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A NOTE ON THE PAPER "REMARKS ON FIXED POINT THEOREMS OF BERINDE"

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Abstract. We analyze the Examples 1, 3 and 4 in the note [S. L. Singh and R. Pant, *Remarks on fixed point theorems of Berinde*, Nonlinear Anal. Forum 12 (2007), No. 2, 231-234] and point out they contain some inaccuracies which make them fail in showing that the fixed point theorems in [V. Berinde, *Approximating fixed point of weak contractions using Picard iteration*, Nonlinear Anal. Forum 9 (2004), No. 1, 43-53] do not hold. We also include some considerations and examples which clarify the problem.

1. Introduction

In a recent note [17], S. L. Singh and R. Pant presented three examples, numbered Example 1, Example 3 and Example 4 and tried to show that the main results in the paper [5] are not valid under the general assumptions given there, i.e., under the conditions

$$0 < \delta < 1 \text{ and } L \geq 0 \tag{1.1}$$

but only under the very restrictive assumption

$$0 < \delta + L < 1. \tag{1.2}$$

The main aim of this note is to show that the examples in [17] are not working and so to point out that the fixed point theorems established in [5] are valid in the general form they were originally stated, i.e., under the assumption (1.1).

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2. Two fixed point theorems on weak (almost) contractions

In the paper [5], see also [3] and [4], the first author introduced a class of contractive mappings initially called *weak contractions*, but for which we later adopted the more suggestive term of *almost contractions* ([7]), in view of the fact that almost contractions practically inherit all the main properties of usual contractions, except for the uniqueness of the fixed point.

Definition 1 ([5]). Let (X,d) be a metric space. A map $T : X \to X$ is called *weak contraction* or *almost contraction* if there exist $\delta \in (0,1)$ and some $L \geq 0$ such that

$$d(Tx,Ty) \leq \delta \cdot d(x,y) + L \cdot d(y,Tx), \text{ for all } x,y \in X. \tag{2.1}$$

Remark 1. As mentioned in [5], due to the symmetry of the distance, the almost contraction condition (2.1) implicitly includes the following dual one

$$d(Tx,Ty) \leq \delta \cdot d(x,y) + L \cdot d(x,Ty), \text{ for all } x,y \in X, \tag{2.2}$$

obtained from (2.1) by interchanging x and y.

By summing up the non symmetric contractive conditions (2.1) and (2.2) we get the following symmetric one

$$d(Tx,Ty) \leq \delta \cdot d(x,y) + \frac{L}{2} \cdot [d(x,Ty) + d(y,Tx)], \text{ for all } x,y \in X, \tag{2.3}$$

that has been used in [17].

Remark 2. Note that (2.1) implies condition (2.3), but the reverse is not true, as we shall see in the next section.

For the sake of completeness we state in the following in a simplified form the main results in [5], i.e., Theorem 1 (an existence theorem) and Theorem 2 (an existence and uniqueness theorem).

Theorem 1. *Let (X,d) be a complete metric space and $T : X \to X$ an almost contraction. Then $Fix\,(T) = \{x \in X : Tx = x\} \neq \emptyset$.*

Theorem 2. *Let (X,d) be a complete metric space and $T : X \to X$ an almost contraction for which there exist $\theta \in (0,1)$ and some $L_1 \geq 0$ such that*

$$d(Tx,Ty) \leq \theta \cdot d(x,y) + L_1 \cdot d(x,Tx), \textit{ for all } x,y \in X. \tag{2.4}$$

Then T has a unique fixed point, i.e., $Fix\,(T) = \{x^*\}$.

Similarly to the case of condition (2.1), by summing up (2.4) and its dual we get the symmetric condition

$$d(Tx,Ty) \leq \theta \cdot d(x,y) + \frac{L_1}{2} \cdot [d(x,Tx) + d(y,Ty)], \text{ for all } x,y \in X. \tag{2.5}$$

Remark 3. Note that condition (2.4) does imply condition (2.5) but the reverse is not true, as we shall see later on.

3. Examples by S.L. Singh and R. Pant

In this section we analyze the examples presented by S.L. Singh and R. Pant in [17]. As we shall see, actually neither of them applies for Theorems 1 and 2 above, as the operators considered are not fulfilling the conditions required in these fixed point theorems.

Example 1. The function $T : X \to X$ given by $T(x) = x/2$ if $0 < x \leq 1$ and $T(0) = 1/2$, where $X = [0, 1]$ with the usual metric, included in Example 1 from [17] and asserted there to satisfy condition (2.1), is actually not an almost contraction.

Indeed, take $x = \epsilon > 0$ and $y = 0$ to get by condition (2.1)

$$\frac{1}{2}\,|1-\epsilon| \leq \delta\epsilon + \frac{L}{2}\epsilon \quad \Leftrightarrow \quad \frac{|1-\epsilon|}{\epsilon} \leq 2\delta + L$$

which is impossible: while the right hand side must be constant, the left hand side tends to $+\infty$ as $\epsilon \to 0$. So, T in this example does not satisfy (2.1) and hence Theorem 1 cannot be applied here.

The problem with this example in the note [17] is a computation error in Case 3 ($x > 0$ and $y = 0$) where we must have $d(x, y) = x$ instead of $d(x, y) = \left|x - \frac{1}{2}\right|$.

Example 2. The function $T : X \to X$ given by $Tx = 2x + 1$, where $X = [0, \infty]$ with the usual metric, included in Example 3 from [17], which is claimed to satisfy condition (2.1), is actually not an almost contraction.

Indeed, as shown in [17], T satisfies condition (2.3) but this does not mean that T would also satisfy (2.1), see Remark 2, because (2.1) and (2.3) are not equivalent. If we take $x = 1$ and $y = 3$, then (2.1) becomes

$$|3-7| \leq \delta\,|1-3| + L\,|3-3| \quad \Leftrightarrow \quad 4 \leq \delta \cdot 2,$$

which is impossible, since $\delta < 1$. So, T in this example does not satisfy (2.1) and hence Theorem 1 cannot be applied.

Example 3. The function $T : X \to X$ given by $Tx = x + 1$, where $X = [0, \infty]$ with the usual metric, included in Example 4 from [17] and claimed to satisfy condition (2.4), actually does not satisfy this condition.

As it is shown in [17], T satisfies condition (2.5) but this does not mean that T would also satisfy (2.4), see Remark 3, because (2.4) and (2.5) are not equivalent.

Indeed, if we take $y = 0$ and $x > 0$, then by condition (2.4) we get

$$|x+1-1| \leq \theta\,|x-0| + L_1\,|x-x-1| \quad \Leftrightarrow \quad x \leq \frac{L_1}{1-\theta},$$

which is impossible: while the right hand side must be constant, the left hand side tends to $+\infty$ as $x \to +\infty$.

This shows that T in this example does not satisfy the assumptions of Theorem 2.

So Examples 1, 3 and 4 in [17] do not apply to our fixed point theorems in [5].

4. Some clarifying examples of almost contractions

We end this note by presenting some examples of almost contraction possessing a unique or two fixed points. First of all let us remind three important classes of almost contractions with a unique fixed point that were given in [5] and discussed in much more details in [4].

Let (X, d) be a metric space.

Example 4 ([5])**.** Any Banach contraction $T : X \to X$ is a continuous almost contraction that satisfies the uniqueness condition (2.4).

Example 5 ([5])**.** Any Kannan contraction ([11]), i.e., any map $T : X \to X$ for which there exists $0 \leq b < 1/2$ such that

$$d(Tx, Ty) \leq b\big[d(x, Tx) + d(y, Ty)\big], \text{ for all } x, y \in X, \tag{4.1}$$

is a (generally, discontinuous) almost contraction that fulfills the uniqueness condition (2.4).

Example 6 ([5])**.** Any Chatterjea contraction ([8]), i.e., any map $T : X \to X$ for which there exists $0 \leq c < 1/2$ such that

$$d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big], \text{ for all } x, y \in X, \tag{4.2}$$

is a (generally, discontinuous) almost contraction that fulfills the uniqueness condition (2.4).

The next example gives an idea on how large the class of almost contractions is.

Example 7. Let $X = [0, 1]$ with the usual metric and let $T : [0, 1] \to [0, 1]$ be defined by $Tx = \frac{2}{3}\,x$, if $0 \leq x < \frac{1}{2}$ and $Tx = \frac{2}{3}\,x + \frac{1}{3}$, if $\frac{1}{2} \leq x \leq 1$. Then T is a discontinuous almost contraction that has two fixed points.

We discuss four possible cases.
Case 1: If x, $y \in [0, \frac{1}{2})$, then $Tx = \frac{2}{3}x$, $T = \frac{2}{3}y$ and condition (2.1) becomes

$$\left|\frac{2}{3}x - \frac{2}{3}y\right| \leq \delta\,|x - y| + L\left|y - \frac{2}{3}x\right|,\ x, y \in [0, \frac{1}{2})$$

which obviously holds for $\delta = \frac{2}{3}$ and any $L \geq 0$.
Case 2: If x, $y \in [\frac{1}{2}, 1]$, then $Tx = \frac{2}{3}x + \frac{1}{3}$, $Ty = \frac{2}{3}y + \frac{1}{3}$, and the rest is analogous to Case 1.
Case 3. If $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1]$, then $Tx = \frac{2}{3}x$, $Ty = \frac{2}{3}y + \frac{1}{3}$ and so condition (2.1) becomes

$$\left|\frac{2}{3}x - \frac{2}{3}y - \frac{1}{3}\right| \leq \delta\,|x - y| + L\left|y - \frac{2}{3}x\right|. \tag{4.3}$$

From $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1]$ we get

$$-1 \leq \frac{2}{3}x - \frac{2}{3}y - \frac{1}{3} < -\frac{1}{3} \quad \Rightarrow \quad \left|\frac{2}{3}x - \frac{2}{3}y - \frac{1}{3}\right| \in \left(\frac{1}{3}, 1\right]$$

and, respectively,

$$\frac{1}{6} < y - \frac{2}{3}x \leq 1 \quad \Rightarrow \quad \left|y - \frac{2}{3}x\right| \in \left(\frac{1}{6}, 1\right],$$

which shows that (4.3) holds for all $x \in [0, \frac{1}{2})$, all $y \in [\frac{1}{2}, 1]$ and any $\delta \in (0, 1)$ if we simply take $L \geq 6$. The dual condition (2.2) obviously follows by Case 4.

Case 4. If $x \in [\frac{1}{2}, 1]$ and $y \in [0, \frac{1}{2})$, then $Tx = \frac{2}{3}x + \frac{1}{3}$, $Ty = \frac{2}{3}y$ and so condition (2.1) becomes

$$\left|\frac{2}{3}x - \frac{2}{3}y + \frac{1}{3}\right| \leq \delta\,|x-y| + L\left|y - \frac{2}{3}x - \frac{1}{3}\right|. \tag{4.4}$$

Since in this case we have

$$\frac{2}{3}x - \frac{2}{3}y + \frac{1}{3} \in \left(\frac{1}{3}, 1\right] \quad \Rightarrow \quad \left|\frac{2}{3}x - \frac{2}{3}y + \frac{1}{3}\right| \in \left(\frac{1}{3}, 1\right]$$

and, respectively,

$$-1 \leq y - \frac{2}{3}x - \frac{1}{3} < -\frac{1}{6} \quad \Rightarrow \quad \left|y - \frac{2}{3}x - \frac{1}{3}\right| \in \left(\frac{1}{6}, 1\right]$$

we conclude that (4.4) holds for all $x \in [\frac{1}{2}, 1]$, all $y \in [0, \frac{1}{2})$ and any $\delta \in (0, 1)$ if the constant L satisfies $L \geq 6$.

Hence, T satisfies (2.1) on X with $\delta = \frac{2}{3}$ and $L = 6$.

Note that T does not satisfy the uniqueness condition (2.4). To show that, simply take $x = 0$ and $y = 1$ to get, by (2.4), $1 \leq \theta \cdot 1$, which is impossible, since $\theta \in (0, 1)$.

Therefore, T is an almost contraction for which Theorem 1 can be applied (but Theorem 2 cannot) and $Fix\,(T) = \{0, 1\}$.

For other recent examples, extensions and developments of the concept of almost (weak) contraction, in both single valued and multi-valued case, see [1], [2] and [9]-[16].

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