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APPROXIMATION OF FIXED POINTS OF SOME NONSELF GENERALIZED φ - CONTRACTIONS

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Abstract. A fixed point theorem for a class of nonself discontinuous mappings is extended to generalized nonself φ - contractions. The approximation of the fixed point by means of the Picard iteration is also discussed.
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1. Introduction

The Banach's fixed point theorem is one of the most useful results in fixed point theory. In a metric space setting it can be briefly stated as follows.

Theorem B. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a strict contraction, i.e., a map satisfying*

$$d(Tx, Ty) \leq \alpha \cdot d(x, y), \quad \forall\, x, y \in X, \tag{1.1}$$

where $0 < \alpha < 1$ is a constant. Then T has a unique fixed point in X.

Theorem B, together with its local variants, has many applications in solving nonlinear functional equations, but has one drawback - the contractive condition (1.1) forces T to be continuous throughout X.

In 1968, Kannan [8] obtained a fixed point theorem for mappings T that need not be continuous:

Theorem K. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a mapping for which there exists $a \in \left(0, \frac{1}{2}\right)$ such that*

$$d(Tx, Ty) \leq a\,[d(x, Tx) + d(y, Ty)]\,, \quad \textit{for all} \quad x, y \in X\,. \tag{1.2}$$

Then T has a unique fixed point in X.

Example 1. Let $X = \mathbb{R}$ be the set of reals with the usual norm and $T : X \longrightarrow X$ given by $Tx = 0$, if $x \in (-\infty, 2]$, and $Tx = -\frac{1}{2}$, if $x \in (2, \infty)$. Then T satisfies (1.2) with $a = \frac{1}{5}$ and T is not continuous.

Following Kannan's theorem, a lot of papers were devoted to obtaining fixed point theorems for various contractive conditions that do not require the continuity of T, see, for example, Rus [13].

One of the most general contractive conditions obtained in this way was given by Ciric [5].

Theorem C1. *Let (X,d) be a complete metric space and $T:X\to X$ a mapping that satisfies*

$$d(Tx,Ty)\leq h\cdot\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}\ , \tag{1.3}$$

for all $x,y\in X$ and for some constant $0<h<1$.

Then T has a unique fixed point in X.

Remarks.

1) As shown by Rhoades ([12], Theorem 2), a contractive mapping satisfying (1.3) is still continuous *at the fixed point*.

2) The fixed point theorems for contractive definitions of the form (1.1) - (1.3) were unified by many authors, see for example Berinde [2], Rus [13].

For any $T:X\to X$ and $x,y\in X$, where X is a metric space, let us denote

$$\begin{aligned} B(x,y) &= d(x,y);\\ K(x,y) &= \frac{1}{2}[d(x,Tx)+d(y,Ty)]\,;\\ C(x,y) &= \max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}\,. \end{aligned}$$

The following theorem formally unifies Banach's, Kannan's and Ciric's fixed point theorems.

Theorem G. *Let (X,d) be a complete metric space and $T:X\to X$ a mapping satisfying*

$$d(Tx,Ty)\leq\lambda\cdot E(x,y),\quad \textit{for all}\quad x,y\in X\,, \tag{1.4}$$

where λ is a constant, $0<\lambda<1$, and $E(x,y)$ is any of the expressions $B(x,y)$, $K(x,y)$ and $C(x,y)$.

Then T has a unique fixed point.

Remarks.

1) Theorem G above can be extended by considering a function $\varphi:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ which preserves some essential properties of the function

$$\varphi(t)=\lambda t,\quad t\in\mathbb{R}\qquad(0<\lambda<1) \tag{1.5}$$

and replacing condition (1.4) by a more general one:

$$d(Tx,Ty)\leq\varphi(E(x,y)),\quad \text{for all}\quad x,y\in X\,. \tag{1.6}$$

2) One of the first results of this kind were obtained by Browder [4]. The function φ involved in such fixed point theorems is usually called

comparison function and is supposed to satisfy at least the following two conditions:

(i_φ) φ is nondecreasing, i.e., $t_1 < t_2 \Rightarrow \varphi(t_1) \le \varphi(t_2)$;

(ii_φ) The sequence $\{\varphi^n(t)\}$ converges to zero, for each $t \in \mathbb{R}_+$, where φ^n stands for the n-th iterate of φ .

Example 2. It is easy to check that a comparison function φ needs to be neither linear, nor continuous, by considering $\varphi_1(t) = \dfrac{t}{1+t}\,, t \in \mathbb{R}_+$ and $\varphi_2(t) = \dfrac{1}{2}\,t$, if $0 \le t < 1$ and $\varphi_2(t) = t - \dfrac{1}{3}\,$, if $t \ge 1$.

To prove our main result we shall need the following Lemma.

Lemma 1. *If φ satisfies (i_φ), (ii_φ) and*

$$t \le \varphi(t)\,, \textit{for a certain } t \in \mathbb{R}_+, \tag{1.7}$$

then $t = 0$.

Proof. Suppose the contrary, i.e., there exists $t > 0$ such that (1.7) is satisfied. Then, by (i_φ) we inductively get

$$t \le \varphi^n(t)\,, \qquad n \ge 1.$$

In view of (ii_φ) this implies

$$t \le \varphi^n(t) \longrightarrow 0 \qquad \text{as} \qquad n \to \infty\,,$$

a contradiction. □

2. Nonself discontinuous contractive type mappings

All fixed points theorems stated in the previous section involve self mappings of a metric space. However, in many applications of the fixed point theorems, a mapping of a closed subset K of X is not generally a self mapping of K into K but into X, or to check the invariance condition $T(K) \subset K$ is very difficult.

Starting from Theorem C1, it was an open problem for more than 20 years to extend it from self maps $T : K \longrightarrow K$ satisfying (1.3) to corresponding nonself mappings $T : K \longrightarrow X$, $K \ne X$, satisfying (1.3).

Recently, Ciric [7] solved this problem by considering an additional boundary condition, but limiting his results to a Banach space setting.

Theorem C2. *Let E be a Banach space, K a nonempty closed subset of E and ∂K the boundary of K. Let $T : K \longrightarrow E$ be a nonself mapping satisfying (1.3) for all $x, y \in K$. If*

$$T(\partial K) \subset K \tag{2.1}$$

then T has a unique fixed point in K.

Such kind of theorems were obtained for other contractive type conditions by Assad [1], Rhoades [11] and Ciric [6]. Very recently, Radovanovic [9] extended the previous quoted results to the following general contractive type condition: there exists a constant h, $0 < h < 1$, such that for all $x, y \in K$

$$d(Tx, Ty) \leq h \cdot R(x, y) \tag{2.2}$$

where

$$R(x,y) = \max\left\{\frac{1}{2}\, d(x,y), d(x,Tx), d(y,Ty), m(x,y), \frac{1}{2}\, M(x,y)\right\}, \tag{2.3}$$

and

$$m(x,y) = \min\left\{d\,(x,Ty), d(y,Tx)\right\}, \tag{2.4}$$

$$M(x,y) = \max\left\{d\,(x,Ty), d(y,Tx)\right\}. \tag{2.5}$$

He also obtained the error estimate for the Picard iteration when approximating the fixed point of mappings satisfying (2.2).

The main aim of this paper is to extend the result of Radovanovic [9] to a more general contractive condition. Finally we also point out that (2.2) implies Ciric's condition (1.3).

3. Main result

To prove our main result we shall make use of the following simple property that holds in a linear normed space: if $x \in K$ and $y \notin K$, then there exists a point $z \in \partial K$ (the boundary of K) such that

$$d(x,z) + d(z,y) = d(x,y),$$

which is equivalent to the fact that $z \in \partial K \cap \operatorname{seg}[x,y]$.
We can state now our main result.

Theorem 1. *Let E be a Banach space, K a nonempty closed subset of E and $T : K \longrightarrow E$ a mapping for which there exists a continuous comparison function φ such that*

$$d(Tx, Ty) \leq \varphi\big(R(x,y)\big), \quad \forall\, x, y \in K, \tag{3.1}$$

where $R(x,y)$ is given by (2.3).

If T satisfies the Rothe's boundary condition (2.1) then T has a unique fixed point in K.

Proof. Starting from $x_0 \in \partial K$ we shall construct a sequence $\{x_n\}$ as follows.
Since , by (2.1), $Tx_0 \in K$, we compute $x_1 = Tx_0$. Now, if $Tx_1 \in K$, then set $x_2 = Tx_1$. If $Tx_1 \notin K$, then choose $x_2 \in \partial K$ such that

$$d(x_1, x_2) + d(x_2, Tx_1) = d(x_1, Tx_1).$$

Continuing in this manner, we obtain a sequence $\{x_n\}$ in K, having the following property: if $Tx_n \in K$, then $x_{n+1} = Tx_n$, otherwise x_{n+1} is a certain point in ∂K for which

$$d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, Tx_n).$$

This enables us to split se sequence $\{x_n\}$ into two disjoint sets A and B, where

$$\begin{aligned} A &= \big\{x_i \in \{x_n\} : x_i = Tx_{i-1}\big\}\,, \\ B &= \big\{x_i \in \{x_n\} : x_i \neq Tx_{i-1},\ x_i \in \partial K \quad \text{and} \\ &\qquad d(x_{i-1}, x_i) + d(x_i, Tx_{i-1}) = d(x_{i-1}, Tx_{i-1})\big\}\,. \end{aligned}$$

Note that, in view of (2.1), if $x_1 \in B$, then both x_{i+1} and x_{i-1} belong to A. Indeed, let suppose that $x_{i-1} \notin A$. Then $x_{i-1} \in B \subset \partial H$ and hence $Tx_{i-1} \in K$, by (2.1). By the rule of constructing $\{x_n\}$, it results $x_i = Tx_{i-1}$, i.e., $x_i \in A$, a contradiction.

We shall prove now that $\{x_n\}$ and $\{Tx_n\}$ are Cauchy sequences. Denote

$$\begin{aligned} S(n,k) &= \{x_j, Tx_j : n \le j \le n+k\};\ b(n,k) = \operatorname{diam}\big(S(n,k)\big) \\ S(n) &= \{x_j, Tx_j : n \le j\}; \qquad\qquad b(n) = \operatorname{diam}\big(S(n)\big)\,. \end{aligned}$$

Then, it is immediately that: a) $b(n,k) \uparrow b(n)$, as $k \to \infty$;

b) $\big\{b(n)\big\}$ is a decreasing sequence of positive terms and hence

$$b = \lim_{n\to\infty} b(n) \quad \text{exists}\,.$$

To prove than $\{x_n\}$ and $\{Tx_n\}$ are Cauchy sequences is suffices to show that $b = 0$. We claim that

$$b(n,k) \le \varphi\big(b\,(n-2, k+2)\big)\,, \quad n,k \ge 2\,. \tag{3.2}$$

To prove (3.32) we consider three cases.

Case 1. $b(n,k) = d(x_i, Tx_j)$ with $n \le i,\ j \le n+k$.
If $x_i = Tx_{i-1}$, that is, $x_i \in A$, then

$$b(n,k) = d(T_{i-1}, Tx_j) \le \varphi\big(R(x_{i-1}, x_j)\big) \le \varphi\big(b\,(n-2, k+2)\big)\,.$$

If $x_i \neq Tx_{i-1}$, then $x_{i-1} = Tx_{i-2}$ and so $x_i \in \partial K$ and $d(x_{i-1}, Tx_{i-1}) = d(x_{i-1}, x_i) + d(x_i, Tx_{i-1})$ or, equivalently, $x_i \in \operatorname{seg}[x_{i-1}, Tx_{i-1}]$.
Thus

$$\begin{aligned} b(n,k) &= d(x_i, Tx_j) \le \max\big\{d(Tx_{i-2}, Tx_i), d(Tx_{i-1}, Tx_i)\big\} \le \\ &\le \varphi\big(\max\{R(x_{i-2}, x_i), R(x_{i-1}, x_i)\}\big) \le \varphi\big(b\,(n-2, k+2)\big)\,. \end{aligned}$$

Case 2. $b(n,k) = d(x_i, x_j)$, with $n \le i,\ j \le n+k$.
Now, if $x_j = Tx_{j-1}$, then Case 2 reduces to Case 1.
If $x_j \neq Tx_{j-1}$, then like in the Case 1 we have $j \ge 2$, $x_{j-1} = Tx_{j-2}$ and

$$Tx_j \in \partial K \cap \operatorname{seg}[Tx_{j-2}, Tx_{j-1}]\,.$$

Hence

$$b(n,k) = d(x_i, x_j) \le \max\big\{d(x_i, Tx_{j-2}), d(x_i, Tx_{j-1})\big\}$$

and so Case 2 also reduces to Case 1.

Case 3. $b(n,k) = d(Tx_i, Tx_j)$, with $n \le i,\ j \le k$.
In view of the previous cases, this is trivial.

Therefore, (3.2) is proved. Since φ is continuous by letting $k \to \infty$ in (3.2) we obtain

$$b(n) \leq \varphi\big(b(n-2)\big)\,, \quad n \geq 2 \tag{3.3}$$

and then, letting $n \to \infty$ in (3.3) we obtain

$$b \leq \varphi(b)\,.$$

In view of Lemma 1, the last inequality implies $b = 0$ and so, both $\{x_n\}$ and $\{Tx_n\}$ are Cauchy sequences.

Since $x_n \in K$ and K is a closed bounded subset of a Banach space, we deduce that $\lim\limits_{n\to\infty} x_n = p \in K$.

Since

$$d(x_n, Tx_n) \leq b_n \longrightarrow 0\,, \quad \text{as} \quad n \to \infty$$

we also have $\lim\limits_{n\to\infty} Tx_n = p$, which shows that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_n = p \in K\,.$$

To prove that p is a fixed point of T we use the triangle inequality

$$d(p, Tp) \leq d(p, x_n) + d(x_n, Tx_n) + d(Tp, Tx_n) \leq$$

$$\leq d(p, x_n) + d(x_n, Tx_n) + \varphi\big(R(p, x_n)\big) \longrightarrow 0, \text{ as} \quad n \to \infty\,.$$

The uniqueness of p follows by (3.1). □

Remarks.

1) For $\varphi(t) = h \cdot t$, $t \in \mathbb{R}_+$, $0 < h < 1$, from Theorem 1 we obtain Theorem 1 in Radovanic [9].

2) Theorem 1 gives information on the existence and uniqueness of the fixed point of T, but does not provide an error estimate for approximating this fixed point, like in the aforementioned paper, where the following estimation is obtained

$$d(p, x_n) \leq \frac{a^{n-1}}{a-1} \cdot \max\big\{d\,(x_0, x_1), d(x_1, x_2)\big\}\,,$$

where $a = h^{1/2}$, with h the contraction coefficient.

3) However, if we consider an additional condition on φ involved in (3.1), that is

$$\sum_{k=0}^{\infty} \varphi^k(t) \quad \text{converges for all} \quad t \in \mathbb{R}_+\,, \tag{3.4}$$

then it is possible to obtain a similar error estimate for approximating the fixed point of T.

A mapping satisfying a condition of the form (3.1) is said to be a *generalized φ-contraction*, see Rus [13].

4) To approximate p, we may consider the subsequence $\{x_{i_0+2n}\}$, where $x_{i_0} \in A$.

5) Following exactly the same steps as in proving Theorem 1, we can prove a more general result.

Theorem 2. *Let E be a Banach space, K a nonempty closed subset of E and ∂K the boundary of K. Let $T : K \longrightarrow E$, $S : E \longrightarrow E$ and $S : K \longrightarrow K$. Suppose $\partial K \neq \varnothing$, S is continuous and let assume that S and T satisfy the following conditions:*

(i) There exists a comparison function φ such that for every $x, y \in K$

$$d(Tx, Ty) \leq \varphi\big(R_1(x, y)\big)\,,$$

where

$$R_1(x,y) = \max\left\{\frac{1}{2}\, d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), m(x,y), \frac{1}{2}\, M(x,y)\right\}\,,$$

and

$$\begin{aligned} m(x,y) &= \min\big\{d(Sx, Ty), d(Sy, Tx)\big\}\,, \\ M(x,y) &= \max\big\{d(Sx, Ty), d(Sy, Tx)\big\}\,. \end{aligned}$$

(ii) S and T are weakly commutative on K, i.e.

$$d(STx, TSx) \leq d(Sx, Tx)\,, \quad \text{for every} \quad x \in K\,.$$

$$(iii) \qquad \begin{aligned} &T(K) \cap K \subset S(K); \\ &T(\partial K) \subset K; \\ &S(\partial K) \supset \partial K\,. \end{aligned}$$

Then S and T have a unique common fixed point in K.

Remarks.

1) It is obvious that in the particular case $S = 1_E$ (the identity map), Theorem 2 reduces to Theorem 1.

2) The assumption "S is continuous" in Theorem 2 can be weakened to "S^m is continuous, for some integer $m > 0$", like in Theorem 3 of Rakocevic [10].

Many other common fixed point theorems originating in Theorem 1 can be obtained, following the other results in Rakocevic [10].

3) In order to show that condition (3.1) implies Ciric's condition (1.3), it suffices to note that

$$R(x,y) \leq C(x,y)\,.$$

References

[1] Assad, N.A., *On a fixed point theorem in Banach spaces*, Tamkang J. Math., 7 (1976), 91-94

[2] Berinde, V., *Generalized contractions and applications* (in Romanian), Editura Cub Press 22, Baia Mare, 1997

[3] Berinde, V., *Iterative Approximation of Fixed Points*, Editura Efemeride, Baia Mare, 2002

[4] Browder, F.E., *On the convergence of successive approximations for nonlinear functional equations* Indagat. Math. 30 (1968), 27-35

[5] Ciric, Lj.B., *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc 45 (1974), 267-273

[6] Ciric, Lj.B., *A remark on Rhoades fixed point theorem for nonself mappings*, Internat. J. Math & Math. Sci., 16 (1993), 397-400

[7] Ciric, Lj.B., *Quasi contraction nonself mappings on Banach spaces*, Bull. Acad. Serbe Sci. Arts, Cl. Sci. Math. Natur. Sci. Math 23 (1998), 25-31

[8] Kannan, R., *Some results on fixed points*, Bull. Calcutta Math. Soc., 60 (1968), 71-76

[9] Radovanovic, R., *Approximation of a fixed point of some nonself mappings*, Math. Balkanika, 15 (2001), 213-218

[10] Rakocevic, V., *Quasi contraction nonself mappings on Banach spaces and common fixed point theorems*, Publ. Math. Debrecen, 58 (2001), No. 3, 451-460

[11] Rhoades, B.E., *A fixed point theorem for some nonself mappings*, Math. Japon., 23(1978), 457-459

[12] Rhoades, B.E., *Contractive definitions and continuity*, Contemporary Math., 72(1988), 233-245

[13] Rus, I.A., *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001