs, mainly taken from the paper [32], which are important in this context.

Example 2.4. If G_0 is the complete graph on X, that is, $E(G_0) = X \times X$, then a G_0-contraction is a usual contraction in the sense of Banach, i.e., it satisfies

$$d(Tx, Ty) \leq \alpha \cdot d(x, y), \text{ for all } x, y \in X.$$

Example 2.5. Let X be a nonempty set endowed with a partial ordering $\preceq$. Consider the graph G_1 on X whose set of vertices is given by

$$E(G_1) = \{(x, y) \in X \times X : x \preceq y\}.$$

As shown by Jachymski [32], Example 2.3, the class of G_1-contractions corresponds to the mappings studied by Nieto and Rogriguez-Lopez [36].

Example 2.6. Let X be a nonempty set endowed with a partial ordering $\preceq$. Consider the graph G_1 on X whose set of vertices is given by

$$E(G_2) = \{(x, y) \in X \times X : x \preceq y \vee y \preceq x\}.$$

Then the mappings studied by Ran and Reurings [42] are G_2-contractions. As shown by Jachymski [32], Example 2.4, the class of G_2-contractions is actually larger and coincides with the class of mappings studied by Petruşel and Rus [41] and Nieto and Rogriguez-Lopez [37].

Note also that Jachymski [32] has showed, see Example 2.4, that there exist G-contractions which are not usual contractions, so Theorems 2.1 and Theorem 2.3 are effective generalizations of the classical contraction mapping principle.

3. Fixed point theorems for non-self contractions in Banach spaces endowed with a graph

Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a non-self mapping. If $x \in K$ is such that $Tx \notin K$, then we can always choose an $y \in \partial K$ (the boundary of K) such that $y = (1 - \lambda)x + \lambda Tx \, (0 < \lambda < 1)$, which actually expresses the fact that

$$d(x, Tx) = d(x, y) + d(y, Tx), \; y \in \partial K, \tag{3.1}$$

where we denoted $d(x, y) = \|x - y\|$.

A related condition to that given by (3.1), called inward condition, has been used by Caristi [26] to obtain a generalization of contraction mapping principle for nonself mappings. The inward condition is more general since it does not require y in (3.1) to belong to ∂K.

Note also that, in general, the set Y of points y satisfying condition (3.1) above may contain more than one element.

For a nonself mapping $T : K \to X$ we shall say that it is (well) defined on the Banach space X endowed with the graph G if it has this property for the subgraph of G induced by K, that is,

$$(x, y) \in E(G) \text{ with } Tx, Ty \in K \text{ implies } (Tx, Ty) \in E(G) \cap (K \times K), \tag{3.2}$$

for all $x, y \in K$.

The next theorem extends Theorem 2.1 and thus establishes a fixed point theorem for non self contractions defined on a Banach space endowed with a graph.

Theorem 3.1. *Let (X, d, G) be a Banach space endowed with a simple directed and weakly connected graph G such that the property (L) holds, i.e., for any sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, there exists a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$ satisfying*

$$(x_{k_n}, x) \in E(G), \forall n \in \mathbb{N}. \tag{3.3}$$

Let K be a nonempty closed subset of X and $T : K \to X$ be a G_K-contraction, that is, there exists a constant $\delta \in [0, 1)$ such that

$$d(Tx, Ty) \leq \delta \cdot d(x, y), \text{ for all } (x, y) \in E(G_K), \tag{3.4}$$

where G_K is the subgraph of G determined by K.

If $K_T := \{x \in \partial K : (x, Tx) \in E(G)\} \neq \emptyset$ and T satisfies Rothe's boundary condition

$$T(\partial K) \subset K, \tag{3.5}$$

then

(i) *$Fix\,(T) = \{x^*\}$;*

(ii) *Picard iteration $\{x_n = T^n x_0\}_{n=1}^{\infty}$ converges to x^*, for all $x_0 \in K_T$, and the following estimates hold*

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta}\, d(x_0, x_1)\,, \quad n = 0, 1, 2, \ldots \tag{3.6}$$

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta}\, d(x_{n-1}, x_n)\,, \quad n = 1, 2, \ldots \tag{3.7}$$

Proof. If $T(K) \subset K$, then T is actually a self mapping of the closed set K and the conclusion follows by Theorem 3.2 in [32] with $X = K$. Therefore, we consider the case $T(K) \not\subset K$. Let $x_0 \in K_T$. This means that $(x_0, Tx_0) \in E(G)$ and in view of (2.1), we have

$$(T^n x_0, T^{n+1} x_0) \in E(G), \forall n \in \mathbb{N}. \tag{3.8}$$

Denote $y_n := T^n x_0$, for all $n \in \mathbb{N}$.

By (3.5) it also follows that $Tx_0 \in K$.

Denote $x_1 := y_1 = Tx_0$. Now, if $Tx_1 \in K$, set $x_2 := y_2 = Tx_1$. If $Tx_1 \notin K$, we can choose an element x_2 on the segment $[x_1, Tx_1]$ which also belong to ∂K, that is,

$$x_2 = (1 - \lambda)x_1 + \lambda T x_1 \, (0 < \lambda < 1).$$

Continuing in this way we obtain two sequences $\{x_n\}$ and $\{y_n\}$ whose terms satisfy one of the following properties:

i) $x_n := y_n = Tx_{n-1}$, if $Tx_{n-1} \in K$;

ii) $x_n = (1 - \lambda)x_{n-1} + \lambda T x_{n-1} \in \partial K \, (0 < \lambda < 1)$, if $Tx_{n-1} \notin K$.

To simplify the argumentation in the proof, let us denote

$$P = \{x_k \in \{x_n\} : \, x_k = y_k = Tx_{k-1}\}$$

and

$$Q = \{x_k \in \{x_n\} : \, x_k \neq Tx_{k-1}\}.$$

Note that $\{x_n\} \subset K$ for all $n \in \mathbb{N}$ and that, if $x_k \in Q$, then both x_{k-1} and x_{k+1} belong to the set P. Moreover, by virtue of (3.5), we cannot have two consecutive terms of $\{x_n\}$ in the set Q (but we can have two consecutive terms of $\{x_n\}$ in the set P) .

We claim that $\{x_n\}$ is a Cauchy sequence. To prove this, we must discuss three different cases: Case I. $x_n,\, x_{n+1} \in P$; Case II. $x_n \in P,\ x_{n+1} \in Q$; Case III. $x_n \in Q,\ x_{n+1} \in P$;

Case I. $x_n,\, x_{n+1} \in P$.

In this case we have $x_n = y_n = Tx_{n-1}$, $x_{n+1} = y_{n+1} = Tx_n$, hence

$$d(x_{n+1}, x_n) = d(y_{n+1}, y_n) = d(Ty_n, Ty_{n-1}).$$

Since, by (3.8), $(y_n, y_{n-1}) \in E(G)$, we have by the contraction condition (3.4)

$$d(Ty_n, Ty_{n-1}) = d(Tx_n, Tx_{n-1}) \leq \delta d(x_n, x_{n-1}),$$

and therefore,

$$d(x_{n+1}, x_n) \leq \delta d(x_n, x_{n-1}). \tag{3.9}$$

Case II. $x_n \in P,\ x_{n+1} \in Q$.

In this case we have $x_n = y_n = Tx_{n-1}$, but $x_{n+1} \neq y_{n+1} = Tx_n$ and

$$d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, Tx_n).$$

Thus $d(x_{n+1}, Tx_n) \neq 0$ and hence

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) - d(x_{n+1}, Tx_n) < d(x_n, Tx_n) = d(Tx_{n-1}, Tx_n),$$

since $x_n \in P$. So, by using (3.4) we get

$$d(x_n, x_{n+1}) < d(Tx_{n-1}, Tx_n) = d(Ty_{n-1}, Ty_n),$$

and using similar arguments to that in Case I, we obtain again inequality (3.9).

Case III. $x_n \in Q,\ x_{n+1} \in P$. In this case we have $x_{n+1} = Tx_n$, $x_n \neq y_n = Tx_{n-1}$ and

$$d(x_n, x_{n-1}) + d(x_n, Tx_{n-1}) = d(x_{n-1}, Tx_{n-1}). \tag{3.10}$$

Hence, by triangle inequality

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, Tx_{n-1}) + d(Tx_{n-1}, x_{n+1}) \\ &= d(x_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_n) \\ &= d(x_n, Tx_{n-1}) + d(Ty_{n-1}, Ty_n). \end{aligned}$$

Since, by (3.8), $(y_{n-1}, y_n) \in E(G)$, we obtain by the contraction condition (3.4)

$$d(Ty_{n-1}, Ty_n) \leq \delta d(y_{n-1}, y_n) = \delta d(x_{n-1}, x_n),$$

Thus, since $0 < \delta < 1$, by using (3.10), we have

$$\leq d(x_n, Tx_{n-1}) + \delta d(x_{n-1}, x_n) < d(x_n, Tx_{n-1}) + d(x_{n-1}, x_n) = d(x_{n-1}, Tx_{n-1}).$$

Since, by (3.8), $(x_{n-2}, x_{n-1}) = (y_{n-2}, y_{n-1}) \in E(G)$, by the contraction condition (3.4) we get

$$d(x_n, x_{n+1}) < d(x_{n-1}, Tx_{n-1}) = d(Tx_{n-2}, Tx_{n-1}) \leq \delta d(x_{n-2}, x_{n-1}). \tag{3.11}$$

Therefore, by summarizing all three cases and using (3.9) and (3.11), it follows that the sequence $\{d(x_n, x_{n-1})\}$ satisfies the inequality

$$d(x_n, x_{n+1}) \leq \delta \max\{d(x_{n-2}, x_{n-1}),\ d(x_{n-1}, x_n)\}, \tag{3.12}$$

for all $n \geq 2$. Now, by induction for $n \geq 2$, from (3.12) one obtains

$$d(x_n, x_{n+1}) \leq \delta^{[n/2]} \max\{d(x_0, x_1),\, d(x_1, x_2)\},$$

where $[n/2]$ denotes the greatest integer not exceeding $n/2$.

Further, for $m > n > N$,

$$d(x_n, x_m) \leq \sum_{i=N}^{\infty} d(x_i, x_{i-1}) \leq 2\frac{\delta^{[N/2]}}{1-\delta} \max\{d(x_0, x_1),\, d(x_1, x_2)\},$$

which shows that $\{x_n\}$ is a Cauchy sequence.

Since $\{x_n\} \subset K$ and K is closed, $\{x_n\}$ converges to some point x^* in K, i.e.,

$$x^* = \lim_{n \to \infty} x_n. \tag{3.13}$$

By property (L), there exists a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$ satisfying

$$(x_{k_n}, x^*) \in E(G),\ \forall n \in \mathbb{N}.$$

and hence, by the contraction condition (3.4),

$$d(Tx_{k_n}, Tx^*) \leq \alpha d(x_{k_n}, x^*). \tag{3.14}$$

Therefore,

$$d(x^*, Tx^*) \leq d(x^*, x_{k_n+1}) + d(x_{k_n+1}, Tx^*) = d(x_{k_n+1}, x^*) + d(Tx_{k_n}, Tx^*)\,.$$

which, by (3.14) yields

$$d(x^*, Tx^*) \leq d(x^*, x_{k_n+1}) + \delta \cdot d(x_{k_n}, x^*)\,, \tag{3.15}$$

for all $n \geq 1$. Letting now $n \to \infty$ in (3.15), we obtain

$$d(x^*, Tx^*) = 0,$$

which shows that x^* is a fixed point of T.

The uniqueness of x^* immediately follows by the contraction condition (3.4). □

Similarly to Theorem 2.3, one obtains the following result.

Theorem 3.2. *Let (X, d, G) be a Banach space endowed with a simple directed and weakly connected graph G. Let K be a nonempty closed subset of X and $T : K \to X$ be a G-contraction on K.*

If $K_T := \{x \in \partial K : (x, Tx) \in E(G)\} \neq \emptyset$, T is orbitally G-continuous and T satisfies Rothe's boundary condition

$$T(\partial K) \subset K,$$

then the conclusion of Theorem 3.1 remains valid.

Theorems 3.1 and 3.2 subsume several important results in the fixed point theory of self and nonself mappings. We illustrate them by the following example.

Example 3.3. Let $X = [0,1] \cup \{3\}$ be endowed with the usual norm and let $K = \{0,1,3\}$. Consider the function $T : K \to X$, defined by $Tx = 0$, for $x \in \{0,1\}$ and $T3 = 0.5$. Let the graph G be defined by

$$E(G) = \{(x,x) : x \in [0,1]\} \cup \{(0,1),(1,3),(3,3),(3,0.5)\}.$$

It is easy to check that T is well defined on the Banach space X endowed with the graph G_K.

Indeed, the subgraph G_K of G determined by K has the set of vertices $E(G_K) = \{(0,0),(0,1),(1,1),(1,3),(3,3)\}$ and it is easy to check that (3.2) holds, that is, for all $x, y \in K$,

$$(x,y) \in E(G) \text{ with } Tx, Ty \in K \text{ implies } (Tx,Ty) \in E(G) \cap (K \times K).$$

In view of (3.2), the edges $(1,3),(3,3)$ has to be removed and for the rest of edges we have

$$(T0,T0) = (T0,T1) = (T1,T1) = (0,0) \in E(G_K).$$

Moreover, G is weakly connected and T is a non self G-contraction on K with contraction coefficient $\alpha = \dfrac{1}{4}$, since

$$|T0 - T3| = \frac{1}{2} < \frac{1}{4} \cdot |0-3| \text{ and } |T1 - T3| = \frac{1}{2} \leq \frac{1}{4} \cdot |1-3|.$$

(for the rest of the edges of $E(G_K)$, the contraction condition (3.4) is obvious, since the quantity in its left hand side is always zero).

Property (L) holds with the only two constant sequences $\{x_n = 0\}$ and $\{x_n = 1\}$ satisfying the property $(x_n, x_{n+1}) \in E(G_K)$ for all $n \in \mathbb{N}$.

Rothe's boundary condition is also satisfied, as $\partial K = \{0,1\}$ and so $T(\partial K) = \{0\} \subset K$.

Finally, since we also have $K_T = \{0,1\} \neq \emptyset$, all assumptions in Theorem 3.1 are satisfied and 0 is the unique fixed point of T.

Remark 3.4. Note that we can obtain the conclusion in Example 3.3 by applying Theorem 3.2, too, since T is in this case orbitally G-continuous. But, since T is a non self mapping, we cannot apply neither Theorem 1 nor Theorem 3.2 in [32].

4. Particular cases, conclusions and further study

Theorems 3.1 and 3.2 established in this paper are, indeed, very general, at least for two main reasons.

First, they extend and unify several important fixed point theorems by considering nonself mappings instead of self mappings.

Secondly, by working on Banach spaces endowed with a graph, our results are valid not only for mappings that satisfy the contraction condition for pairs (x,y) of the whole space $X \times X$, but only for the vertices (x,y) of a simple directed and weakly connected graph $G = (X, E(G))$, with $E(G) \subset X \times X$.

Amongst the most important particular cases of Theorem 3.1 and Theorem 3.2, we mention in the following just a few of them.

1. If G is the graph G_0 in Example 2.4, then by Theorem 3.1 we obtain an extension of the Jachymski's theorem ([32], Theorem 3.2), in the simplified form

given by us in the present paper (Theorem 2.1) and restricted to Banach spaces instead of complete metric spaces.

2. If G is the graph G_1 in Example 2.5, then by Theorem 3.1 and Theorem 3.2 we obtain extensions of the results established by [36] to the case of nonself contractions (and also restricted to Banach spaces instead of complete metric spaces).

3. If G is the graph G_2 in Example 2.5, then by Theorem 3.1 and Theorem 3.2 we obtain extensions of the results established by Ran and Reurings [42], Petruşel and Rus [41], Nieto and Rogriguez-Lopez [37] etc., to the case of nonself contractions.

4. If $X = A \cup B$, $T(A) \subset B$, $T(B) \subset A$ and consider the graph $G = (X, E(G))$ with

$$E(G) = \{(x, y) \in X \times X : x \in A, \, y \in B\},$$

then by Theorem 3.1 and Theorem 3.2 we obtain extensions of the results established by Kirk, Srinivasan and Veeramani [33] to the case of nonself contractions, see also [39], [40].

Note also that Theorem 3.1 and Theorem 3.2, stated and proven here in the setting of a Banach space, could be extended to convex metric spaces without any conceptual difficulty.

Our further study will focus on establishing similar results but in the case of other important contraction conditions from the metrical fixed point theory.

Acknowledgements

The first author thanks the Cyprus Mathematical Society for the excellent conditions offered during the research visit in Nicosia, where this paper has been mainly written.

References

[1] M. Abbas, B. Ali and G. Petruşel, *Fixed points of set-valued contractions in partial metric spaces endowed with a graph*, Carpathian J. Math. **30** (2014), 129–137.

[2] R. P. Agarwal, M. A. El-Gebeily and D. O'Regan, *Generalized contractions in partially ordered metric spaces*, Appl. Anal. **87** (2008), 1–8.

[3] M. A. Alghamdi, V. Berinde and N. Shahzad, *Fixed points of multi-valued non-self almost contractions*, J. Appl. Math. Volume 2013, Article ID 621614, 6 pages http://dx.doi.org/10.1155/2013/621614.

[4] M. A. Alghamdi, V. Berinde and N. Shahzad, *Fixed points of non-self almost contractions*, Carpathian J. Math. **30** (2014), 1–8.

[5] N. A. Assad, *On a fixed point theorem of Iséki*, Tamkang J. Math. **7** (1976), 19–22.

[6] N. A. Assad, *On a fixed point theorem of Kannan in Banach spaces*, Tamkang J. Math. **7** (1976), 91–94.

[7] N. A. Assad, *On some nonself nonlinear contractions*, Math. Japon. **33** (1988), 17–26.

[8] N. A. Assad, *On some nonself mappings in Banach spaces*, Math. Japon. **33** (1988), 501–515.

[9] N. A. Assad, *Approximation for fixed points of multivalued contractive mappings*, Math. Nachr. **139** (1988), 207–213.

[10] N. A. Assad, *A fixed point theorem in Banach space*, Publ. Inst. Math. (Beograd) (N.S.) **47** (1990), 137–140.

[11] N. A. Assad, *A fixed point theorem for some non-self-mappings*, Tamkang J. Math. **21** (1990), 387–393.

[12] N. A. Assad and W. A. Kirk, *Fixed point theorems for set-valued mappings of contractive type*, Pacific J. Math. **43** (1972), 553–562..

[13] N. A. Assad and S. Sessa, *Common fixed points for nonself compatible maps on compacta*, Southeast Asian Bull. Math. **16** (1992), 91–95

[14] S. Banach, *Theorie des Operations Lineaires*,. Monografie Matematyczne, Warszawa-Lwow, 1932.

[15] V. Berinde, *A common fixed point theorem for nonself mappings.* Miskolc Math. Notes 5 (2004), no. 2, 137–144

[16] V. Berinde, *Approximation of fixed points of some nonself generalized ϕ-contractions.* Math. Balkanica (N.S.) 18 (2004), no. 1-2, 85–93

[17] V. Berinde, *Iterative Approximation of Fixed Points*, 2nd Ed., Springer Verlag, Berlin Heidelberg New York, 2007

[18] V. Berinde, *Convergence theorems for fixed point iterative methods defined as admissible perturbations of a nonlinear operator*, Carpathian J. Math. **29** (2013), 9–18.

[19] V. Berinde and M. Păcurar, *Fixed point theorems for nonself single-valued almost contractions*, Fixed Point Theory **14** (2013), 301–311.

[20] V. Berinde and M. Păcurar, *From a Dieudonné theorem concerning the Cauchy problem to an open problem in the theory of weakly Picard operators*, Carpathian J. Math. **30** (2014), 283–292.

[21] F. Bojor, *Fixed point of φ-contraction in metric spaces endowed with a graph*, Ann. Univ. Craiova, Math. Comput. Sci. **37** (2010), 85–92.

[22] F. Bojor,*Fixed points of Bianchini mappings in metric spaces endowed with a graph*, Carpathian J. Math. **28** (2012), 207–214.

[23] F. Bojor, *Fixed points of Kannan mappings in metric spaces endowed with a graph*, An. Stiint. Univ. "Ovidius" Constanta Ser. Mat. **20** (2012), 31–40.

[24] F. Bojor, *Fixed point theorems for Reich type contractions on metric spaces with a graph*, Nonlinear Anal. **75** (2012), 3895–3901.

[25] F. Bojor, *Fixed point theorems in in metric spaces endowed with a graph* (in Romanian), PhD Thesis, North University of Baia Mare, 2012.

[26] J. Caristi,*Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc. **215** (1976), 241–251.

[27] C. Chifu and G. Petruşel, *Generalized contractions in metric spaces endowed with a graph*, Fixed Point Theory Appl. 2012, 2012:161, 9 pp.

[28] L. B. Ćirić, *A remark on Rhoades' fixed point theorem for non-self mappings*, Internat. J. Math. Math. Sci. **16** (1993), 397–400.

[29] L. B. Ćirić, *Quasi contraction non-self mappings on Banach spaces*, Bull. Cl. Sci. Math. Nat. Sci. Math. (1998), 25–31.

[30] L. B. Ćirić, J. S. Ume, M. S. Khan and H. K. Pathak, *On some nonself mappings*, Math. Nachr. **251** (2003), 28–33.

[31] S.-H. Cho, *A fixed point theorem for a Ćirić-Berinde type mapping in orbitally complete metric spaces*, Carpathian J. Math. **30** (2014), 63–70.

[32] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc. **136** (2008), 1359–1373.

[33] W. A. Kirk, P.S. Srinivasan and P. Veeramani, *Fixed points for mappings satisfying cyclical contractive conditions*, Fixed Point Theory **4** (2003), 79–89.

[34] A. Mureşan, *The theory of some asymptotic fixed point theorems*, Carpathian J. Math. **30** (2014), 361–368.

[35] A. Nicolae, D. O'Regan and A. Petruşel, *Fixed point theorems for singlevalued and multivalued generalized contractions in metric spaces endowed with a graph*, Georgian Math. J. **18** (2011), 307–327.

[36] J. J. Nieto and R. Rodriguez-Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order **22** (2006), 223–239.

[37] J. J. Nieto and R. Rodriguez-Lopez, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta. Math. Sin., (Engl. Ser.) **23** (2007), 2205–2212. .

[38] J. J. Nieto, R. L. Pouso, and R. Rodriguez-Lopez, *Fixed point theorems in ordered abstract spaces*, Proc. Amer. Math. Soc. **135** (2007), 2505–2517.
[39] M. Petric, *Some results concerning cyclical contractive mappings*, Gen. Math. **18** (2010), 213–226.
[40] M. Petric, *Best proximity point theorems for weak cyclic Kannan contractions*, Filomat **25** (2011), 145–154.
[41] A. Petruşel and I. A. Rus, *Fixed point theorems in ordered L-spaces*, Proc. Amer. Math. Soc. **134** (2006), 411–418.
[42] A. C. M. Ran and M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. **132** (2004), 1435–1443.
[43] R. E. Rhoades, *A fixed point theorem for some non-self-mappings*, Math. Japon. **23** (1978/79), 457–459.
[44] I. A. Rus, *Generalized contractions*, Seminar on Fixed Point Theory **3** (1983) 1–130.
[45] I. A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.
[46] I. A. Rus, *Picard operators and applications*, Sci. Math. Jpn. **58** (2003), 191–219.
[47] I. A. Rus, A. Petruşel and G. Petruşel, *Fixed Point Theory*, Cluj University Press, Cluj-Napoca, 2008.
[48] I. A. Rus, M.-A. Şerban, *Basic problems of the metric fixed point theory and the relevance of a metric fixed point theorem*, Carpathian J. Math. **29** (2013), 239–258.

Manuscript received ,
revised ,

Vasile Berinde
Department of Mathematics and Computer Science North University of Baia Mare Victoriei 76, 430072 Baia Mare Romania and Department of Mathematics and Statistics King Fahd University of Petroleum and Minerals Dhahram, Kingdom of Saudi Arabia
E-mail address: vberinde@ubm.ro; vasile_berinde@yahoo.com; vasile.berinde@gmail.com

Mădălina Păcurar
Department of Statistics, Analysis, Forecast and Mathematics Faculty of Economics and Bussiness Administration Babeş-Bolyai University of Cluj-Napoca 56-60 T. Mihali St., 400591 Cluj-Napoca Romania
E-mail address: madalina.pacurar@econ.ubbcluj.ro