

APPROXIMATING FIXED POINTS OF WEAK CONTRACTIONS USING THE PICARD ITERATION

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ABSTRACT. The concept of weak contraction is introduced and compared to some known metrical contractive type maps and then two fixed point theorems for this class of operators in complete metric spaces are proven. Our results extend well known fixed point theorems due to Banach, Kannan, Chatterjea, Zamfirescu and many others. The main merit of weak contractions is that they unify large classes of contractive type operators, whose fixed points can be obtained by means of the Picard iteration and for which both a priori and a posteriori error estimates are also available.

1. Introduction

The classical Banach's contraction principle is one of the most useful results in fixed point theory. In a metric space setting it can be briefly stated as follows.

Theorem B. Let (X, d) be a complete metric space and $T : X \rightarrow X$ a strict contraction, i.e. a map satisfying

$$d(Tx, Ty) \leq a d(x, y), \quad \text{for all } x, y \in X, \quad (1.1)$$

where $0 < a < 1$ is constant. Then

- (p1) T has a unique fixed point p in X ;
- (p2) The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \dots \quad (1.2)$$

converges to p , for any $x_0 \in X$.

Note. A map satisfying (p1) and (p2) is said to be a *Picard operator*, see Rus [29, 31].

Theorem B, together with its direct generalizations have many applications in solving nonlinear equations, but suffer from one drawback - the contractive condition (1.1) forces T be continuous on X . It is then natural to ask if there exist contractive conditions which do not imply the continuity of T . This was answered in

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the affirmative by R. Kannan [14] in 1968, who proved a fixed point theorem which extends Theorem B to mappings that need not be continuous, by considering instead of (1.1) the next condition: there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)], \quad \text{for all } x, y \in X. \quad (1.3)$$

Following the Kannan's theorem, a lot of papers were devoted to obtaining fixed point theorems for various classes of contractive type conditions that do not require the continuity of T , see for example, Rus [29, 32], Taskovic [34], and references therein.

One of them, actually a sort of dual of Kannan fixed point theorem, due to Chatterjea [5], is based on a condition similar to (1.3): there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)], \quad \text{for all } x, y \in X. \quad (1.4)$$

It is well known, see Rhoades [24], that the contractive conditions (1.1) and (1.3), as well as (1.1) and (1.4), are independent.

In 1972, Zamfirescu [35] obtained a very interesting fixed point theorem, by combining (1.1), (1.3) and (1.4).

Theorem Z. Let (X, d) be a complete metric space and $T : X \rightarrow X$ a map for which there exist real numbers a, b and c satisfying $0 \leq a < 1$, $0 < b, c < 1/2$ such that for each pair x, y in X , at least one of the following is true:

- (z_1) $d(Tx, Ty) \leq a d(x, y)$;
- (z_2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$;
- (z_3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Then T is a Picard operator.

One of the most general contraction condition for which the map satisfying it is still a Picard operator, has been obtained by Ćirić [8] in 1974: there exists $0 < h < 1$ such that

$$d(Tx, Ty) \leq h \cdot \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \quad \text{for all } x, y \in X. \quad (1.5)$$

Remark. A map satisfying (1.5) is commonly called quasi contraction.

It is obvious that each of the conditions (1.3), (1.4) and (z_1)-(z_3) implies (1.5);

There exist many other fixed theorems based on contractive conditions of this type, see for example, Rhoades [24, 26] and the monographs Berinde [2], Rus [29], Taskovic [34].

Motivated by the extensive literature devoted to the nonlinear contractive operators aforementioned, it is our main aim in this paper to obtain fixed point theorems based on a contraction condition more general than (z_1)-(z_3) and that does not require the continuity of the map as well.

2. Weak contractions

Definition 1. Let (X, d) be a metric space. A map $T : X \rightarrow X$ is called *weak contraction* if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx), \quad \text{for all } x, y \in X. \quad (2.1)$$

Remark 1. Due to the symmetry of the distance, the weak contraction condition (2.1) implicitly includes the following dual one

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + L \cdot d(x, Ty), \quad \text{for all } x, y \in X, \quad (2.2)$$

obtained from (2.1) by formally replacing $d(Tx, Ty)$ and $d(x, y)$ by $d(Ty, Tx)$ and $d(y, x)$, respectively, and then interchanging x and y .

Consequently, in order to check the weak contractiveness of T , it is necessary to check both (2.1) and (2.2).

Obviously, any strict contraction satisfies (2.1), with $\delta = a$ and $L = 0$, and hence is a weak contraction (that possesses a unique fixed point).

Other examples of weak contractions are given by the next propositions.

Proposition 1. Any Kannan map, i.e., any map satisfying the contractive condition (1.3), is a weak contraction.

Proof. By condition (1.3) and triangle rule, we get

$$\begin{aligned} d(Tx, Ty) &\leq b[d(x, Tx) + d(y, Ty)] \\ &\leq b\{[d(x, y) + d(y, Tx)] + [d(y, Tx) + d(Tx, Ty)]\}, \end{aligned}$$

which yields

$$(1 - b)d(Tx, Ty) \leq bd(x, y) + 2b \cdot d(y, Tx)$$

and which implies

$$d(Tx, Ty) \leq \frac{b}{1-b} d(x, y) + \frac{2b}{1-b} d(y, Tx), \quad \text{for all } x, y \in X,$$

i.e., in view of $0 < b < \frac{1}{2}$, (2.1) holds with $\delta = \frac{b}{1-b}$ and $L = \frac{2b}{1-b}$.

Since (1.3) is symmetric with respect to x and y , (2.2) also holds.

Proposition 2. Any map T satisfying the contractive condition (1.4) is a weak contraction.

Proof. Using $d(x, Ty) \leq d(x, y) + d(y, Tx) + d(Tx, Ty)$ by (1.4) we get after simple computations,

$$d(Tx, Ty) \leq \frac{c}{1-c} d(x, y) + \frac{2c}{1-c} d(y, Tx),$$

which is (2.1), with $\delta = \frac{c}{1-c} < 1$ (since $c < 1/2$) and $L = \frac{2c}{1-c} \geq 0$.

The symmetry of (1.4) also implies (2.2).

An immediate consequence of Propositions 1 and 2 is the following

Corollary 1. Any Zamfirescu map, i.e., any mapping satisfying the assumptions in Theorem Z, is a weak contraction.

Proposition 3. Any quasi contraction with $0 < h < 1/2$ is a weak contraction.

Proof. Let $T : X \rightarrow X$ be a quasi-contraction, i.e. a map for which there exists $0 < h < 1$ such that

$$d(Tx, Ty) \leq h \cdot M(x, y), \quad \text{for all } x, y \in X, \quad (2.3)$$

where

$$M(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Let $x, y \in X$ be arbitrary taken. We have to discuss five possible cases.

Case 1. $M(x, y) = d(x, y)$, when, in virtue of (2.3), condition (2.1) and (2.2) are obviously satisfied (with $\delta = h$ and $L = 0$).

Note. Since $M(x, y) = M(y, x)$, for Cases 2 and 3 and Cases 4 and 5, respectively, it suffices to prove that at least one of the relations (2.1) and (2.2) holds. (We sometimes however prove the both inequalities).

Case 2. $M(x, y) = d(x, Tx)$, when by (2.3) and triangle rule

$$d(Tx, Ty) \leq h d(x, Tx) \leq h [d(x, y) + d(y, Tx)],$$

and so (2.1) holds with $\delta = h$ and $L = h$.

Since $d(x, Tx) \leq d(x, Ty) + d(Ty, Tx)$, we get

$$d(Tx, Ty) \leq \frac{1}{1-h} d(x, Ty) \leq \delta d(x, y) + \frac{1}{1-h} d(x, Ty),$$

for all $\delta \in (0, 1)$. So (2.2) also holds.

Case 3. $M(x, y) = d(y, Ty)$, when (2.1) and (2.2) follow by Case 2, in virtue of the symmetry of $M(x, y)$.

Case 4. $M(x, y) = d(x, Ty)$, when (2.2) is obviously true and (2.1) is obtained only if $h < \frac{1}{2}$. Indeed, since by (2.3), $d(Tx, Ty) \leq h \cdot d(x, Ty)$ and

$$d(x, Ty) \leq d(x, y) + d(y, Tx) + d(Tx, Ty),$$

one obtains

$$d(Tx, Ty) \leq \frac{h}{1-h} d(x, y) + \frac{h}{1-h} d(y, Tx),$$

which is (2.1) with $\delta = \frac{h}{1-h} < 1$ (since $h < \frac{1}{2}$) and $L = \frac{h}{1-h} > 0$.

Case 5. $M(x, y) = d(y, Tx)$, which reduces to Case 4. The proof is complete.

Remark 2. Proposition 3 shows that the quasi-contractions with $0 < h < 1/2$ are certainly weak contractions. However, there exists quasi-contractions with $q \geq \frac{1}{2}$, as shown by Example 2, which are still weak contractions. It appears then that $h < \frac{1}{2}$ is not a necessary condition for a quasi-contraction to be a weak contraction, see Problem 1.

There are many other examples of contractive conditions which implies the weak contractiveness condition, see for example, Taskovic [34], Rus [30] and Berinde [2], for some of them.

Having in view the fact that the class of weak contractions properly includes large classes of quasi contractions and weak contractions and quasi contractions are independent, on the one hand, and the extensive literature related to quasi contractions, see for example [7-18, 22, 24-34], and references therein, on the other hand, it is the aim of this paper to prove two fixed point theorems in the class of weak contractions: an existence theorem (Theorem 1) as well as an existence and uniqueness theorem (Theorem 2). Their merit is that extend Theorem Z and offer a method for approximating fixed points, for which both a priori and a posteriori estimates are available.

3. Two fixed point theorems

The main result of this paper is given by

Theorem 1. Let (X, d) be a complete metric space and $T : X \rightarrow X$ a weak contraction, i.e., a map satisfying (2.1) with $\delta \in (0, 1)$ and some $L \geq 0$. Then

- 1) $F(T) = \{x \in X : Tx = x\} \neq \emptyset$;
- 2) For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.2) converges to some $x^* \in F(T)$;
- 3) The following estimates

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), \quad n = 0, 1, 2, \dots \quad (3.1)$$

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n), \quad n = 1, 2, \dots \quad (3.2)$$

hold, where δ is the constant appearing in (2.1).

Proof. We shall prove that T has at least a fixed point in X . To this end, let $x_0 \in X$ be arbitrary and $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration defined by (1.2).

Take $x := x_{n-1}$, $y := x_n$ in (2.1) to obtain

$$d(Tx_{n-1}, Tx_n) \leq \delta \cdot d(x_{n-1}, x_n),$$

which shows that

$$d(x_n, x_{n+1}) \leq \delta \cdot d(x_{n-1}, x_n). \quad (3.3)$$

Using (3.3) we obtain by induction

$$d(x_n, x_{n+1}) \leq \delta^n d(x_0, x_1), \quad n = 0, 1, 2, \dots$$

and then

$$\begin{aligned} d(x_n, x_{n+p}) &\leq \delta^n (1 + \delta + \dots + \delta^{p-1}) d(x_0, x_1) \\ &= \frac{\delta^n}{1 - \delta} (1 - \delta^p) \cdot d(x_0, x_1), \quad n, p \in \mathbb{N}, p \neq 0. \end{aligned} \quad (3.4)$$

Since $0 < \delta < 1$, (3.4) shows that $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence and hence is convergent. Denote

$$x^* = \lim_{n \rightarrow \infty} x_n. \quad (3.5)$$

Then

$$d(x^*, Tx^*) \leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) = d(x_{n+1}, x^*) + d(Tx_n, Tx^*).$$

By (2.1) we have

$$d(Tx_n, Tx^*) \leq \delta d(x_n, x^*) + L d(x^*, Tx_n)$$

and hence

$$d(x^*, Tx^*) \leq (1 + L) d(x^*, x_{n+1}) + \delta \cdot d(x_n, x^*), \quad (3.6)$$

valid for all $n \geq 0$. Letting $n \rightarrow \infty$ in (3.6) we obtain

$$d(x^*, Tx^*) = 0$$

i.e., x^* is a fixed point of T .

The estimate (3.1) is obtained from (3.4) letting $p \rightarrow \infty$.

In order to obtain (3.2), observe that by (3.3) we inductively obtain

$$d(x_{n+k}, x_{n+k+1}) \leq \delta^{k+1} \cdot d(x_{n-1}, x_n), \quad k, n \in \mathbb{N},$$

and hence, similarly to deriving (3.4) we obtain

$$d(x_n, x_{n+p}) \leq \frac{\delta(1 - \delta^p)}{1 - \delta} d(x_{n-1}, x_n), \quad n \geq 1, p \in \mathbb{N}^*. \quad (3.7)$$

Now letting $p \rightarrow \infty$ in (3.7), (3.2) follows. The proof is complete.

Remark 3.

1) Theorem 1 is a significant extension of Theorem B, Theorem Z and many other related results.

2) Note that, although the fixed point theorems mentioned at 1) actually forces the uniqueness of the fixed point, the weak contractions need not have a unique fixed point, as shown by Example 1.

3) However, the weak contractions possess other important properties, amongst which we mention

a) In the class of weak contractions a method for constructing the fixed points - i.e., the Picard iteration - is always available;

b) Moreover, for this method of approximating the fixed points, both *a priori* and *a posteriori* error estimates are available. These are very important from a practical point of view, since they provide stopping criteria for the iterative process;

c) Last, but not least, the weak contractive condition (2.1) (and (2.2)) may be easily be handled and checked in concrete applications.

4) The fixed point x^* attained by the Picard iteration depends on the initial guess $x_0 \in X$. Therefore, the class of weak contractions provides a large class of weakly Picard operators.

Recall, see Rus [31, 32], that an operator $T : X \rightarrow X$ is said to be a *weakly Picard operator* if the sequence $\{T^n x_0\}_{n=0}^\infty$ converges for all $x_0 \in X$ and the limits are fixed points of T .

5) It is easy to see that condition (2.1) implies the so called Banach orbital condition

$$d(Tx, T^2x) \leq \alpha d(x, Tx), \quad \text{for all } x \in X,$$

studied by various authors in the context of fixed point theorems, see for example Hicks and Rhoades [12], Ivanov [13], Rus [29] and Taskovic [34].

It is possible to force the uniqueness of the fixed point of a weak contraction, by imposing an additional contractive condition, quite similar to (2.1), as shown by the next theorem.

Theorem 2. Let (X, d) be a complete metric space and $T : X \rightarrow X$ a weak contraction for which there exist $\theta \in (0, 1)$ and some $L_1 \geq 0$ such that

$$d(Tx, Ty) \leq \theta \cdot d(x, y) + L_1 \cdot d(x, Tx), \quad \text{for all } x, y \in X. \quad (3.8)$$

Then

- 1) T has a unique fixed point, i.e., $F(T) = \{x^*\}$;
- 2) The Picard iteration $\{x_n\}_{n=0}^\infty$ given by (1.2) converges to x^* , for any $x_0 \in X$;
- 3) The a priori and a posteriori error estimates

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), \quad n = 0, 1, 2, \dots$$

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n), \quad n = 1, 2, \dots$$

hold.

- 4) The rate of convergence of the Picard iteration is given by

$$d(x_n, x^*) \leq \theta d(x_{n-1}, x^*), \quad n = 1, 2, \dots \quad (3.9)$$

Proof. Assume T has two distinct fixed points $x^*, y^* \in X$. Then by (3.8), with $x := x^*, y := y^*$ we get

$$d(x^*, y^*) \leq \theta \cdot d(x^*, y^*) \iff (1 - \theta) d(x^*, y^*) \leq 0,$$

so contradicting $d(x^*, y^*) > 0$.

Letting $y := x_n, x := x^*$ in (3.8), we obtain the estimate (3.9). The rest of proof follows by Theorem 1.

Remark 4.

1) Note that, by the symmetry of the distance, (3.8) is satisfied for all $x, y \in X$ if and only if

$$d(Tx, Ty) \leq \theta d(x, y) + L_1 d(y, Ty), \quad (3.10)$$

also holds, for all $x, y \in X$.

So, similarly to the case of the dual conditions (2.1) and (2.2), in concrete applications it is necessary to check that both conditions (3.8) and (3.10) are satisfied.

2) Note that condition (3.8) has been used by Osilike [19-21] to prove stability results for certain fixed point iteration procedures.

3) It is known, see Osilike [19, 20], that condition (3.8) alone does not imply T has a fixed point. But if T satisfying (3.8) has a fixed point, it is certainly unique.

4) It is a simple task to prove that any operator T satisfying one of the conditions (1.1), (1.3), (1.4), or the conditions in Theorem Z, also satisfies the uniqueness conditions (3.8) and (3.10). Therefore, in view of Example 1, Theorem 2 (and also Theorem 1) properly generalizes Theorem Z.

Moreover, any quasi contraction with $0 < h < \frac{1}{2}$ also satisfies (3.8) and (3.10). This shows that Theorem 2 unifies and generalizes the fixed point theorems of Banach, Kannan, Chatterjea and Zamfirescu and partially covers the Ćirić's fixed point theorem.

The same Example 2 also shows that a quasi contraction generally does not satisfy condition (3.8).

5) As it can be seen, Theorem 2 (as well as Theorem 1, except for the uniqueness of the fixed point) preserves all conclusions in the Banach contraction principle in its complete form, see for example Berinde [3], under significantly weaker contractive conditions. Indeed, the metrical contractive conditions known in literature (see Rhoades [24] and Meszaros [18]) that involve in the right-hand side the displacements

$$d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)$$

with the nonnegative coefficients

$$a(x, y), b(x, y), c(x, y), d(x, y), e(x, y),$$

respectively, are commonly based on the restrictive assumption

$$0 < a(x, y) + b(x, y) + c(x, y) + d(x, y) + e(x, y) < 1,$$

while, our condition (2.1) do not require $\delta + L$ be less than 1, thus providing a large class of contractive type maps.

The previous remarks raise the next open problems:

Problem 1. *Is any quasi contraction a weak contraction ?*

Problem 2. *Find a contractive type condition different of (3.8), that ensures the uniqueness of fixed points of weak contractions.*

4. Examples

We end this paper with some examples that have an illustrative purpose.

Let $[0, 1]$ be the unit interval with the usual norm.

Example 1. Let $T : [0, 1] \rightarrow [0, 1]$ be the identity map, i.e., $Tx = x$, for all $x \in [0, 1]$.

Then

- 1) T does not satisfy the Ćirić's contractive condition (1.5), since $M(x, y) = |x - y|$ and

$$|x - y| > h \cdot |x - y|, \quad \text{for all } x \neq y \text{ and } 0 < h < 1.$$

- 2) T satisfies condition (2.1) with $\delta \in (0, 1)$ arbitrary and $L \geq 1 - \delta$. Indeed conditions (2.1) and (2.2) lead to

$$|x - y| \leq \delta |x - y| + L \cdot |y - x|,$$

which is true for all $x, y \in [0, 1]$ if we take $\delta \in (0, 1)$ arbitrary and $L \geq 1 - \delta$.

- 3) The set of fixed points of T is the entire interval $[0, 1]$. i.e., $F(T) = [0, 1]$.

Example 2. Let $T : [0, 1] \rightarrow [0, 1]$ be given by $Tx = \frac{2}{3}$, for $x \in [0, 1]$ and $T1 = 0$.

Then

- 1) T satisfies (1.5) with $h \in [\frac{2}{3}, 1)$.
- 2) T satisfies (2.1) with $\delta \geq \frac{2}{3}$ and $L \geq \delta$.
- 3) T has a unique fixed point, $x^* = \frac{2}{3}$.
- 4) T does not satisfy (3.8).

Example 3. Let $T : [0, 1] \rightarrow [0, 1]$ be given by $Tx = \frac{1}{2}$ for $x \in [0, 1]$ and $T1 = 1$.

Then

- 1) T does not satisfy neither Ćirić's condition (1.5), nor Zamfirescu's condition.
- 2) T has two fixed points, $F(T) = \{\frac{1}{2}, 1\}$.
- 3) T does not satisfy (2.1).

Indeed, for $x, y \in [0, 1]$ or $x = y = 1$, (2.1) is obvious. When $x \in [0, 1]$ and $y = 1$, (2.1) reduces to

$$\frac{1}{2} \leq \delta \cdot |x - 1| + L \cdot \frac{1}{2},$$

which is true for all $\delta \in (0, 1)$, if $L \geq 1$. When $x = 1$ and $y \in [0, 1]$, then (2.1) holds if and only if

$$\frac{1}{2} \leq \delta \cdot |1 - y| + L|1 - y|$$

which is equivalent to $(\delta + L)(1 - y) \geq 1/2$, which could not be true if L is constant, since $1 - y \rightarrow 0$, as $y \rightarrow 1$, ($y < 1$). Note that, despite the fact that T does not satisfies (2.1), any limit of the Picard iteration is a fixed point of T and that T does not satisfy (3.8), as well.

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