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COUPLED AND TRIPLE FIXED POINT THEOREMS FOR MIXED MONOTONE ALMOST CONTRACTIVE MAPPINGS IN PARTIALLY ORDERED METRIC SPACES

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Abstract. In this paper we obtain general coupled and triple fixed point theorems for mixed monotone almost contractive operators by considering a more general contractive condition and thus extending several previous results in literature.

1. Introduction

Let $(X, \leq)$ be a partially ordered set and endow the product space $X \times X$ with the following partial order:

$$\text{for } (x,y), (u,v) \in X \times X, (u,v) \leq (x,y) \Leftrightarrow x \geq u, y \leq v.$$

A mapping $F : X \times X \to X$ is called *mixed monotone* (see [22]) if $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and, respectively,

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

A pair $(x,y) \in X \times X$ is called a *coupled fixed point* of the mapping F if

$$F(x,y) = x \text{and } F(y,x) = y.$$

Here is the main existence result in [16].

Theorem 1.1 (Bhaskar and Lakshmikantham [16]). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a constant $k \in [0,1)$ with*

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)], \text{for each } x \geq u, y \leq v. \tag{1.1}$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \text{ and } y_0 \geq F(y_0, x_0),$$

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then there exist $x, y \in X$ *such that*

$$x = F(x, y) \text{ and } y = F(y, x).$$

In [16] Bhaskar and Lakshmikantham also established some uniqueness results for coupled fixed points and existence results for fixed points of F (in this context, x is a called a *fixed point* of F if $F(x, x) = x$).

The results of Bhaskar and Lakshmikantham in [16] combine in the context of coupled fixed point theory the main fixed point results obtained previously by Nieto and Rodriguez-Lopez in [25] and [26]. The last two papers, in turn, continue the fixed point theorem established in the seminal paper of Ran and Reurings [33], which has the merit to combine a metrical fixed point theorem (the contraction mapping principle) and an order theoretic fixed point result (Tarski's fixed point theorem).

Coupled fixed point theory itself is generally (but incorrectly) considered to start in 1987 with the paper by Guo and Lakshmikantham [22], who studied coupled fixed points of monotone continuous and discontinuous mappings in partially ordered Banach spaces. In fact, the first author who studied coupled fixed points of nonlinear mappings, even in conjunction with a contractive type condition (nonexpansiveness), appears to be Opoitsev, who in a series of papers [27–30], considered and studied the concept of coupled fixed point of a map $\widehat{T}$ defined as a pair (v, w) satisfying: $\widehat{T}(v, w) = v$; $\widehat{T}(w, v) = w$. (see [30], Chapter III, par. 2, pp. 128).

Anyway, one of the merits of the paper by Bhaskar and Lakshmikantham [16] is the fact that it relaunched the interest for the study of coupled fixed points of mixed monotone or monotone nonlinear mappings in partially ordered sets by considering this problem in conjunction with a contractive type condition more suitable than the one considered by Opoitsev [27–30]. There is an abundant literature on this topic developed in the last decade that cannot be cited completely here.

Amongst the most significant problems in nonlinear analysis, to which the coupled fixed point and / or triple fixed point theorems have found relevant applications, we mention the study of (periodic) two point boundary value problems, see [10, 16].

Starting from this background, our main aim in this paper is to obtain some general coupled and triple fixed point theorems for mixed monotone almost contractive operators, by considering a more general contractive condition than the ones considered in [8, 11, 16] and [23], thus extending and unifying several important theoretical results in this area of research.

2. Preliminaries

The starting point in writing the present paper consists of some coupled and triple fixed point theorems established in the papers [8, 11] and [23]. Our primary aim is to unify and generalize results in [8, 11, 16] and [23] and in many other related papers.

The first main result in [8] is the following coupled fixed point result which generalizes Theorem 1.1 (Theorem 2.1 in [16]).

Theorem 2.1 ([8]). *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Let* $F : X \times X \to X$

be a mixed monotone mapping for which there exists a constant $k \in [0,1)$ such that for each $x \geq u, y \leq v$,

$$d\left(F\left(x,y\right),F\left(u,v\right)\right)+d\left(F\left(y,x\right),F\left(v,u\right)\right)\leq k\left[d\left(x,u\right)+d\left(y,v\right)\right]. \tag{2.1}$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F\left(x_0,y_0\right) \text{ and } y_0 \leq F\left(y_0,x_0\right), \tag{2.2}$$

or

$$x_0 \geq F\left(x_0,y_0\right) \text{ and } y_0 \leq F\left(y_0,x_0\right), \tag{2.3}$$

then there exist $\overline{x}, \overline{y} \in X$ such that

$$\overline{x} = F\left(\overline{x},\overline{y}\right) \text{ and } \overline{y} = F\left(\overline{y},\overline{x}\right).$$

Berinde and Borcut [11] introduced and studied the concept of triple fixed point, as follows.

Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X,d) is a complete metric space. Consider on the product space $X \times X \times X$ the following partial order: for $(x,y,z), (u,v,w) \in X \times X \times X$,

$$(u,v,w) \leq (x,y,z) \Leftrightarrow x \geq u, y \leq v, z \geq w.$$

Definition 2.2. Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \rightarrow X$. We say that F has the mixed monotone property if $F(x,y,z)$ is monotone nondecreasing in x and z, and is monotone nonincreasing in y, that is, for any $x,y,z \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F\left(x_1,y,z\right) \leq F\left(x_2,y,z\right),$$

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F\left(x,y_1,z\right) \geq F\left(x,y_2,z\right),$$

and

$$z_1, z_2 \in X, z_1 \leq z_2 \Rightarrow F\left(x,y,z_1\right) \leq F\left(x,y,z_2\right).$$

Definition 2.3. An element $(x,y,z) \in X \times X \in X$ is called a triple fixed point of $F : X \times X \times X \rightarrow X$ if

$$F\left(x,y,z\right) = x, F\left(y,x,y\right) = y, \text{ and } F\left(z,y,x\right) = z.$$

The following theorem is the first main result in [11].

Theorem 2.4 ([11])**.** *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \times X \rightarrow X$ be a continuous mapping having the mixed monotone property on X^3. Assume that there exist $j,k,l \in [0,1)$ with $j+k+l < 1$ for which*

$$d\left(F\left(x,y,z\right),F\left(u,v,w\right)\right) \leq jd\left(x,u\right)+kd\left(y,v\right)+ld\left(z,w\right), \tag{2.4}$$

$\forall x \geq u, y \leq v, z \geq w$. If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F\left(x_0,y_0,z_0\right), y_0 \geq F\left(y_0,x_0,y_0\right) \text{ and } z_0 \leq F\left(z_0,y_0,x_0\right),$$

then there exist $x,y,z \in X$ such that

$$x = F\left(x,y,z\right), y = F\left(y,x,y\right) \text{ and } z = F\left(z,y,x\right).$$

Karapinar [23] generalized Theorem 2.4 by considering instead of the Banach type contraction condition (2.4), an almost contraction type condition, as first considered by Berinde [2]. For the case of single valued mappings $T : X \to X$, this condition is described in the following definition.

Definition 2.5. Let (X, d) be a metric space. A mapping $T : X \to X$ is called an almost contraction if there exist a constant $\delta \in [0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx)\,, \quad \text{for all } x, y \in X. \tag{2.5}$$

As shown by the next theorem, Karapinar [23] actually used another version of condition (2.5), i.e.,

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + L \min\{d(y, Tx), d(x, Ty)\}\,, \quad \text{for all } x, y \in X. \tag{2.6}$$

Theorem 2.6 ([23]). *Let $(X, d, \leq)$ be a partially ordered complete metric space. Let $F : X \times X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exist non-negative numbers $a, b, c \in [0, 1)$ with $a + b + c < 1$ and $L \geq 0$ for which we have*

$$d\left(F\left(x, y, z\right), F\left(u, v, w\right)\right) \leq ad\left(x, u\right) + bd\left(y, v\right) + cd\left(z, w\right) + \tag{2.7}$$

$$+L \cdot \min \left\{ \begin{array}{l} d(F(x, y, z), x), d(F(x, y, z), y), d(F(x, y, z), z), d(F(x, y, z), u), \\ d(F(x, y, z), v), d(F(x, y, z), w), d(F(u, v, w), x), d(F(u, v, w), y), \\ d(F(u, v, w), z), d(F(u, v, w), u), d(F(u, v, w), v), d(F(u, v, w), w), \end{array} \right\}$$

$\forall x \geq u, y \leq v, z \geq w$.

Assume also that either

1. *F is continuous or*
2. *X has the following properties:*
   (a) *if a non-decreasing sequence $\{x_n\}$ converges to x, then $x_n \leq x$, for all n;*
   (b) *if a non-increasing sequence $\{y_n\}$ converges to y, then $y_n \geq y$, for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F\left(x_0, y_0, z_0\right), y_0 \geq F\left(y_0, x_0, y_0\right) \text{ and } z_0 \leq F\left(z_0, y_0, x_0\right),$$

then there exist $x, y, z \in X$ such that

$$x = F\left(x, y, z\right), y = F\left(y, x, y\right) \text{ and } z = F\left(z, y, x\right).$$

In the next section we obtain some coupled fixed point results and, then, a triple fixed point theorem that unify the ideas in Theorem 2.1 (a symmetric contraction condition) and Theorem 2.6 (an almost contractive type condition). Thus, our results generalize and extend the main results in the papers [8, 11, 16] and [23] and many other related ones.

3. Main results

First, we start with the case of coupled fixed points, for the sake of simplicity of the exposition.

Theorem 3.1. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X^2. Assume that there exist $k \in [0, 1)$ and $L \geq 0$ such that*

$$d\left(F\left(x, y\right), F\left(u, v\right)\right) + d\left(F\left(y, x\right), F\left(v, u\right)\right) \leq k[d\left(x, u\right) + d\left(y, v\right)]+ \tag{3.1}$$

$$+L \cdot \min \begin{Bmatrix} d(F(x,y),x), d(F(x,y),y), d(F(x,y),u), d(F(x,y),v), \\ d(F(y,x),x), d(F(y,x),y), d(F(y,x),u), d(F(y,x),v), \\ d(F(u,v),x), d(F(u,v),y), d(F(u,v),u), d(F(u,v),v), \\ d(F(v,u),x), d(F(v,u),y), d(F(v,u),u), d(F(v,u),v) \end{Bmatrix},$$

$\forall x \geq u, y \leq v$.

Assume also that either

(1) *F is continuous or*
(2) *X has the following properties:*
  (a) *if a non-decreasing sequence $\{x_n\}$ converges to x, then $x_n \leq x$, for all n;*
  (b) *if a non-increasing sequence $\{y_n\}$ converges to y, then $y_n \geq y$, for all n.*

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F\left(x_0, y_0\right) \text{ and } y_0 \geq F\left(y_0, x_0\right), \tag{3.2}$$

then there exist $\overline{x}, \overline{y} \in X$ such that

$$\overline{x} = F\left(\overline{x}, \overline{y}\right) \text{ and } \overline{y} = F\left(\overline{y}, \overline{x}\right).$$

Proof. Let $x_0, y_0 \in X$ and satisfy (3.2). We construct inductively the sequences $\{x_n\}$ and $\{y_n\}$ by the formulas

$$x_{n+1} = F\left(x_n, y_n\right), n \geq 0, \tag{3.3}$$

and

$$y_{n+1} = F\left(y_n, x_n\right), n \geq 0, \tag{3.4}$$

respectively. By using the mixed monotony of F and (3.2), we obtain that, for all $n \geq 1$,

$$x_{n+1} = F\left(x_n, y_n\right) \geq x_n \geq \cdots \geq x_1 \geq x_0,$$

and

$$y_{n+1} = F\left(y_n, x_n\right) \leq y_n \leq \cdots \leq y_1 \leq y_0.$$

Now, by letting $x := x_{n-1}$, $y := y_{n-1}$, $u := x_n$, $v := y_n$ in (3.1), one obtains

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq k \cdot [d(x_{n-1}, x_n) + d(y_{n-1}, y_n)]+$$

$$L \cdot \min \left\{\ldots, d(F\left(x_{n-1}, y_{n-1}\right), x_n), \ldots, \right\},$$

and since $d(F(x_{n-1}, y_{n-1}), x_n) = 0$, we get

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \le k \cdot [d(x_{n-1}, x_n) + d(y_{n-1}, y_n)], n \ge 1.$$

The previous inequality shows that, for all $n \ge 0$,

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \le k^n \cdot c,$$

where $c := d(x_0, x_1) + d(y_0, y_1)$. So, for all $n \ge 0$, we also have

$$d(x_n, x_{n+1}) \le k^n \cdot c,$$

and

$$d(y_n, y_{n+1}) \le k^n \cdot c.$$

To prove that $\{x_n\}$ is a Cauchy sequence, for $n > m$, we evaluate $d(x_n, x_m)$ and find out that

$$\begin{aligned} d(x_n, x_m) &\le d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\le k^m \frac{1 - k^{n-m}}{1 - k}, \end{aligned}$$

for all natural numbers n, m with $n > m$. This proves that $\{x_n\}$ is indeed a Cauchy sequence. Similarly, $\{y_n\}$ is a Cauchy sequence, too.

As (X, d) is a complete metric space, there exist $\overline{x}, \overline{y} \in X$ such that

$$\lim_{n\to\infty} x_n = \overline{x} \text{ and } \lim_{n\to\infty} y_n = \overline{y}. \tag{3.5}$$

Now, suppose that assumption (1) holds. This means that F is continuous and therefore, by letting $n \to \infty$ in (3.3) and (3.4) we get

$$\overline{x} = F(\overline{x}, \overline{y}) \text{ and } \overline{y} = F(\overline{y}, \overline{x}),$$

respectively, which shows that $(\overline{x}, \overline{y})$ is a coupled fixed point of F.

Now, suppose that alternative (2) holds. Since $\{x_n\}$ is non-decreasing and $x_n \to \overline{x}$, and $\{y_n\}$ is non-increasing and $y_n \to \overline{y}$, it follows that

$$x_n \le \overline{x}, \text{ for all } n \ge 0,$$

and

$$y_n \ge \overline{y}, \text{ for all } n \ge 0,$$

respectively. Now, in view of the above inequalities, by letting $x := x_n$, $y := y_n$, $u := \overline{x}$, $v := \overline{y}$ in (3.1), one obtains

$$\begin{aligned} &d(F(x_n, y_n), F(\overline{x}, \overline{y})) + d(F(y_n, x_n), F(\overline{y}, \overline{x})) \le \\ &\le k \cdot [d(x_n, \overline{x}) + d(y_n, \overline{y})] + L \cdot \min\{\ldots, d(F(x_n, y_n), \overline{x}), \ldots, \}, \end{aligned} \tag{3.6}$$

for all $n \ge 0$.

By letting $n \to \infty$ in (3.6) and using (3.5), we get

$$d(F(\overline{x}, \overline{y}), \overline{x}) + d(F(\overline{y}, \overline{x}), \overline{y}) \le 0$$

which shows that

$$d(F(\overline{x}, \overline{y}), \overline{x}) = 0 \text{ and } d(F(\overline{y}, \overline{x}), \overline{y}) = 0,$$

that is,

$$\overline{x} = F(\overline{x}, \overline{y}) \text{ and } \overline{y} = F(\overline{y}, \overline{x}),$$

which shows again that $(\overline{x}, \overline{y})$ is a coupled fixed point of F. □

Remark 3.2. (1) Theorem 2.4, which is the main result in [8], can now be obtained as a particular case of Theorem 3.1, if we take $L = 0$ in (3.1);

(2) Following all steps involved in the proof of Theorem 3.1, we note that a more general result, stated as Theorem 3.3 below, also holds. We note that the contraction condition (3.7) in Theorem 3.3 is somehow much more closed to the almost contraction condition (2.5), while (2.4) in Theorem 2.4 is similar to the classical contraction condition.

Theorem 3.3. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X^2. Assume that there exist $k \in [0, 1)$ and $L \geq 0$ such that*

$$d\left(F\left(x, y\right), F\left(u, v\right)\right) + d\left(F\left(y, x\right), F\left(v, u\right)\right) \leq \tag{3.7}$$

$$\leq k[d\left(x, u\right) + d\left(y, v\right)] + L \cdot d(F(x, y), u),$$

$\forall x \geq u, y \leq v$.

Assume also that either

(1) *F is continuous or*
(2) *X has the following properties:*
  (a) *if a non-decreasing sequence $\{x_n\}$ converges to x, then $x_n \leq x$, for all n;*
  (b) *if a non-increasing sequence $\{y_n\}$ converges to y, then $y_n \geq y$, for all n.*

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F\left(x_0, y_0\right) \text{ and } y_0 \geq F\left(y_0, x_0\right), \tag{3.8}$$

then there exist $\overline{x}, \overline{y} \in X$ such that

$$\overline{x} = F\left(\overline{x}, \overline{y}\right) \text{ and } \overline{y} = F\left(\overline{y}, \overline{x}\right).$$

Remark 3.4. Theorem 2.4, which is the main result in [8], can also be obtained as a particular case of Theorem 3.3, if we take $L = 0$ in (3.7);

Since the contractivity conditions (3.1) and (3.7) are valid only for comparable elements in X^2, Theorems 3.1 and 3.3 cannot guarantee in general the uniqueness of the coupled fixed point.

It is therefore our interest now to find additional conditions to ensure that the coupled fixed point in Theorem 3.1 and Theorem 3.3 is in fact unique. Such a condition is the one used in Theorem 2.2 of Bhaskar and Lakshmikantham [16] or in Theorem 1.1 of Ran and Reurings [33]:

every pair of elements in X^2 has either a lower bound or an upper bound, which is known, see [16], to be equivalent to the following condition: for all $Y = (x, y)$, $\overline{Y} = (\overline{x}, \overline{y}) \in X^2$,

$$\exists Z = (z_1, z_2) \in X^2 \text{that is comparable to } Y \text{and } \overline{Y}. \tag{3.9}$$

Theorem 3.5. *Assume that all hypotheses of Theorem 3.1 are fulfilled. If, additionally, condition* (3.9) *is satisfied, then F has a unique coupled fixed point.*

Proof. Proof is similar to that of Theorem 4 in [8] and so we omit it. □

4. Conclusions

Existence and uniqueness results similar to the ones in [23], as well as tripled point results with identical components, like the ones in [16], could also be obtained by using slightly weaker contraction conditions than the contractive conditions in [8–11, 16] and [23]. For some other related results that could serve as starting points for similar approaches, see also [3, 4, 7, 9, 14, 15] and references cited there.

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