Mathematical
Inequalities
& Applications
Volume 16, Number 1 (2013), 279–288 doi:10.7153/mia-16-21

NEW SHARP ESTIMATES OF THE GENERALIZED EULER–MASCHERONI CONSTANT

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(*Communicated by N. Elezović*)

Abstract. The aim of this paper is to establish new sequences which converge towards the Euler-Mascheroni constant. Our results solve some open problems posed by Berinde [A new generalization of Euler's constant Creat. Math. Inform. 18 (2009) no. 2 123–128] and extend some results of DeTemple, [A quicker convergence to Euler's constant Amer. Math. Monthly 100 (1993) 468–470] and Sîntămărian [A generalization of Euler's constant, Numer. Algorithms 46 (2007), 141–151].

1. Introduction

One of the most important sequences in analysis of the form

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n,$$

considered by Leonhard Euler in 1735, is known to converge towards the limit $\gamma = 0.577215...$, which is now called the Euler-Mascheroni constant. First of all, we recall that the sequence $(\gamma_n)_{n\geqslant 1}$ converges to its limit like n^{-1}, since

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n}, \tag{1.1}$$

(see, *e.g.,* Alzer [1], or Young [14]). Tóth [13] proved

$$\frac{1}{2n+2/5} < \gamma_n - \gamma \leqslant \frac{1}{2n+1/3}, \quad n \geqslant 1, \tag{1.2}$$

then Qiu and Vuorinen [11] showed the double inequality

$$\frac{1}{2n} - \frac{1}{2n^2} < \gamma_n - \gamma \leqslant \frac{1}{2n} - \frac{\gamma - 1/2}{n^2}, \quad n \geqslant 1. \tag{1.3}$$

Questions on the fast approximations of the Euler-Mascheroni constant γ were also discussed by Karatsuba [4] and the following inequalities were obtained

$$\frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{126n^6} \leqslant \gamma_n - \gamma \leqslant \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4}. \tag{1.4}$$

Mathematics subject classification (2010): 33B15, 41A10, 42A16.
Keywords and phrases: Euler-Mascheroni constant, inequalities, approximations.

For every $a > 0$, the numbers of the form

$$\gamma(a) = \lim_{n\to\infty}\left(\frac{1}{a} + \frac{1}{a+1} + \dots + \frac{1}{a+n-1} - \ln\frac{a+n-1}{a}\right)$$

were introduced in the monograph by Knopp [5]. There are known now as the generalized Euler-Mascheroni constant, since $\gamma(1) = \gamma$. In the recent past, many authors were preoccupied to give increasingly accurate estimates for $\gamma(a)$, similar to those given for γ, like (1.1)–(1.4).

In this sense, we mention the following sequences

$$x_n = \frac{1}{a} + \frac{1}{a+1} + \dots + \frac{1}{a+n-1} - \ln\frac{a+n}{a}$$

and

$$y_n = \frac{1}{a} + \frac{1}{a+1} + \dots + \frac{1}{a+n-1} - \ln\frac{a+n-1}{a}$$

which converge to $\gamma(a)$ like n^{-1}, since Sîntămărian [12] proved that for every integer $n \geqslant 1$,

$$\frac{1}{2(n+a)} < \gamma(a) - x_n < \frac{1}{2(n+a)-2}$$

and

$$\frac{1}{2(n+a)} < y_n - \gamma(a) < \frac{1}{2(n+a)-2}.$$

We give new better bounds for these sequences, showing the following

THEOREM 1. *For every $a > 0$, and integer $n \geqslant 2$, we have*

$$\frac{1}{2(n+a)-\frac{1}{4}} < \gamma(a) - x_n < \frac{1}{2(n+a)-\frac{1}{3}} \tag{1.5}$$

and

$$\frac{1}{2(n+a)-\frac{4}{3}} < y_n - \gamma(a) < \frac{1}{2(n+a)-\frac{5}{3}}. \tag{1.6}$$

In some sense, the constants $\frac{1}{3}$ and $\frac{5}{3}$ are sharp in (1.5)–(1.6), as we can see from the following:

THEOREM 2. *a) For every $a \geqslant \frac{13}{30}$ and every integer $n \geqslant 1$, we have*

$$\frac{1}{2(n+a)-\frac{1}{3}+\frac{1}{18n}} < \gamma(a) - x_n.$$

b) For every $a \geqslant \frac{17}{30}$ and every integer $n \geqslant 1$, we have

$$\frac{1}{2(n+a)-\frac{5}{3}+\frac{1}{18n}} < y_n - \gamma(a).$$

Very recently, Berinde [2, Theorem 2.2] introduced the sequences

$$z_n(a,b)=\frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a+n-1}-\ln\left(\frac{a+n}{a}+b\right)$$

and

$$t_n(a,b)=\frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a+n-1}-\ln\left(\frac{a+n-1}{a}+b\right)$$

convergent to $\gamma(a)$, and proved that for every integer $n \geqslant 1$,

$$z_n(a,b)<z_{n+1}(a,b)<\gamma(a)<t_n(a,b)<t_{n+1}(a,b)$$

and

$$0<\frac{1}{a}-\ln\left(1+b+\frac{1}{a}\right)<\gamma(a)<\frac{1}{a}-\ln b.$$

It is introduced in [7] the following general class of sequences

$$\mu_n(a,b,c)=\frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a+n-2}+\frac{c}{a+n-1}-\ln\left(\frac{a+n-1}{a}+b\right),$$

depending on parameters $a,b,c,$ with $a>0$ and $b>-(a+1)/a$. Remark that $\mu_n(a,b,1)=t_n(a,b)$. A particular case of [7, Theorem 2.1] solves an open problem posed by Berinde [2] about the sequence $(t_n(a,b))_{n\geqslant 1}$. This answer is gathered in the following

THEOREM 3. *Let $a,\ b\in\mathbb{R}$ be given and satisfy $a>0$ and $b>-(a+1)/a$.*
a) If $b\neq\frac{1}{2a}$, the speed of convergence of the sequence $(t_n(a,b))_{n\geqslant 2}$ is equal to n^{-1}, since

$$\lim_{n\to\infty} n\left(t_n(a,b)-\gamma(a)\right)=\frac{1}{2}-ab\neq 0.$$

b) If $b=\frac{1}{2a}$, the speed of convergence of the sequence

$$\beta_n=\frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a+n-2}+\frac{1}{a+n-1}-\ln\left(\frac{a+n-1}{a}+\frac{1}{2a}\right)$$

equals n^{-2}, since

$$\lim_{n\to\infty} n^2\left(\beta_n-\gamma(a)\right)=\frac{1}{24}.$$

The proof of this Theorem 3 is based on the following result, which was first used in [6]–[10] to accelerate some convergences and to construct asymptotic expansions.

LEMMA 1. *If $(x_n)_{n\geqslant 1}$ is convergent to x and if there exists the limit*

$$\lim_{n\to\infty} n^k(x_n-x_{n+1})=l\in\overline{\mathbb{R}},$$

with $k>1$, then there exists the limit

$$\lim_{n\to\infty} n^{k-1}(x_n-x)=\frac{l}{k-1}.$$

For proof, see [9]. The following result gives a similar answer for the sequence $(z_n)_{n\geqslant 1}$.

THEOREM 4. *Let* $a,\ b\in\mathbb{R}$ *be given and satisfy* $a>0$ *and* $b>-(a+1)/a$.
a) If $b\neq -\frac{1}{2a}$, *the speed of convergence of the sequence* $(z_n(a,b))_{n\geqslant 2}$ *is equal to* n^{-1}, *since*

$$\lim_{n\to\infty} n\left(z_n(a,b)-\gamma(a)\right)=-\frac{1}{2}-ab\neq 0.$$

b) If $b=-\frac{1}{2a}$, *the speed of convergence of the sequence*

$$\delta_n=\frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a+n-2}+\frac{1}{a+n-1}-\ln\left(\frac{a+n}{a}-\frac{1}{2a}\right)$$

equals n^{-2}, *since*

$$\lim_{n\to\infty} n^2\left(\delta_n-\gamma(a)\right)=\frac{1}{24}.$$

We have

$$z_n(a,b)-z_{n+1}(a,b)=-\frac{1}{a+n}-\ln\left(\frac{a+n}{a}+b\right)+\ln\left(\frac{a+n+1}{a}+b\right),$$

or, using a computer software, such as Maple,

$$z_n(a,b)-z_{n+1}(a,b)=\left(-\frac{1}{2}-ab\right)\frac{1}{n^2}+\left(a+ab+2a^2b+a^2b^2+\frac{1}{3}\right)\frac{1}{n^3}+O\left(\frac{1}{n^4}\right). \tag{1.7}$$

Now, we have

$$\lim_{n\to\infty} n^2\left(z_n(a,b)-z_{n+1}(a,b)\right)=-\frac{1}{2}-ab,$$

and if $ab=-\frac{1}{2}$, then

$$\lim_{n\to\infty} n^3\left(z_n(a,b)-z_{n+1}(a,b)\right)=\frac{1}{12}.$$

and Theorem 4 follows using Lemma 1.

Next we give some estimates of the sequences $(\beta_n)_{n\geqslant 1}$ and $(\delta_n)_{n\geqslant 1}$.

THEOREM 5. *a) For every integer* $n\geqslant 1$, *we have*

$$\frac{1}{24(n+a)^2}<\delta_n-\gamma(a)<\frac{1}{24(n+a-1)^2}$$

and

$$\frac{1}{24(n+a)^2}<\beta_n-\gamma(a)<\frac{1}{24(n+a-1)^2}.$$

This result is an extension of DeTemple's work [3] who defined the sequence

$$R_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right)$$

and proved

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}.$$

(case $a = 1$). Also Theorem 5 responses to an open problem posed by Berinde [2].

2. The proofs

Proof of Theorem 1. The sequences

$$x_n' = x_n + \frac{1}{2(n+a) - \frac{1}{4}}, \quad x_n'' = x_n + \frac{1}{2(n+a) - \frac{1}{3}}$$

are convergent to $\gamma(a)$. Our inequalities (1.5) follows, if we prove that the sequence $(x_n')_{n \geqslant 1}$ is strictly increasing and the sequence $(x_n'')_{n \geqslant 1}$ is strictly decreasing. We have $x_{n+1}' - x_n' = c(n)$, where

$$c(x) = \frac{1}{a+x} - \ln\frac{a+x+1}{a} + \ln\frac{a+x}{a} + \frac{1}{2(x+1+a) - \frac{1}{4}} - \frac{1}{2(x+a) - \frac{1}{4}},$$

with the derivative

$$c'(x) = -\frac{P(x)}{(x+a+1)(x+a)^2(8x+8a-1)^2(8x+8a+7)^2},$$

where

$$P(x) = 512x^3 + (1536a - 128)x^2 + \left(1536a^2 - 256a - 672\right)x$$
$$+512a^3 - 128a^2 - 672a + 49.$$

As the polynomial $P(x+2)$ has all coefficients positive, it results that $P(n) > 0$, for every $n \geqslant 2$. Now, $c(x)$ is strictly decreasing, with $c(\infty) = 0$, so $c > 0$, on $[2, \infty)$. Thus $(x_n')_{n \geqslant 2}$ is strictly increasing and consequently, $x_n' < \gamma(a)$.

Let $x_{n+1}'' - x_n'' = d(n)$, where

$$d(x) = \frac{1}{a+x} - \ln\frac{a+x+1}{a} + \ln\frac{a+x}{a} + \frac{1}{2(x+1+a) - \frac{1}{3}} - \frac{1}{2(x+a) - \frac{1}{3}},$$

with the derivative

$$d'(x) = \frac{216x^2 + (432a + 240)x + 240a + 216a^2 - 25}{(x+a+1)(x+a)^2(6x+6a-1)^2(6x+6a+5)^2} > 0.$$

Now, $d(x)$ is strictly increasing, with $d(\infty) = 0$, so $d < 0$, on $[2, \infty)$. Thus $(x_n'')_{n \geqslant 2}$ is strictly decreasing and consequently, $x_n'' > \gamma(a)$.

The sequences

$$y_n' = y_n - \frac{1}{2(n+a) - \frac{4}{3}}, \quad y_n'' = y_n - \frac{1}{2(n+a) - \frac{5}{3}}$$

are convergent to $\gamma(a)$. Our inequalities (1.6) follows, if we prove that the sequence $(y_n')_{n\geqslant 1}$ is strictly decreasing and the sequence $(y_n'')_{n\geqslant 1}$ is strictly increasing. We have $y_{n+1}' - y_n' = e(n)$, where

$$e(x) = \frac{1}{a+x} - \ln\frac{a+x}{a} + \ln\frac{a+x-1}{a} - \frac{1}{2(x+1+a) - \frac{4}{3}} + \frac{1}{2(x+a) - \frac{4}{3}},$$

with the derivative

$$e'(x) = \frac{Q(x)}{2(x+a-1)(x+a)^2(3x+3a-2)^2(3x+3a+1)^2},$$

where

$$Q(x) = 81x^3 + (243a-81)x^2 + \left(243a^2 - 162a + 24\right)x + 24a - 81a^2 + 81a^3 + 8.$$

As the polynomial $Q(x+2)$ has all coefficients positive, it results that $Q(n) > 0$, for every $n \geqslant 2$. Now, $e(x)$ is strictly increasing, with $e(\infty) = 0$, so $e < 0$, on $[2,\infty)$. Thus $(y_n')_{n\geqslant 2}$ is strictly decreasing and consequently, $y_n' > \gamma(a)$.

Let $y_{n+1}'' - y_n'' = j(n)$, where

$$j(x) = \frac{1}{a+x} - \ln\frac{a+x}{a} + \ln\frac{a+x-1}{a} - \frac{1}{2(x+1+a) - \frac{5}{3}} + \frac{1}{2(x+a) - \frac{5}{3}},$$

with the derivative

$$j'(x) = -\frac{216x^2 + (432a-240)x + 216a^2 - 240a - 25}{(x+a-1)(x+a)^2(6x+6a-5)^2(6x+6a+1)^2} < 0.$$

Now, $j(x)$ is strictly decreasing, with $j(\infty) = 0$, so $j > 0$, on $[2,\infty)$. Thus $(y_n'')_{n\geqslant 2}$ is strictly increasing and consequently, $y_n'' < \gamma(a)$. □

Proof of Theorem 2. As in the proof of Theorem 1, we define the sequences

$$u_n = x_n + \frac{1}{2(n+a) - \frac{1}{3} + \frac{1}{18n}}, \quad v_n = y_n - \frac{1}{2(n+a) - \frac{5}{3} + \frac{1}{18n}}$$

and we prove that $(u_n)_{n\geqslant 1}$ is strictly increasing and $(v_n)_{n\geqslant 1}$ is strictly decreasing.

First, we have $u_{n+1} - u_n = k(n)$, and $v_{n+1} - v_n = l(n)$, where

$$k(x) = \frac{1}{x+a} - \ln\frac{a+x+1}{a} + \ln\frac{a+x}{a}$$

$$+\frac{1}{2(x+1+a)-\frac{1}{3}+\frac{1}{18(x+1)}}-\frac{1}{2(x+a)-\frac{1}{3}+\frac{1}{18x}}$$

and

$$l(x)=\frac{1}{x+a}-\ln\frac{a+x}{a}+\ln\frac{a+x-1}{a}$$
$$-\frac{1}{2(x+1+a)-\frac{5}{3}+\frac{1}{18(x+1)}}+\frac{1}{2(x+a)-\frac{5}{3}+\frac{1}{18x}},$$

with the derivatives

$$k'(x)=-\frac{R(x)}{(x+a+1)(x+a)^2(36x^2-6x+36ax+1)^2(36x^2+66x+36ax+36a+31)^2},$$

respective

$$l'(x)=\frac{S(x)}{(x+a-1)(x+a)^2(36x^2-30x+36ax+1)^2(36x^2+42x+36ax+36a+7)^2},$$

where

$$R(x)=15552(30a-13)x^5+\left(653\,184a+1166\,400a^2-250\,128\right)x^4$$
$$+\left(413\,424a+1842\,912a^2+979\,776a^3+37\,800\right)x^3$$
$$+\left(272\,808a+1131\,408a^2+1251\,936a^3+326\,592a^4+79\,200\right)x^2$$
$$+\left(85\,248a+349\,272a^2+498\,960a^3+287\,712a^4+46\,656a^5-7440\right)x$$
$$+2232a+19\,224a^2+58\,104a^3+63\,504a^4+23\,328a^5+961$$

and

$$S(x)=15\,552(30a-17)x^5+\left(1166\,400a^2-653\,184a+247\,536\right)x^4$$
$$+\left(979\,776a^3-116\,640a^2-239\,760a+360\,072\right)x^3$$
$$+\left(267\,624a-501\,552a^2+381\,024a^3+326\,592a^4+13\,104\right)x^2$$
$$+\left(132\,192a^4-26\,568a^2-60\,912a^3-26\,208a+46\,656a^5-2352\right)x$$
$$+504a-216a^2-7560a^3-14\,256a^4+23\,328a^5+49.$$

If we put $a=\frac{13}{30}+a'$, with $a'\geqslant 0$, then $R(n)$ becomes a polynomial with all coefficients positive. If we put $a=\frac{17}{30}+a''$, with $a''\geqslant 0$, then $S(n)$ becomes a polynomial with all coefficients positive. In consequence, $R>0$ and $S>0$, for every positive integer n and $a\geqslant\frac{13}{30}$, respective $a\geqslant\frac{17}{30}$.

Now, the function k is strictly decreasing, the function l is strictly increasing and the conclusion follows using the same arguments of Theorem 1. $\square$

Proof of Theorem 5. Let us define the sequences

$$\delta_n' = \delta_n - \frac{1}{24\,(n+a-1)^2}, \quad \delta_n'' = \delta_n - \frac{1}{24\,(n+a)^2}.$$

It suffices to show that $(\delta_n')_{n\geqslant 1}$ is strictly increasing and $(\delta_n'')_{n\geqslant 1}$ is strictly decreasing. In this sense, let us put $\delta_{n+1}' - \delta_n' = m(n)$ and $\delta_{n+1}'' - \delta_n'' = p(n)$, where

$$m(x) = \frac{1}{a+x} - \ln\left(\frac{a+x+1}{a} - \frac{1}{2a}\right) + \ln\left(\frac{a+x}{a} - \frac{1}{2a}\right)$$
$$-\frac{1}{24\,(x+a)^2} + \frac{1}{24\,(x+a-1)^2}$$

and

$$p(x) = \frac{1}{a+x} - \ln\left(\frac{a+x+1}{a} - \frac{1}{2a}\right) + \ln\left(\frac{a+x}{a} - \frac{1}{2a}\right)$$
$$-\frac{1}{24\,(x+a+1)^2} + \frac{1}{24\,(x+a)^2},$$

with the derivatives

$$m'(x) = -\frac{24x^3 + (72a-35)x^2 + \left(72a^2 - 70a + 15\right)x + 15a - 35a^2 + 24a^3 - 1}{12\,(2x+2a+1)(2x+2a-1)(x+a)^3(x+a-1)^3},$$

respective

$$p'(x) = \frac{24x^3 + (72a+35)x^2 + \left(70a + 72a^2 + 15\right)x + \left(70a + 72a^2 + 15\right) + 1}{12\,(2x+2a+1)(2x+2a-1)(x+a)^3(x+a+1)^3}.$$

Now, the function m is strictly decreasing, p is strictly increasing, with $m(\infty) = p(\infty) = 0$, so $m > 0$ and $p < 0$. Consequently, $(\delta_n')_{n\geqslant 1}$ is strictly increasing and $(\delta_n'')_{n\geqslant 1}$ is strictly decreasing.

Let us define the sequences

$$\beta_n' = \beta_n - \frac{1}{24\,(n+a-1)^2}, \quad \beta_n'' = \beta_n - \frac{1}{24\,(n+a)^2}.$$

It suffices to show that $(\beta_n')_{n\geqslant 1}$ is strictly increasing and $(\beta_n'')_{n\geqslant 1}$ is strictly decreasing. In this sense, let us put $\beta_{n+1}' - \beta_n' = q(n)$ and $\delta_{n+1}'' - \delta_n'' = r(n)$, where

$$q(x) = \frac{1}{a+x} - \ln\left(\frac{a+x}{a} + \frac{1}{2a}\right) + \ln\left(\frac{a+x-1}{a} + \frac{1}{2a}\right)$$
$$-\frac{1}{24\,(x+a)^2} + \frac{1}{24\,(x+a-1)^2}$$

and

$$r(x) = \frac{1}{a+x} - \ln\left(\frac{a+x}{a} + \frac{1}{2a}\right) + \ln\left(\frac{a+x-1}{a} + \frac{1}{2a}\right) - \frac{1}{24(x+a+1)^2} + \frac{1}{24(x+a)^2},$$

with the derivatives

$$q'(x) = -\frac{24x^3 + (72a-35)x^2 + \left(72a^2 - 70a + 15\right)x + 15a - 35a^2 + 24a^3 + 24n^3 - 1}{12(2x+2a-1)(2x+2a+1)(x+a)^3(x+a-1)^3},$$

respective

$$r'(x) = \frac{24x^3 + (72a+35)x^2 + \left(70a + 72a^2 + 15\right)x + 15a + 35a^2 + 24a^3 + 1}{12(2x+2a-1)(2x+2a+1)(x+a)^3(x+a-1)^3}.$$

Now, the function q is strictly decreasing, r is strictly increasing, with $q(\infty) = r(\infty) = 0$, so $q > 0$ and $r < 0$. Consequently, $(\beta_n')_{n\geqslant 1}$ is strictly increasing and $(\beta_n'')_{n\geqslant 1}$ is strictly decreasing. □

Finally, we are convinced that our new method is suitable for establishing other new estimates for the gamma and polygamma functions, or for the generalized harmonic sums.

Acknowledgement. This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS – UEFISCDI, project number PN-II-ID-PCE-2011-3-0087.

REFERENCES

[1] H. ALZER, *Inequalities for the gamma and polygamma functions*, Abh. Math. Sem. Univ. Hamb. **68** (1998), 363–372.
[2] V. BERINDE, *A new generalization of Euler's constant*, Creat. Math. Inform. **18**, 2 (2009), 123–128.
[3] D. W. DE TEMPLE, *A quicker convergence to Euler's constant*, Amer. Math. Monthly **100**, 5 (1993), 468–470.
[4] E. A. KARATSUBA, *On the computation of the Euler constant γ*, Computational methods from rational approximation theory (Wilrijk, 1999), Numer. Algorithms **24**, 1-2 (2000), 83–97.
[5] K. KNOPP, *Theory and Applications of Infinite Series*, Blackie, London, **453** (1951).
[6] C. MORTICI, *Best estimates of the generalized Stirling formula*, Appl. Math. Comput. **215**, 11 (2010), 4044–4048.
[7] C. MORTICI, *Product approximations via asymptotic integration*, Amer. Math. Monthly **117**, 5 (2010), 434–441.
[8] C. MORTICI, *An ultimate extremely accurate formula for approximation of the factorial function*, Arch. Math. (Basel) **93**, 1 (2009), 37–45.
[9] C. MORTICI, *New approximations of the gamma function in terms of the digamma function*, Appl. Math. Lett. **23**, 1 (2010), 97–100.
[10] C. MORTICI, *Complete monotonic functions associated with gamma function and applications*, Carpathian J. Math. **25**, 2 (2009), 186–191.

[11] S.-L. QIU, M. VUORINEN, *Some properties of the gamma and psi functions with applications*, Math. Comp. **74**, 250 (2005), 723–742.
[12] A. SÎNTĂMĂRIAN, *A generalization of Euler's constant*, Numer. Algorithms **46**, 2 (2007), 141–151.
[13] L. TÓTH, *Problem E 3432*, Amer. Math. Monthly **98**, 3 (1991), 264.
[14] R. M. YOUNG, *Euler's constant*, Math. Gaz. **75**, 472 (1991), 187–190.

(Received May 26, 2010)

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