TRIPLED FIXED POINT THEOREMS FOR MIXED MONOTONE CHATTERJEA TYPE CONTRACTIVE OPERATORS

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ABSTRACT. Starting from the papers [Berinde, V., Borcut, M., *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*, Nonlinear Anal., **74** (2011), 4889-4897], [Borcut, M., Berinde, V., *Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces* , Appl. Math. Comput., **218 (10)** (2012), 5929-5936] and [Borcut, M., *Tripled coincidente point theorems for contractive type mappings in partially ordered metric spaces*, Appl. Math. Comput., **218** (2012), 7339-7346.], we present new results on the existence and uniqueness of tripled fixed points for nonlinear mappings in partially ordered complete metric spaces satisfying more general contractive conditions.

1. INTRODUCTION

In some very recent papers, Berinde and Borcut [6], Borcut and Berinde [7], Borcut [8] have introduced and studied the concept of *tripled fixed point*, respectively *tripled coincidence point* for nonlinear contractive mappings $F : X^3 \to X$, in partially ordered complete metric spaces and obtained existence as well as existence and uniqueness theorems of tripled fixed points, respectively of tripled coincidence points, for some classes of contractive type mappings.

The presented theorems in [6], [7], [8], extend several existing results in the literature: [14], [18], [15]. For the sake of completeness, we recall the main concepts and results from [6] which are needed for the present paper.

Let$(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Consider on the product space X^3 the following partial order: for $(x, y, z), (u, v, w) \in X^3$,

$$(u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u, y \leq v, z \geq w.$$

Definition 1. [6] *Let $(X, \leq)$ be a partially ordered set and $F : X^3 \to X$ a mapping. We say that F has the mixed monotone property if $F(x, y, z)$ is nondecreasing in x and z, and is nonincreasing in y, that is, for any $x, y, z \in X$,*

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z),$$

and

$$z_1, z_2 \in X, z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2).$$

Definition 2. [6] *An element $(x, y, z) \in X^3$ is called a tripled fixed point of $F : X^3 \to X$ if*

$$F(x, y, z) = x, F(y, x, y) = y, \text{ and } F(z, y, x) = z.$$

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Let (X, d) be a metric space. The mapping $\overline{d} : X \times X \times X \to X$, given by

$$\overline{d}\left[(x, y, z), (u, v, w)\right] = d(x, u) + d(y, v) + d(z, w),$$

defines a metric on $X \times X \times X$, which will be denoted for convenience by d, too.

Definition 3. *Let X, Y, Z be nonempty sets and $F : X \times X \times X \to Y$, $G : Y \times Y \times Y \to Z$. We define the symmetric composition (or, the s-composition, for short) of F and G, $G * F : X \times X \times X \to Z$, by*

$$(G * F)(x, y, z) = G\left(F(x, y, z), F(y, x, y), F(z, y, x)\right) (x, y, z \in X).$$

For each nonempty set X, denote by P_x the projection mapping

$$P_X : X \times X \times X \to X, P(x, y, z) = x \text{ for } x, y, z \in X.$$

The symmetric composition has the following properties:

Proposition 1. *(Associativity). If $F : X \times X \times X \to Y$, $G : Y \times Y \times Y \to Z$ and*

$$H : Z \times Z \times Z \to W, \text{ then } (H * G) * F = H * (G * F).$$

Proposition 2. *(Identity Element). If $F : X \times X \times X \to Y$, then*

$$F * P_X = P_Y * F = F.$$

Proposition 3. *(Mixed Monotonicity). If $(X, \leq)$, $(Y, \leq)$, $(Z, \leq)$ are partially ordered sets and the mappings $F : X \times X \times X \to Y, G : Y \times Y \times Y \to Z$ are mixed monotone, then $G * F$ is mixed monotone.*

Proposition 4. *If $(X, \leq)$ is a partially ordered set and F is mixed monotone, then $F^n = F * F^{n-1} = F^{n-1} * F$ is mixed monotone, for every $n \geq 1$.*

The first main result in [6] is given by the following theorem.

Theorem 1. [6] *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist the constants $j, k, l \in [0, 1)$ with $j + k + l < 1$ for which*

$$d\left(F(x, y, z), F(u, v, w)\right) \leq jd(x, u) + kd(y, v) + ld(z, w), \tag{1.1}$$

$\forall x \geq u, y \leq v, z \geq w$. If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), y_0 \geq F(y_0, x_0, y_0) \text{ and } z_0 \leq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ such that

$$x = F(x, y, z), y = F(y, x, y) \text{ and } z = F(z, y, x).$$

Remark 1. *If we take $j = k = l = \frac{\alpha}{3}$ in Theorem 1, then the contraction condition (1.1) can be written in a slightly simplified form*

$$d\left(F(x, y, z), F(u, v, w)\right) \leq \frac{\alpha}{3}[d(x, u) + d(y, v) + d(z, w)]. \tag{1.2}$$

Theorem 2. [6] *By adding to the hypotheses of Theorem 1 the condition: for every $(x, y, z), (x_1, y_1, z_1) \in X^3$, there exists a $(u, v, w) \in X^3$ that is comparable to (x, y, z) and (x_1, y_1, z_1), then the tripled fixed point of F is unique.*

Theorem 3. [6] *In addition to the hypotheses of Theorem 1, suppose that $x_0, y_0, z_0 \in X$ are comparable. Then $x = y = z$.*

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2. Main results

Based on the notions and results presented in the first section, we will prove new existence and uniqueness theorems for operators which verify a Chatterjea contraction type condition, adapted to the case X^3.

Theorem 4. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^3 \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ such that*

$$(2.1) \qquad d(F(x,y,z), F(u,v,w)) \leq \frac{k}{8}[d(x, F(u,v,w)) + d(y, F(v,u,v)) +$$

$$+d(z, F(w,v,u)) + d(u, F(x,y,z)) + d(v, F(y,x,y)) + d(w, F(z,y,x))].$$

Also suppose either

(a) F is continuous or

(b) X has the following property:

$$(i) \;\; \textit{if a nondecreasing sequence } \{x_n\} \to x, \textit{ then } x_n \leq x \textit{ for all } n,$$

$$(ii) \;\; \textit{if a nonincreasing sequence } \{y_n\} \to y, \textit{ then } y_n \geq y \textit{ for all } n.$$

If there exist $x_0, y_0, z_0 \in X$ such that,

$$(2.2) \qquad x_0 \leq F(x_0, y_0, z_0), y_0 \geq F(y_0, x_0, y_0) \textit{ and } z_0 \leq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ such that,

$$x = F(x,y,z), y = F(y,x,y) \textit{ and } z = F(z,y,x).$$

Proof. Let the sequences $\{x_n\}, \{y_n\}, \{z_n\} \subset X$ be defined by

$$x_{n+1} = F(x_n, y_n, z_n) = F^{n+1}(x_0, y_0, z_0), y_{n+1} = F(y_n, x_n, y_n) = F^{n+1}(y_0, x_0, y_0),$$

$$z_{n+1} = F(z_n, y_n, x_n) = F^{n+1}(z_0, y_0, x_0), (n = 0, 1, ...).$$

Since F^n is mixed monotone for every n (by Proposition 4), it follows by (2.2) that $\{x_n\}$ and $\{z_n\}$ are nondecreasing and $\{y_n\}$ is nonincreasing. Indeed, due to the mixed monotone property of F, it is easy to show that

$$x_2 = F(x_1, y_1, z_1) \geq F(x_0, y_0, z_0) = x_1$$

$$y_2 = F(y_1, x_1, y_1) \leq F(y_0, x_0, y_0) = y_1$$

$$z_2 = F(z_1, y_1, x_1) \geq F(z_0, y_0, x_0) = z_1$$

and thus we obtain three sequences satisfying the following conditions

$$x_0 \leq x_1 \leq ... \leq x_n \leq ...,$$

$$y_0 \geq y_1 \geq ... \geq y_n \geq ...,$$

$$z_0 \leq z_1 \leq ... \leq z_n \leq ....$$

Now, for $n \in \mathbb{N}$, denote

$$D_{x_{n+1}} = d(x_{n+1}, x_n), D_{y_{n+1}} = d(y_{n+1}, y_n), D_{z_{n+1}} = d(z_{n+1}, z_n)$$

and

$$D_{n+1} = D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}.$$

Using (2.1), we get

$$D_{x_{n+1}} = d(x_{n+1}, x_n) = d(F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1}))$$

$$\leq \frac{k}{8}\left[d\left(x_n, F_{x_{n-1}}\right) + d\left(y_n, F_{y_{n-1}}\right) + d\left(z_n, F_{z_{n-1}}\right)\right.$$
$$\left. + d\left(x_{n-1}, F_{x_n}\right) + d\left(y_{n-1}, F_{y_n}\right) + d\left(z_{n-1}, F_{z_n}\right)\right]$$
$$= \frac{k}{8}\left[d\left(x_n, x_n\right) + d\left(y_n, y_n\right) + d\left(z_n, z_n\right)\right.$$
$$\left. + d\left(x_{n-1}, x_{n+1}\right) + d\left(y_{n-1}, y_{n+1}\right) + d\left(z_{n-1}, z_{n+1}\right)\right]$$
$$= \frac{k}{8}\left[d\left(x_{n-1}, x_{n+1}\right) + d\left(y_{n-1}, y_{n+1}\right) + d\left(z_{n-1}, z_{n+1}\right)\right]$$
$$\leq \frac{k}{8}\left[d\left(x_{n-1}, x_n\right) + d\left(y_{n-1}, y_n\right) + d\left(z_{n-1}, z_n\right)\right.$$
$$\left. + d\left(x_n, x_{n+1}\right) + d\left(y_n, y_{n+1}\right) + d\left(z_n, z_{n+1}\right)\right]$$
$$= \frac{k}{8}\left[D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}\right],$$

and therefore

$$D_{x_{n+1}} \leq \frac{k}{8}\left[D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}\right]. \tag{2.3}$$

Similarly, we obtain for the sequences $\left\{D_{y_{n+1}}\right\}, \left\{D_{z_{n+1}}\right\}$

$$D_{y_{n+1}} = d\left(y_{n+1}, y_n\right) = d(F(y_n, x_n, y_n), F(y_{n-1}, x_{n-1}, y_{n-1}))$$
$$\leq \frac{k}{8}\left[d\left(y_n, F_{y_{n-1}}\right) + d\left(x_n, F_{x_{n-1}}\right) + d\left(y_n, F_{y_{n-1}}\right)\right.$$
$$\left. + d\left(y_{n-1}, F_{y_n}\right) + d\left(x_{n-1}, F_{x^n}\right) + d\left(y_{n-1}, F_{y_n}\right)\right]$$
$$= \frac{k}{8}\left[d\left(y_n, y_n\right) + d\left(x_n, x_n\right) + d\left(y_n, y_n\right)\right.$$
$$\left. + d\left(y_{n-1}, y_{n+1}\right) + d\left(x_{n-1}, x_{n+1}\right) + d\left(y_{n-1}, y_{n+1}\right)\right]$$
$$= \frac{k}{8}\left[2d\left(y_{n-1}, y_{n+1}\right) + d\left(x_{n-1}, x_{n+1}\right)\right]$$
$$\leq \frac{k}{8}\left[2d\left(y_{n-1}, y_n\right) + d\left(x_{n-1}, x_n\right) + d\left(x_n, x_{n+1}\right) + 2d\left(y_n, y_{n+1}\right)\right]$$
$$= \frac{k}{8}\left[D_{x_n} + 2D_{y_n} + D_{x_{n+1}} + 2D_{y_{n+1}}\right],$$

and so

$$D_{y_{n+1}} \leq \frac{k}{8}\left[D_{x_n} + 2D_{y_n} + D_{x_{n+1}} + 2D_{y_{n+1}}\right] \tag{2.4}$$

and

$$D_{z_{n+1}} = d\left(z_{n+1}, z_n\right) = d(F(z_n, y_n, x_n), F(z_{n-1}, y_{n-1}, x_{n-1}))$$
$$\leq \frac{k}{8}\left[d\left(z_n, F_{z_{n-1}}\right) + d\left(y_n, F_{y_{n-1}}\right) + d\left(x_n, F_{x_{n-1}}\right)\right.$$
$$\left. + d\left(z_{n-1}, F_{z_n}\right) + d\left(y_{n-1}, F_{y_n}\right) + d\left(x_{n-1}, F_{x_n}\right)\right]$$
$$= \frac{k}{8}\left[d\left(x_n, x_n\right) + d\left(y_n, y_n\right) + d\left(z_n, z_n\right)\right.$$
$$\left. + d\left(x_{n-1}, x_{n+1}\right) + d\left(y_{n-1}, y_{n+1}\right) + d\left(z_{n-1}, z_{n+1}\right)\right]$$
$$= \frac{k}{8}\left[d\left(x_{n-1}, x_{n+1}\right) + d\left(y_{n-1}, y_{n+1}\right) + d\left(z_{n-1}, z_{n+1}\right)\right]$$
$$\leq \frac{k}{8}\left[d\left(x_{n-1}, x_n\right) + d\left(y_{n-1}, y_n\right) + d\left(z_{n-1}, z_n\right)\right.$$
$$\left. + d\left(x_n, x_{n+1}\right) + d\left(y_n, y_{n+1}\right) + d\left(z_n, z_{n+1}\right)\right]$$

$$= \frac{k}{8}\left[D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}\right],$$

and therefore

$$D_{z_{n+1}} \leq \frac{k}{8}\left[D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}\right]. \tag{2.5}$$

By using relations (2.3), (2.4) and (2.5), we get

$$D_{n+1} \leq \frac{k}{8}\left[3D_{x_n} + 4D_{y_n} + 2D_{z_n} + 3D_{x_{n+1}} + 4D_{y_{n+1}} + 2D_{z_{n+1}}\right]$$

$$\leq \frac{k}{8}\left[4D_{x_n} + 4D_{y_n} + 4D_{z_n} + 4D_{x_{n+1}} + 4D_{y_{n+1}} + 4D_{z_{n+1}}\right]$$

$$\leq \frac{k}{2}\left[D_n + D_{n+1}\right].$$

Therefore, for all $n \geq 1$, we have

$$D_{n+1} \leq \alpha \cdot D_n \leq ... \leq \alpha^n \cdot D_1, \text{ where } \alpha = \frac{k}{2-k} \in [0,1), \text{ when } k \in [0,1).$$

Because $D_{x_{n+1}} \leq D_{n+1}, D_{y_{n+1}} \leq D_{n+1}$ and $D_{z_{n+1}} \leq D_{n+1}$, then we have

$$D_{x_{n+1}} \leq \alpha^n \cdot D_1, D_{y_{n+1}} \leq \alpha^n \cdot D_1 \text{ and } D_{z_{n+1}} \leq \alpha^n \cdot D_1 \tag{2.6}$$

This implies that $\{x_n\}, \{y_n\}, \{z_n\}$ are Cauchy sequences in X. Indeed, let $m \geq n$, then

$$d(x_m, x_n) \leq D_{x_m} + D_{x_{m-1}} + ... + D_{x_{n+1}} \leq$$

$$\leq \left[\alpha^{m-1} + \alpha^{m-2} + ... + \alpha^n\right] \cdot D_1 = \frac{\alpha^n - \alpha^m}{1-\alpha} \cdot D_1 < \frac{\alpha^n}{1-\alpha} \cdot D_1.$$

Similarly, we can verify that $\{y_n\}$ and $\{z_n\}$ are also Cauchy sequences. Since X is a complete metric space, there exist $x, y, z \in X$ such that,

$$\lim_{x\to\infty} x_n = x, \lim_{x\to\infty} y_n = y, \lim_{x\to\infty} z_n = z.$$

Finally, we claim that

$$x = F(x,y,z), y = F(y,x,y) \text{ and } z = F(z,y,x).$$

Assume the first assumption (a) holds. This means F is continuous at (x,y,z), and hence, for a given $\frac{\epsilon}{2} > 0$, there exists a $\delta > 0$ such that,

$$d((x,y,z),(u,v,w)) = d(x,u) + d(y,v) + d(z,w) < \delta$$

$$\Rightarrow d(F(x,y,z), F(u,v,w)) < \frac{\epsilon}{2}.$$

Since

$$\lim_{x\to\infty} x_n = x, \lim_{x\to\infty} y_n = y, \lim_{x\to\infty} z_n = z,$$

for $\eta = \min\left(\frac{\epsilon}{2}, \frac{\delta}{2}\right)$, there exist n_0, m_0, p_0 such that, for $n \geq n_0, m \geq m_0, p \geq p_0$,

$$d(x_n, x) < \eta, d(y_n, y) < \eta, d(z_n, z) < \eta.$$

Now, for $n \in \mathbb{N}, n \geq max\{n_0, m_0, p_0\}$, we have

$$d(F(x,y,z), x) \leq d(F(x,y,z), x_{n+1}) + d(x_{n+1}, x)$$

$$= d(F(x,y,z), F(x_n, y_n, z_n)) + d(x_{n+1}, x) < \frac{\epsilon}{2} + \eta \leq \epsilon,$$

and this implies that $x = F(x,y,z)$. Similarly, we can show that

$$y = F(y,x,y) \text{ and } z = F(z,y,x).$$

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Suppose now that (b) holds. Since $\{x_n\}, \{z_n\}$ are non-decreasing and $x_n \to x$, $z_n \to z$, and also $\{y_n\}$ is non-increasing and $y_n \to y$, from (b) we have $x_n \leq x, y_n \geq y$ and $z_n \leq z$, for all n. Then by triangle inequality and (2.1), we get

$$\begin{aligned} d(x, F(x,y,z)) &\leq d(x, x_{n+1}) + d(x_{n+1}, F(x,y,z)) \\ &= d(x, x_{n+1}) + d(F(x_n, y_n, z_n), F(x,y,z)) \\ &\leq d(x, x_{n+1}) + \frac{k}{8}[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) \\ &\quad + d(x, F(x,y,z)) + d(y, F(y,x,y)) + d(z, F(z,y,x))], \end{aligned} \tag{2.7}$$

$$\begin{aligned} d(y, F(y,x,y)) \leq d(y, y_{n+1}) + \frac{k}{8}[d(x_n, x_{n+1}) + 2d(y_n, y_{n+1}) \\ + d(x, F(x,y,z)) + 2d(y, F(y,x,y))], \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} d(z, F(z,y,x)) \leq d(z, z_{n+1}) + \frac{k}{8}[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) \\ + d(x, F(x,y,z)) + d(y, F(y,x,y)) + d(z, F(z,y,x))]. \end{aligned} \tag{2.9}$$

By summing (2.7), (2.8), (2.9) we obtain

$$\begin{aligned} d(x, F(x,y,z)) + d(y, F(y,x,y)) + d(z, F(z,y,x)) \\ \leq \frac{2}{2-k}[d(x, x_{n+1}) + d(y, y_{n+1}) + d(z, z_{n+1})] \\ + \frac{k}{4(2-k)}[3d(x_n, x_{n+1}) + 4d(y_n, y_{n+1}) + 2d(z_n, z_{n+1})], \end{aligned}$$

and let $n \to \infty$ one obtains

$$d(x, F(x,y,z)) + d(y, F(y,x,y)) + d(z, F(z,y,x)) \leq 0,$$

that is, $x = F(x,y,z), y = F(y,x,y), z = F(z,y,x)$. □

3. Uniqueness of tripled fixed points

In [6], [7] and [8] the authors also considered some additional conditions to ensure the uniqueness of the tripled fixed point and also appropriate conditions to ensure that for such a tripled fixed point (x,y,z) we have all components equal: $x = y = z$.

Similarly, one can prove that the tripled fixed point ensured by Theorem 4 is in fact unique, provided that the product space $X \times X \times X$ endowed with the partial order mentioned earlier possesses an additional property.

Theorem 5. *If, in addition to the hypotheses of Theorem 4, the condition: for every $(x,y,z), (x_1,y_1,z_1) \in X \times X \times X$, there exists a $(u,v,w) \in X \times X \times X$ that is comparable to (x,y,z) and (x_1,y_1,z_1), is satisfied, then the tripled fixed point of F is unique.*

Proof. If $(x^*, y^*, z^*) \in X \times X \times X$ is another tripled fixed point of F, then we show that

$$d((x,y,z), (x^*, y^*, z^*)) = 0,$$

where

$$\lim_{x\to\infty} x_n = x, \lim_{x\to\infty} y_n = y, \lim_{x\to\infty} z_n = z.$$

We consider two cases.

Case 1. If (x,y,z) is comparable to (x^*,y^*,z^*) with respect to the ordering in $X\times X\times X$ then, for every $n=0,1,2,\dots$, the triple

$$\left(F^n(x,y,z),F^n(y,x,y),F^n(z,y,x)\right)=(x,y,z) \text{ is comparable to}$$
$$\left(F^n(x^*,y^*,z^*),F^n(y^*,x^*,y^*),F^n(z^*,y^*,x^*)\right)=(x^*,y^*,z^*).$$

Also, using the process for obtaining (2.6), we get

$$\begin{aligned}d\left((x,y,z),(x^*,y^*,z^*)\right)&=d(x,x^*)+d(y,y^*)+d(z,z^*)\\ =d\left(F^n(x,y,z),F^n(x^*,y^*,z^*)\right)&+d\left(F^n(y,x,y),F^n(y^*,x^*,y^*)\right)\\ &+d\left(F^n(z,y,x),F^n(z^*,y^*,x^*)\right)\\ \le\alpha^n\left[d(x,x^*)+d(y,y^*)+d(z,z^*)\right]&=\alpha^n d\left((x,y,z),(y^*,x^*,z^*)\right),\alpha\in[0,1).\end{aligned}$$

By letting $n\to\infty$, this implies that $d\left((x,y,z),(y^*,x^*,z^*)\right)=0$.

Case 2 : If (x,y,z) are not comparable to (x^*,y^*,z^*), then there exists an upper bound or a lower bound $(u,v,w)\in X\times X\times X$ of (x,y,z) and (x^*,y^*,z^*). Then, for all $n=1,2,...,$

$$\left(F^n(u,v,w),F^n(v,u,v),F^n(w,v,u)\right) \text{ is comparable to}$$
$$\left(F^n(x,y,z),F^n(y,x,y),F^n(z,y,x)\right)=(x,y,z) \text{ and to}$$
$$\left(F^n(x^*,y^*,z^*),F^n(y^*,x^*,y^*),F^n(z^*,y^*,x^*)\right)=(x^*,y^*,z^*).$$

We have,

$$\begin{aligned}d\left(\begin{pmatrix}x\\y\\z\end{pmatrix},\begin{pmatrix}x^*\\y^*\\z^*\end{pmatrix}\right)&=d\left(\begin{pmatrix}F^n(x,y,z)\\F^n(y,x,y)\\F^n(z,y,x)\end{pmatrix},\begin{pmatrix}F^n(x^*,y^*,z^*)\\F^n(y^*,x^*,y^*)\\F^n(z^*,y^*,x^*)\end{pmatrix}\right)\\ &\le d\left(\begin{pmatrix}F^n(x,y,z)\\F^n(y,x,y)\\F^n(z,y,x)\end{pmatrix},\begin{pmatrix}F^n(u,v,w)\\F^n(v,u,v)\\F^n(w,v,u)\end{pmatrix}\right)\\ &+d\left(\begin{pmatrix}F^n(u,v,w)\\F^n(v,u,v)\\F^n(w,v,u)\end{pmatrix},\begin{pmatrix}F^n(x^*,y^*,z^*)\\F^n(y^*,x^*,y^*)\\F^n(z^*,y^*,x^*)\end{pmatrix}\right)\\ \le\alpha^n\{[d(x,u)+d(y,v)&+d(z,w)]+[d(u,x^*)+d(v,y^*)+d(w,z^*)]\}\to 0\\ \text{as } n\to\infty, \text{ and so } &d\left(\begin{pmatrix}x\\y\\z\end{pmatrix},\begin{pmatrix}x^*\\y^*\\z^*\end{pmatrix}\right)=0.\end{aligned}$$

□

Theorem 6. *In addition to the hypotheses of Theorem 4, suppose that $x_0,y_0,z_0\in X$ are comparable. Then $x=y=z$.*

Proof. Recall that $x_0,y_0,z_0,\in X$ are such that

$$x_0\le F(x_0,y_0,z_0),y_0\ge F(y_0,x_0,y_0),z_0\le F(z_0,y_0,x_0).$$

Now, if $x_0\le y_0$, and $z_0\le y_0$ we claim that, for all $n\in\mathbb{N}$, $x_n\le y_n$ and $z_n\le y_n$. Indeed, by the mixed monotone property of F,

$$x_1=F(x_0,y_0,z_0)\le F(y_0,x_0,y_0)=y_1$$

and

$$z_1=F(z_0,y_0,x_0)\le F(y_0,x_0,y_0)=y_1.$$

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Assume that $x_n \leq y_n$ and $z_n \leq y_n$ for some n. Then

$$x_{n+1} = F^{n+1}(x_0, y_0, z_0) = F(F^n(x_0, y_0, z_0), F^n(y_0, x_0, y_0), F^n(z_0, y_0, x_0)) \\ = F(x_n, y_n, z_n) \leq F(y_n, x_n, y_n) = y_{n+1},$$

and similarly for z_n. Now,

$$d(x, y) \leq d(x, x_{n+1}) + d(y, x_{n+1}) \leq d(x, x_{n+1}) + d(x_{n+1}, y_{n+1}) + d(y, y_{n+1}) \\ = d(x, F^{n+1}(x_0, y_0, z_0)) + d\left[F(F^n(x_0, y_0, z_0), F^n(y_0, x_0, y_0), F^n(z_0, y_0, x_0)),\right. \\ \left., F(F^n(y_0, x_0, y_0), F^n(x_0, y_0, x_0), F^n(y_0, x_0, y_0))\right] + d(y, y_{n+1}) \to 0$$

as $n \to \infty$.

This implies that $d(x, y) = 0$ and hence we have $x = y$.

Similarly, we obtain that $d(x, z) = 0$ and $d(y, z) = 0$. The other cases for x_0, y_0, z_0 are similar. □

4. Example and final remarks

Let $X = [0, 1]$ be endowed with the usual metric $d(x, y) = |x - y|$ and let $F : X^3 \to X$ be given by $F(x, y, z) = \frac{1}{20}$, for $(x, y, z) \in \left[0, \frac{4}{5}\right] \times [0, 1]^2$ and $F(x, y, z) = \frac{11}{80}$, for $(x, y, z) \in \left[\frac{4}{5}, 1\right] \times [0, 1]^2$.

Then F satisfies Chatterjea's contractive condition (2.1) with $k = \frac{14}{15} < 1$ but does not satisfy the Banach type contractive condition (1.1).

Let us first prove the first part of the assertion above. It suffices to completely cover the following limit case.

Case 1. $x \in \left[\frac{4}{5}, 1\right]$, and $u, y, z, v, w \in \left[0, \frac{4}{5}\right)$

In this case $F(x, y, z) = \frac{11}{80}$, $F(u, v, w) = \frac{1}{20}$ and so condition (2.1) reduces to

$$\left|\frac{11}{80} - \frac{1}{20}\right| \leq \frac{k}{8}\left[\left|x - \frac{1}{20}\right| + \left|y - \frac{1}{20}\right| + \left|z - \frac{1}{20}\right| + \left|u - \frac{11}{80}\right| + \left|v - \frac{1}{20}\right| + \left|w - \frac{1}{20}\right|\right]. \tag{4.1}$$

For $x \in \left[\frac{4}{5}, 1\right]$, we have

$$\left|x - \frac{1}{20}\right| \geq \left|\frac{4}{5} - \frac{1}{20}\right| = \frac{3}{4}$$

and hence the minimum value of the right hand side of (4.1) is greater or equal to $\frac{k}{8} \cdot \frac{3}{4}$.

Therefore, in order to have (4.1) satisfied for all $x \in \left[\frac{4}{5}, 1\right]$ and $u, y, z, v, w \in \left[0, \frac{4}{5}\right)$, with $x \geq u$, $y \leq v$, $z \geq w$, i.e.,

$$\left|\frac{1}{20} - \frac{11}{80}\right| \leq \frac{k}{8} \cdot \frac{3}{4},$$

it suffices to take k such that $\frac{14}{15} \leq k < 1$.

Note that for the remaining cases to be discussed, the right hand side of (2.1) will be greater than the value obtained in Case 1.

For example, in the **Case 2.** $x, v \in \left[\frac{4}{5}, 1\right]$ and $u, y, z, w \in \left[0, \frac{4}{5}\right)$, the minimum value of the right hand side of (2.1) will be greater or equal to $\frac{k}{8} \cdot \frac{6}{4}$.

Note also that in the cases $x, u \in \left[\frac{4}{5}, 1\right]$ or $x, u \in \left[0, \frac{4}{5}\right)$, the left hand side of (2.1) is always zero and so (1.2) is satisfied for all values of $y, z, v, w \in [0, 1]$.

This proves that, indeed, F satisfies (2.1) with $k = \frac{14}{15} < 1$.

F is not continuous but X satisfies property (b) in Theorem 4. Moreover, by taking $x_0 = 0$, $y_0 = \frac{1}{5}$ and $z_0 = \frac{1}{8}$, one can check that (2.2) is fulfilled. Thus, all assumptions in Theorem 4 are satisfied and hence F does admit tripled fixed points. By Theorem 5 we actually conclude that F has a unique tripled fixed point, $\left(\frac{1}{20}, \frac{1}{20}, \frac{1}{20}\right)$.

Now let us show that F does not satisfy (1.1).

Assume the contrary, that is, that F does satisfy (1.1) and take $\epsilon > 0$ such that $u = \frac{4}{5} - \epsilon \in \left[0, \frac{4}{5}\right)$, $x = \frac{4}{5}$ and $y = z$, $v = w \in [0, 1]$ arbitrary in (1.1) to obtain

$$\frac{7}{80} \leq i \cdot \epsilon, \epsilon > 0. \tag{4.2}$$

Now letting $\epsilon \to 0$ in (4.2) we reach to a contradiction. This proves that, indeed, F does not satisfy (1.1).

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