ON THE EXTENDED NEWTON'S METHOD

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Abstract The purpose of this paper is to state a convergence theorem for the Newton's method under weak non-Kantorovich assumptions. These assumptions may be appropriate for various nonlinear systems of difference equations that we obtain by discretization in solving a system of ordinary differential equations.

For computational reasons a stopping criterion for the Newton's iterative scheme may be obtained from (6). In order to ensure a well-defined iterative process we use a natural prolongation of the function involved in the problem to furnish the so-called *extended Newton's method*.

1. INTRODUCTION

At each step in applying the implicit Euler's discretization for solving a system of ordinary differential equations we must solve an equation of the form

$$F(y) \equiv y - y^{\circ} - hf(y) = 0, \tag{1}$$

where $y^0 \in \mathbf{R}^n$, $h > 0$ is a real constant and $f : D \subset \mathbf{R}^n \to \mathbf{R}^n$.

The Newton's iterations for equation (1) are

$$F'(y^k)\,(y^{k+1} - y^k) = -F(y^k)\,,\ k = 0, 1, 2, \ldots \tag{2}$$

that is

$$(I - hf'(y^k))(y^{k+1} - y^k) = -(y^k - y^0 - kf(y^k)),\ k = 0, 1, 2, \ldots, \tag{2'}$$

Deuflhard (see [9]) gave sufficient conditions which provide that Newton's method can be applied to equation (1).One of the basic assumptions in obtaining the convergence of the Newton's iterations is that $f'(.)$ is Lipschitz continuous, i.e.

$$\| f'(u) - f'(v) \| \le L \cdot \| u-v \|, \quad \text{for all} \quad u, v \in D. \tag{3}$$

Recently, Pavaloiu [11] and Lazar [9] proved the convergence of the Newton's method under more general conditions.

They only assume that $f'(\circ)$ is Hölder continuous, that is

$$\| f'(u) - f'(v) \| \le L_1 \cdot \| u-v \|^p, \quad \text{for all} \quad u, v \in D, \tag{4}$$

where $p \in (0,1]$. is aconstant.

All these assumptions and results are in fact of Kantorovich type and, in particular,for $p=1$ these results can be reduced to the Kantorovich theorem.

In our opinion,from a practical point of view it is useful to offer convergence theorems for the Newton's method based on an alternative type of assumptions.Consequently,the main goal of this paper is to give a new result on the convergence of the Newton's method,stated in terms of non-Kantorovich conditions,using some recent results [1 -7].

2. THE CONVERGENCE THEOREM

For the sake of simplicity of exposition, we restrict ourself to the one dimensional case, since all results can be easily derived in the $\mathbb{R}^m$ case by a simple component-wise extension.The following theorem is proved in [4]:

Theorem 2.1.*Let* $F: [a,b] \to \mathbb{R}$ *be a real function defined on the interval* $I=[a,b]$ *satisfying the following conditions*

(F_1) $F(a)\,F(b) < 0$;

(F_2) $F \in C^1[a,b]$ *and* $F'(x) \neq 0$, *for each* $x \in [a,b]$;

(F_3) $2m \ge M$, *where*

$$m = \min_{x \in I} |F'(x)|, \quad M = \max_{x \in I} |F'(x)|, \quad .$$

Then the Newton's iterations given by

$$x_{n+1}=x_n-\frac{F(x_n)}{F'(x_n)},\ n\geq 0 \tag{5}$$

converges to α, *the unique solution of* $F(x)=0$ *in* I, *for each* $x_0\in I$, *with*

$$|x_n-\alpha|\leq\frac{|F'(x_n)|}{m}|x_n-x_{n+1}|,\ n\geq 0. \tag{6}$$

Proof. By applying the mean value theorem (see [3], [4], for a more detailed proof) we obtain

$$x_{n+1}-\alpha=\left[1-\frac{F'(c_n)}{F'(x_n)}\right]\cdot(x_n-\alpha),\ n\geq 0 \tag{7}$$

and respectively

$$x_{n+1}-x_n=\frac{F'(c_n)}{F'(x_n)}(x_n-\alpha),\ n\geq 0, \tag{8}$$

where $c_n=\alpha+\theta(x_n-\alpha),\ 0<\theta<1.$

Now, from (F_3) and the compactness of I it results that

$$k=\max_{x,y\in I}\left|1-\frac{F'(y)}{F'(x)}\right|\leq 1,$$

which together with (7) and by an induction process yields

$$|x_{n+1}-\alpha|\leq k^n\cdot|x_0-\alpha|,\ n\geq 0.$$

Therefore

$$x_n\to\alpha\ ,\text{ as }\ n\to\infty,$$

for each $x_o\in[a,b]$.

However, condition (F_2), which is weaker than the corresponding one in [10],does not guarantee that $I=[a,b]$ is an invariant set with respect to the iteration (5),that is,it is possible to reach at a certain step p to the situation $x_p\notin I$.

Due to these possible difficulties,we consider a prolongation of F to the whole real axis (denoted by convenience by F too and) defined as follows

$$F(x)=\begin{cases} F'(a)\cdot(x-a)+F(a), & if\ x<a\\ F(x), & if\ x\in I\\ F'(b)\cdot(x-b)+F(b), & if\ x>b.\end{cases}$$

Now, if for some iterate x_p we have $x_p<a$, then

$$x_{p+1}=x_p-\frac{F(x_p)}{F'(x_p)}=x_p-\frac{F'(a)\,(x_p-a)+F(a)}{F'(a)}=a-\frac{F(a)}{F'(a)}>a,$$

because from $(F_1)-(F_2)$ it results $F(a)\cdot F'(a)<0$.

For $x_p>b$, we analogously deduce

$$x_{p+1}=x_p-\frac{F(x_p)}{F'(x_p)}=x_p-\frac{F'(b)\,(x_p-b)+F(b)}{F'(a)},$$

hence

$$x_{p+1}=b-\frac{F(b)}{F'(b)}<b,$$

because $F(b)\,F'(b)>0$. Having in view that (x_n) converges to α and $a<\alpha<b$, we deduce that, beginning from a certain rank p_o, we necessarily shall have

$$x_n\in I.$$

Finally, estimation (6) may be obtained from (8).

Remarks. 1)In the presence of Kantorovich type conditions as (3) or (4) the convergence of the Newton's method is quadratic, while our estimation (6) however shows a linear convergence;

2)For the m-dimensional case, let

$$[a,b]=[a_1,b_1]\times[a_2,b_2]\times\ldots\times[a_m,b_m] \quad \text{and} \quad F=(F_1,F_2,\ldots,F_m).$$

The conditions $(F_2)-(F_3)$ become respectively (condition (F_1), which is a condition of Miranda type, see [12], may be avoided for our purposes):

(F_2^m) $F\in C^1[a;b]$, ,det $[F'(x)]\neq 0$, for each $x\in[a;b]$

and $[F'(a)]^{-1}\cdot F(a)<0$, $[F'(b)^{-1}\cdot F(b)>0$;

(F_3^m) $\|I-[F^{-1}(x)]^{-1}\cdot F^{-1}(y)\|\le 1$,

for each $x\in[a;b]$ and $y\in(a;b$

dimensional case too.

For the proof of the correspondi

refer to [8].

3)It is easy to see that fr

$|1-\frac{f'}{f'}$

which is an equivalent form of (F_3)

3. EXAMPLES AND CONCLUSIO

Let us consider a system of the form

$\begin{cases} y'(t)=f_1(y(\\ z'(t)=f_2(y(\end{cases}$

with the initial values

$\begin{cases} y(0)=-1/2 \\ z(0)=-1/2, \end{cases}$

and let us denote $f=(f_1,f_2)$, $D=[$

$X(t)=(y(t),\ z(t))^T$, $t>0$ an

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and the implicit Euler's discretization

difference equation

$X_{n+1}=X_n+h_n f$

where $X_o=\overline{\eta}$ and $0=t_o<t_1<\ldots<$

for each $x \in [a;b]$ and $y \in (a;b)$, and an estimation similar to (6) holds for the m-dimensional case too.

For the proof of the corresponding convergence theorem in the m-dimensional case, we refer to [8].

3)It is easy to see that from (F_3^m) we obtain for m=1

$$|1-\frac{f'(y)}{f'(x)}|\le 1,\ x,y\in[a,b]$$

which is an equivalent form of (F_3) in Theorem 2.1.

3. EXAMPLES AND CONCLUSIONS

Let us consider a system of the form

$$\begin{cases} y'(t)=f_1(y(t),z(t)) \\ z'(t)=f_2(y(t),z(t)) \end{cases} \tag{9}$$

with the initial values

$$\begin{cases} y(0)=-1/2 \\ z(0)=-1/2, \end{cases} \tag{10}$$

and let us denote $f=(f_1,f_2)$, $D=[-1,1]\,x\,[-1,1]$,

$X(t)=(y(t),\ z(t))^T$, $t>0$ and $\bar{\eta}=(-1/2,-1/2)^T$.

Then,problem (9) - (10) can be written as

$$\begin{cases} X'(t)=f(X(t)),\ t\ge 0, \\ X(0)=\bar{\eta}, \end{cases} \tag{11}$$

and the implicit Euler's discretization for (11) on the interval $[0,T]$ leads to the following difference equation

$$X_{n+1}=X_n+h_n f(X_{n+1}),\ n=0,1,\dots,N-1 \tag{12}$$

where $X_o=\bar{\eta}$ and $0=t_o<t_1<\dots<t_N=T$ is a division of $[0,T]$ and

$h_n = t_{n+1} - t_n, \ n=0,1,\dots,N-1.$

We shall approximate $X(t_n)$ by X_n, therefore at each step in (12),the Newton iterations (according to (2')) are the following

$$(I - h_n f'(X^k))(X^{k+1} - X^k) = -(X^k - X_n - h_n f(X^k)), \ k=0,1,2,\dots, \tag{13}$$

where $X^o = X_n$. We shall approximate the exact solution X_{n+1} of (12) by X^k, for a suitable $k>0$.

The difference equation (11) shows we need in fact to apply the Newton's method for the equation $F(X)=0$, with $F(X) = X - X^o - hf(X)$, $h>0, X^o \in D$ given.

For our present purposes let us consider the problem (9)-(10) with $f(X)$ such that $F(X)$ defined on D is given by

$F(u,v) = (-u^2+u, -v^2+v)^T$, for $u, v \in [-1,0)$,

$F(u,v) = (-u^2+u, v^2+v)^T$, for $u \in [-1,0)$ and $v \in [0,1)$,

$F(u,v) = (u^2+u, -v^2+v)^T$, for $u \in [0,1]$ and $v \in [-1,0)$

and

$F(u,v) = (u^2+u, v^2+v)^T$, for $u, v \in [0,1]$.

It is easy to see that $F(X)=0$ has a unique solution $X=(0,0) \in D$.

Firstly, let us verify condition (F_2^m) in the case $u, v \in [-1,0)$ (the other three cases may be analogously treated).

If $X=(u,v)$, then

$$F'(X) = \begin{bmatrix} -2u+1 & 0 \\ 0 & -2v+1 \end{bmatrix}$$

and det $F'(X) = (1-2u)(1-2v) \neq 0$, for all $u, v \in [-1,0)$, so the first condition in (F_2^m) is true.

For the last two conditions in (F_2^m), let us observe that $a=(-1,-1)$ and $b=(1,1)$, so

$$[F'(a)]^{-1}F(a) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \dots$$

and

$$[F'(b)]^{-1}F(b) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

Let $\|\cdot\|$ be the Euclidian norm and ...

Then

$$\| I - [F'(X)]^{-1} \cdot F(Y) \| = 2 \left[\frac{(s \dots}{(1 \dots} \right.$$

for $(u,v), (s,t) \in [-\frac{1}{2}, 0) \times (-\frac{1}{2} \dots$

Indeed

$$2|s-u| < 1 \leq |1-2u|, \text{ for } \dots$$

hence $\frac{(s-u)^2}{(1-2u)^2} < \frac{1}{4}$, and so on.

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insistence made this paper in final form.

$$[F'(a)]^{-1}F(a)=\begin{bmatrix}3&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}-2\\-2\end{bmatrix}=\begin{bmatrix}-6\\-6\end{bmatrix}<\begin{bmatrix}0\\0\end{bmatrix}=0_{\mathbb{R}}^2$$

and

$$[F'(b)]^{-1}F(b)=\begin{bmatrix}3&0\\0&3\end{bmatrix}\cdot\begin{bmatrix}2\\2\end{bmatrix}=\begin{bmatrix}6\\6\end{bmatrix}>0.$$

Let $\|\cdot\|$ be the Euclidian norm and $X=(u,v)$, $Y=(s,t)$.
Then

$$\|I-[F'(X)]^{-1}\cdot F(Y)\|=2\left[\frac{(s-u)^2}{(1-2u)^2}+\frac{(v-t)^2}{(1-2t)^2}\right]^{1/2}<1,$$

for $(u,v),(s,t)\in[-\frac{1}{2},0)\,x\,(-\frac{1}{2},0)\subset D.$

Indeed

$$2|s-u|<1\leq|1-2u|,\quad \text{for}\quad u\in[-\frac{1}{2},0)\quad\text{and}\quad s\in(-\frac{1}{2},0),$$

hence $\frac{(s-u)^2}{(1-2u)^2}<\frac{1}{4}$, and so on.

Therefore the Newton's iterations converge to the unique solution of $F(X)=0$.
Due to the expression of F, it seems that is more difficult to verify a Kantorovoch condition of type (3) or (4) than our condition (F_3) or (F_3^m).

So, we think our convergence theorem (Theorem 2.1) and its corresponding result in the m-dimensional case may be useful in solving various nonlinear difference systems.

A detailed treatment of these aspects in the m-dimensional case will be soon published by the author.

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Abstract. We prov
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1. INTRODUCTI

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