A METHOD FOR SOLVING SECOND ORDER DIFFERENCE EQUATIONS

VASILE BERINDE
Department of Mathematics and Computer Science
North University of Baia Mare, Victoriei 76,
4800 Baia Mare ROMANIA
e-mail:vberinde@univer.ubm.ro

Abstract For a class of non homogeneous linear second order difference equations with non-constant coefficients, an elementary method for obtaining the general solution is given.

The basic idea of our method is to reduce the solution of a second order difference equation to the solution of two first order nonhomogeneous difference equations.

Two examples, taken from a recent monograph [1], are also treated in order to illustrate the simplicity as well as the limits of our method in comparison with the general method described and used in [2]. The last one is based on the discrete Green functions and on the a priori knowledge of two linearly independent solutions of the homogeneous equation associated to the given difference equation.

Having in view the analogy between difference and differential equations our method may also be adopted to solve a special class of differential equations.

1.INTRODUCTION

We are concerned with the non homogeneous second order linear difference equation

$$a_1(n)u(n+2) + a_2(n)u(n+1) + a_3(n)u(n) = a_4(n),\ n \geq n_0, \qquad (1)$$

where $a_i(n), i = 1, 2, 3, 4$ are given functions defined on $\mathbf{N}$ and

$$a_1(n) \cdot a_3(n) \neq 0 \qquad for \quad each \quad n \in \mathbf{N}, n \geq n_0. \tag{2}$$

If $v_1(n), v_2(n)$ are two linearly independent solutions of the associated homogeneous equation

$$a_1(n)u(n+2) + a_2(n)u(n+1) + a_3(n)u(n) = 0, \tag{3}$$

and $G(k,l)$ is the Green's function of (3) , that is

$$G(k,l) = \begin{vmatrix} v_1(l) & v_2(l) \\ v_1(k) & v_2(k) \end{vmatrix} \Big/ \begin{vmatrix} v_1(l) & v_2(l) \\ v_1(l+1) & v_2(l+1) \end{vmatrix},$$

then (see[1], Section 2.10, pp. 71-75) the general solution of (1) is given by

$$u(n) = c_1 v_1(n) + c_2 v_2(n) + \sum_{l=n_0+1}^{n-1} G(n,l) \frac{a_4(l-1)}{a_2(l-1)}, \quad n \geq n_0, \tag{4}$$

where c_1, c_2 are arbitrary constants.

Therefore, in order to apply formula (4) it is necessary to know two linearly independent solutions of (2).

2. ON A CLASS OF DIFFERENCE EQUATIONS

The main goal of this paper is to show that, for a particular class of second order linear difference equations, we can apply an elementary method, without assuming the knowledge of particular solutions.

First of all, let us observe that, if (2) is fulfilled we can equivalently write (1) in the form

$$u(n+2) - a(n)u(n+1) + b(n)u(n) = c(n), \qquad n \geq n_0 \tag{1'}$$

where

$$a(n) = -\frac{a_2(n)}{a_1(n)}, \quad b(n) = \frac{a_3(n)}{a_1(n)} \quad and \quad c(n) = \frac{a_4(n)}{a_1(n)}. \tag{5}$$

Let us take $n_0 = 1$ without any loss of generality and assume that $a(n)$ and $b(n)$ are so that

$$d(n) = (a(n))^2 - 4b(n) \geq 0, \quad \forall n \geq 1. \tag{6}$$

Then the quadratic equ
(1), which plays an impo
with constant coefficients)

does possess real roots, le
by r_1 and r_2 . Here r_1, r_2

$r_1 = r_1($

If at least one root, say r_1

then we can always solve t
(7) are fulfilled, then we
form

$u(n+2) - (r_1 + r$

or equivalently

$u(n+2) - r_1 u(n +$

Now, if we denote

$v(n$

it results that $v(n)$ is the
equation

$v(n +$

which can be directly solv

$v(n) = \prod_{k=n_0}^{n-1} r$

for each $n \geq n_0, n \in \mathbf{N}$.

Now, in order to obtain
(1), we have to solve the
that is to solve the differe

Then the quadratic equation (actually the characteristic equation of (1'), which plays an important role in the case of difference equations with constant coefficients):

$$r^2 - a(n)r + b(n) = 0, \tag{7}$$

does possess real roots, let us denote these (generally different) roots by r_1 and r_2. *Here r_1, r_2 are functions,*

$$r_1 = r_1(n) \quad \text{and} \quad r_2 = r_2(n).$$

If at least one root, say r_1, is a constant function, that is

$$r_1 = r_1(const), \tag{8}$$

then we can always solve the difference equation (1). Indeed, if (5) and (7) are fulfilled, then we can write the difference equation (1') in the form

$$u(n+2) - (r_1 + r_2(n))u(n+1) + r_1 r_2(n)u(n) = c(n),$$

or equivalently

$$u(n+2) - r_1 u(n+1) - r_2(n)\left[u(n+1) - r_1 u(n)\right] = c(n).$$

Now, if we denote

$$v(n) = u(n+1) - r_1 u(n), \tag{9}$$

it results that $v(n)$ is the solution of the linear first order difference equation

$$v(n+1) - r_2(n)v(n) = c(n)$$

which can be directly solved, see[2], and its general solution is given by

$$v(n) = \prod_{k=n_0}^{n-1} r_2(k) \cdot \left[v(n_0) + \sum_{k=n_0}^{n-1} \frac{c(k)}{\prod_{i=n_0}^{k} r_2(i)} \right], \tag{10}$$

for each $n \geq n_0, n \in \mathbf{N}$.

Now, in order to obtain the general solution of (1'), and therefore of (1), we have to solve the difference equation (9) with respect to $u(n)$, that is to solve the difference equation

$$u(n+1) - r_1 u(n) = v(n).$$

Using (10) with r_1 and $v(n)$ instead of $r_2(n)$ and $c(n)$, respectively, we obtain the general solution of (1')

$$u(n) = r_1^{n-n_0}\left[u(n_0) + \sum_{k=n_0}^{n-1} \frac{v(k)}{r_1^{k-n_0+1}}\right], \qquad n \geq n_0, \tag{11}$$

where $v(k)$ is given by (10).

Consequently , we have just proved

Theorem 1. *If conditions (6) and (8) are satisfied, then the second order linear difference equation (1') may always be solved, its general solution being given by (11).*

Applications. Let us consider the two difference equations in Example 2.10.1, pp. 72, Example 2.10.2 from[1], pp. 74, respectively.

Example 1. The difference equation

$$(2n+1)u(n+2) - 4(n+1)u(n+1) + (2n+3)u(n) = (2n+1)(2n+3), \quad n \in \mathbf{N} \tag{12}$$

can be written as

$$u(n+2) - \frac{4(n+1)}{2n+1}u(n+1) + \frac{2n+3}{2n+1}u(n) = 2n+3.$$

Using the previous notations we have

$$a(n) = \frac{4(n+1)}{2n+1}, \quad b(n) = \frac{2n+3}{2n+1} \quad \text{and} \quad c(n) = 2n+3, \quad n \in \mathbf{N}.$$

Since

$$d(n) = (a(n))^2 - 4b(n) = 1 > 0, \qquad \forall n \in \mathbf{N},$$

condition (6) holds. Moreover,

$$r_1 = 1 \qquad \text{and} \qquad r_2(n) = \frac{2n+3}{2n+1}$$

are the two different roots of the characteristic equation (7). Therefore , from (10) we have

$v(n) = \prod$ …

that is

$v(n) =$ …

Then by (11) we obtain …

$u(n) =$ …

which leads to the genera…

$u(n) = \alpha$ …

where α, β are arbitrary …

Example 2. For the …

$((n+1)^3 - n^3)u(n+2) - (($ …

$= [(n+1)^3 -$ …

we have

$a(n) = \frac{6n^2+12n+8}{3n^2+3n+1},$ …

Although condition (6) is …

$d(n) = \Big($ …

the real roots of the char…

$r_1(n) = 1 +$ …

and condition (8) is not …

$$v(n) = \prod_{k=0}^{n-1} \frac{2k+3}{2k+1} \left[v(0) + \sum_{k=0}^{n-1} \frac{2k+3}{\prod_{i=0}^{k} \frac{2i+3}{2i+1}} \right],$$

that is

$$v(n) = (2n+1)[v(0)+n], \qquad n \geq 0.$$

Then by (11) we obtain the general solution of (12)

$$u(n) = u(0) + \sum_{k=0}^{n-1} (2k+1)[v(0)+k],$$

which leads to the general solution

$$u(n) = \alpha + \beta \cdot n^2 + \frac{n(n-1)(4n+1)}{6},$$

where α, β are arbitrary constants.

Example 2. For the difference equation

$$((n+1)^3 - n^3)u(n+2) - ((n+2)^3 - n^3)u(n+1) + ((n+2)^3 - (n+1)^3)u(n) =$$
$$= [(n+1)^3 - n^3] \cdot [(n+2)^3 - (n+1)^3], \quad n \in \mathbf{N}, \qquad (13)$$

we have

$$a(n) = \frac{6n^2+12n+8}{3n^2+3n+1}, \quad b(n) = \frac{3n^2+9n+7}{3n^2+3n+1}, \quad c(n) = 3n^2+9n+7.$$

Although condition (6) is satisfied,

$$d(n) = \left(\frac{3n+3}{3n^2+3n+1}\right)^2 > 0, \qquad \forall n \in \mathbf{N},$$

the real roots of the characteristic equation are

$$r_1(n) = 1 + \frac{9n+7}{6n^2+6n+2}, \quad r_2(n) = 1 + \frac{3n+3}{6n^2+6n+2},$$

and condition (8) is not satisfied.

So, our method fails, while the general formula (4) can be applied because it is possible to find two linearly independent solutions of the homogeneous equation associated to (13), see[1]:

$$v_1(n) = 1, \qquad v_2(n) = n^3,$$

the general solution being

$$u(n) = \alpha + \beta k^3 + \frac{1}{9}(k-1)(3k+1).$$

3. ANALOGOUS DIFFERENTIAL EQUATIONS

The main merit of our method rests on its applicative feature: we do not apply just a formula, as in the general case - but we follow the steps of a practical algorithm.

We can solve by an analogous method a similar class of differential equations as, for example, the following one:

$$(2x+1)y'' - 4(x+1)y' + (2x+3)y = (2x+1)(2x+3),$$

which is the continuous analogous of (12).

We can write it on $D = R\setminus\{-1\}$ as

$$y'' - \frac{4(x+1)}{2x+1}y' + \frac{2x+3}{2x+1}y = 2x+3 \Longleftrightarrow y'' - y' + \frac{2x+3}{2x+1}(y'-y) = 2x+3.$$

If we denote $z = y' - y$ we have to solve the first order linear equation

$$z' + \frac{2x+3}{2x+1}z = 2x+3,$$

and, with $z(x)$ so obtained, to solve again a first order differential equation

$$y' - y = z(x).$$

4. ANOTHER CLASS OF DIFFERENCE EQUATIONS

The method used in section 2 may be easily extended to the class of difference equations (1') for which there exist $r_1(n), r_2(n)$ such that

In this case (1') may be

$u(n+2) - (r_1(n) +$

and then

$u(n+2) - (r_1(n)u(n +$

If we denote

$v(n)$

we are lead to a first ord

$v(n$

an so on.

Example 3. If we c

equation in Example 2,

($a(n)$ and $c(n)$ being the

$r_1(n) = \frac{3n^2 + 15n}{3n^2 + 3n}$

and thus our method do

Remarks. 1) For

$r_2(n)$ such that (14) hold

order difference equation

2) The sa

of the form

$u(n + 2q) -$

and also for analogous di

$$\begin{cases} a(n) = r_1(n) + r_2(n) \\ b(n) = r_1(n-1) \cdot r_2(n) \end{cases} \tag{14}$$

In this case (1') may be written as

$$u(n+2) - (r_1(n) + r_2(n))u(n+1) + r_1(n-1)r_2(n)u(n) = c(n)$$

and then

$$u(n+2) - (r_1(n)u(n+1) - r_2(n)[u(n+1) - r_1(n-1)u(n)] = c(n).$$

If we denote

$$v(n) = u(n+1) - r_1(n-1)u(n),$$

we are lead to a first order difference equation

$$v(n+1) - r_2(n)v(n) = c(n),$$

an so on.

Example 3. If we consider a slight modification of the difference equation in Example 2, by putting

$$b(n) = \frac{3n^2 + 9n - 5}{3n^2 + 3n + 1}$$

($a(n)$ and $c(n)$ being the same) we find that (12) holds with

$$r_1(n) = \frac{3n^2 + 15n + 7}{3n^2 + 3n + 1} \qquad \textit{and} \qquad r_2(n) = \frac{3n^2 + 9n - 5}{3n^2 + 3n + 1},$$

and thus our method does work.

Remarks. **1)** For $a(n), b(n)$ given, in order to obtain $r_1(n)$ and $r_2(n)$ such that (14) holds, we have to solve generally a *nonlinear* second order difference equation;

2) The same method does work for difference equations of the form

$$u(n+2q) - a(n)u(n+q) + b(n)u(n) = c(n),$$

and also for analogous differential equations.

REFERENCES

[1] Agarwal P. Ravi, *Difference equations and inequalities. Theory, Methods and Applications,* Marcel Dekker, Inc., New York, 1992

[2] Berinde Vasile, On a class of nonlinear recurrences (in Romanian), *Lucr. Sem. Creativ. Mat.*, vol. 4 (1994-1995), 89-94

[3] Berinde Vasile, On an elementary method for solving first order difference equations (in preparation)

ON THE

EQUATIO

W.J. BRIDI

G. LADAS

Department of M

University of Rh

Kingston, RI 028

Abstract We s

of the solutions of

where $\{A_n\}_{n=0}^{\infty}$ an

and where the initi

We also investigat

where $\{A_n\}_{n=0}^{\infty}$ is

and where the initi