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Tartalom/Content

ON THE RATE OF CONVERGENCE OF SOME NEWTON-TYPE ITERATIVE METHODS

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Abstract. A new convergence factor as well as a new type of rate of convergence for an iterative process are derived, in a similar way to those of quotient convergence factors and root-convergence factors, given by Ortega & Rheinboldt in [9].

This new concept of convergence rate, called exit rate of convergence, is suggested by some author's results concerning the Newton's iterative process [1] -[6] or certain Newton-type methods [7]. It is also motivated by the fact that estimates of the form (3) often arise naturally in the study of certain iterative processes of Newton's type, under various diferentiability conditions.

INTRODUCTION

For a nonlinear equation

$$f(x)=0 \tag{1}$$

the Newton method consists in constructing a sequence (x_n) given by

$$x_{n+1}=x_n-\left[f'(x_n)\right]^{-1}f(x_n). \tag{2}$$

If f satisfies some specific differentiability conditions then the Newton iteration given by (2) converges to x^*, the unique solution of (1).

From a computing point of wiew it is very important to know an *exit criterion* for the iterative process, that is a *stopping inequality* of the form

$$\|x_n-x^*\|\le c\cdot\|x_n-x_{n-1}\|^p. \tag{3}$$

If we are able to deduce such exit criteria for the Newton method (or for some Newton type methods) we can stop the iterative process to a certain step so that the solution is obtained with the desired precision.

Indeed, if we accept an approximation error $\epsilon>0$, a priori given, in order to obtain

$$\|x_n-x^*\|<\varepsilon,$$

we need - in view of the exit criterion (3) - to stop the iterative process to a certain step n for which we have

$$\|x_n-x_{n-1}\|<\left(\frac{\varepsilon}{C}\right)^{1/p} \tag{4}$$

(since (x_n) is convergent, this is always possible).

Unfortunately, there exist a few convergence results in the literature which give such exit criteria for the Newton method or for the Newton type methods.

Only the convergence order of the methods is usually given by an inequality of the form

$$\|x_{n+1}-x^*\|\le\overline{C}\cdot\|x_n-x^*\|^p \tag{5}$$

These exit criteria are usually obtained under strong differentiability conditions on the operator f.
For example, in the scalar case, that is $f:[a,b]\to\mathbb{R}$ is a real function defined on the interval $[a,b]$, if $f\in C^2[a,b]$ and $f''(x)\neq 0$ on $[a,b]$ and f satisfies some additional conditions, we have the following stopping inequality for the Newton method:

$$|x_n - x^*| \le C\,|x_n - x_{n-1}|^2,\quad C = \frac{M_2}{2m_1}.$$

The convergence order is given in this case by

$$|x_{n+1} - x^*| \le \overline{C}\,|x_n - x^*|^2,$$

this means, the convergence of the Newton's method is quadratic.
In the paper [7] we proved a general convergence theorem for a class of Newton type methods under weak differentiability conditions on the involved function f.

1. A CONVERGENCE THEOREM FOR CERTAIN NEWTON-TYPE METHODS

Let us now consider for the scalar equation

$$f(x) = 0,$$

where $f:[a,b]\to\mathbb{R}$, a Newton type method, that is, the iterative process is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\cdot g(x_n), \tag{6}$$

where $g:[a,b]\to\mathbb{R}$ is a given nonzero function.(The Newton method is obtained for $g\equiv 1$).
We need a result from [7]

THEOREM 1.
Let $f:[a,b]\to\mathbb{R}$ be a real function defined on the real interval $[a,b]$, $a<b$, satisfying

$$(f_1)\ \ f(a)f(b) < 0\ ;\ \ (f_2)\ \ f\in C^1[a,b],\ f'(x)\neq 0,\ x\in[a,b];$$

and let denote

$$m = \min_{x\in[a,b]} |f'(x)|,\qquad M = \max_{x\in[a,b]} |f'(x)|.$$

If g satisfies the following conditions

$$(g_1)\ g\in C[a,b],\ g(x)>0,\ x\in[a,b];\qquad (g_2)\ \max_{x\in[a,b]} g(x) \le \frac{2m}{M};$$

(g_3) The sequence (x_n) given by (6) remains in $[a,b]$, for each $x_0\in[a,b]$.

Then (x_n) converges to α, the unique solution of the equation $f(x)=0$ and the stopping inequality

$$|x_n - \alpha| \le \frac{M}{Km}\,|x_n - x_{n-1}|,\ n\ge 0$$

holds, where

$$k = \min_{x\in[a,b]} g(x).$$

Remark.

1) For $g \equiv 1$, from Theorem 1 we obtain a weak exit criterion for the classical Newton method, because we assume only

$$f \in C^1[a,b],\ f'(x) \neq 0,\ x \in [a,b],$$

instead of the usual condition

$$f \in C^2[a,b],\ f''(x) \neq 0,\ x \in [a,b].$$

If f'' however exists the stopping inequality may be improved to the usual estimation

$$|x_n - x^*| \le C|x_n - x_{n-1}|^2.$$

2)For

$$g(x) = \frac{f'(x)}{f(x) - f(b)}(x - b),$$

from Theorem 1 we obtain a result concerning the *regula falsi method*.

3)For $g(x) = \dfrac{f'(x_0)}{f(x)}$, where x_0 is the initial approximation we obtain a stopping inequality for *the modified Newton method*;

4)For

$$g(x) = \frac{f(x)f'(x)}{f(x + f(x)) - f(x)},$$

we obtain a stopping inequality for *the Steffensen's method*, and so on.

2.THE QUOTIENT AND ROOT-RATE OF CONVERGENCE

Let $(x_n) \subset \mathbb{R}^m$ be a convergent sequence to x* and generated by an iterative process.Motivated by an estimate of the form (5), Ortega & Rheinboldt introduced (in [9], chapter 9, pp. 281-298) the notions of quotient convergence factors and root-convergence factors, denoted by $Q_p(x_n)$ and $R_p(x_n)$, respectively, and defined by

$$Q_p(x_n) = \begin{cases} 0, \ if \ \ x_n = x^*, \ forall \ but \ finitely \ many \ n, \\ \lim\limits_{n \to \infty} sup \dfrac{\|x_{n-1} - x^*\|}{\|x_n - x^*\|^p}, \ if \ x_n \neq x^*, \ for \ all \ but \ finitely \ many \ n, \\ +\infty, \ otherwise, \end{cases} \tag{7}$$

respectively by

$$R_p(x_n) = \begin{cases} \lim\limits_{n \to \infty} sup \ \|x^n - x^*\|^{1/n}, \ if \ p = 1 \\ \lim\limits_{n \to \infty} sup \ \|x^n - x^*\|^{1/p^n}, \ if \ p > 1. \end{cases} \tag{8}$$

If $C(\mathcal{P}, x^*)$ denote the set of all sequence with limit x* generated by an iterative process $\mathcal{P}$, then $Q_p(\mathcal{P}, x^*) = sup \{Q_p(x_n) \mid (x_n) \in C(\mathcal{P}, x^*)\}$, $1 \le p < +\infty$ are called the **Q- factors** of $\mathcal{P}$ at x* with respect to the norm in which the $Q_p(x_n)$ are computed.

For a given iterative process $\mathcal{P}$ and limit point x^*, $Q_p(\mathcal{P}, x^*)$ as a function of p exhibitis some basic properties: Q_p is an isotone function of p which takes on only the values 0 and ∞ except at possibly one point. In a similar manner is introduced the R- factor of $\mathscr{P}$.

The primary motivation for introducing Q-factors of an iterative process is to have a precise mean of comparing the rate of convergence of different iterations. by the following.

Definition ([9]). Let $\mathscr{P}_1$ and $\mathscr{P}_2$ denote two iterative processes with the same limit point x*. and let $Q_p(\mathscr{P}_1, x^*)$ and $Q_p(\mathscr{P}_2, x^*)$ be the corresponding Q-factors computed in the same norm on R^m. Then we say that $\mathscr{P}_1$ is **Q-faster** than $\mathscr{P}_2$ at x* if there is a $p \in [1, \infty)$ such that

$$Q_p(\mathscr{P}_1, x^*) < Q_p(\mathscr{P}_2, x^*).$$

Example. ([9]) Let m= 2. e = (1,0). u = (1,1) and (x_n). (y_n) the sequences defined by

$$x_n = \begin{cases} \left(\frac{1}{2}\right)^n e, \ for\ n\ odd \\ \left(\frac{1}{2}\right)^{n-1} u, \ for\ n\ even, \end{cases} \qquad y_n = \left(\frac{5}{8}\right)^n u, \quad n=1,2,\dots .$$

Then $Q_1(x_n) < Q_1(y_n)$ under the euclidian norm, when (x_n) is Q-faster than (y_n). while $Q_1(x_n) < Q_1(y_n)$. under the norm $\|a\|_\infty = max(|a_1|, |a_2|)$, $a=(a_1, a_2)$. when (y_n) is Q-faster than (x_n).

3.THE EXIT RATE OF CONVERGENCE

In the sequel, we shall restrict ourself to the one dimensional case, although all these results could be stated in the m-dimensional case.

Let f, g be as in section 1 and let (x_n) be a given sequence generated by an iterative method of Newton type (6).

Definition 3.1. Let $(x_n) \subset R$ be a convergent with limit x*. Then the quatities

$$E_p(x_n) = \begin{cases} 0, \ if\ x_n = x^*, \ for\ but\ finitely\ many\ n, \\ \lim\sup_{n\to\infty} \dfrac{|x_n - x^*|}{|x_{n-1} - x_n|^p}, \ if\ x_n \neq x^*, \ for\ all\ but\ finitely\ many\ n \\ \infty, \ otherwise, \end{cases}$$

defined for all $p \in [1, \infty)$, are called the **exit convergence factors** , or **E-factors**, for short, of (x_n).

Remarks. 1) If $E_p = E_p(x_n) < +\infty$ for some $p \in [1, \infty)$, then for any ε>0, there exists a n_0 such that (3) holds with $C = E_p + \varepsilon$;

2) When considering not just one sequence but an iterative process $\mathscr{P}$, it is desirable that the rate-of-convergence indicator measures the worst possible asymptotic rate of convergence of any sequence of $\mathscr{P}$ with the same limit point.

So, we are led to introduce

Definition 3.2. Let $C(\mathscr{P}, x^*)$ denote the set of all sequences with timit x^* generated by an iterative process $\mathscr{P}$. Then

$$E_p(\mathscr{P}, x^*) = sup\{E_p(x_n) / (x_n) \in C(\mathscr{P}, x^*)\}, \quad 1 \le p < +\infty,$$

are called the **E-factors** of $\mathscr{P}$ at x^*.

Definition 3.3. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ denote two iterative processes with the same limit point x^*, and let $E_p(\mathscr{P}_1, x^*)$ and $E_p(\mathscr{P}_2, x^*)$ be the corresponding **E-factors**. Then $\mathscr{P}_1$ is **E-faster** than $\mathscr{P}_2$ at x^* if there is a $p \in [1, \infty)$ such that

$$E_p(\mathscr{P}_1, x^*) < E_p(\mathscr{P}_2, x^*).$$

If, for a given process $\mathscr{P}$, we have $E_1(\mathscr{P}, x^*)=0$, we say that the process has **E- superlinear** convergence at x^*, while, if $0 < E_1(\mathscr{P}, x^*) < \infty$, the convergence is called **E-linear**. Any process $\mathscr{P}$ for which $E_1(\mathscr{P}, x^*) \ge 1$ is called **E-sublinear**.

The main result of this paper is given by

THEOREM 2. Let all conditions in Theorem 1 be satisfied.
Then any Newton-type process $\mathscr{P}$ associated to equation (1) is not **E-sublinear.**

Proof. It results from Theorem 1.

Example. For the sequence (x_n), $x_n = \left(\frac{1}{2}\right)^n \cdot a$, a=const (wich is the Newton iteration for $f(x) = x^2$) we have $E_1(x_n) = 2$, hence $E_1(\mathcal{N}, 0)$ is superlinear ($\mathcal{N}$ = stands for the Newton's process).

Note. It is important to establish which of the usual Newton-type processes is E-faster. This is the subject of some forthcoming papers.

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