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North University Center at Baia Mare
Faculty of Sciences

Ph.D. Thesis

ACCELERATION TECHNIQUES FOR APPROXIMATING FIXED POINTS

Scientific advisor:
Senior researcher VASILE BERINDE

Ph.D. Student:
OANA BUMBARIU

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Contents

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Introduction

The study of the convergence acceleration methods has become an important domain of numerical analysis, the idea of applying suitable sequence transformation for the acceleration of the convergence of series or for the summation of a divergent series is almost so old as analysis itself.

In recent years an impressive number of papers have been published regarding on the development or improvement of existing sequence acceleration methods for solving nonlinear equations of the form $f(x) = 0$.

It is well known that in numerical analysis many methods produce sequences, for example iterative methods for solving systems of equations, methods involving series expansions, perturbation methods, discretization methods, etc. The convergence of those sequences may be slow and their effective use is quite limited. In order to accelerate them we need to use an convergence acceleration process, T, that consists in transforming the slowly converging sequence, $\{S_n\}$, into a new one, $\{T_n\}$, which under some assumptions converges to the same limit, S, as the first one but faster. The transformation T accelerates the convergence speed of the sequence $\{S_n\}$ if it satisfies the relation

$$\lim_{n\to\infty} \frac{T_n - S}{S_n - S} = 0.$$

If the above relation is fulfilled we say that $\{T_n\}$ converges to S faster than $\{S_n\}$ converges to S.

One of the most popular and well known sequence acceleration methods was given by one of the greatest New Zealand's mathematicians, A.C. Aitken, the method consists in

$$A_1^{(n)} = S_n - \frac{(S_{n+1} - S_n)^2}{S_{n+2} - 2S_{n+1} + S_n} = S_n - \frac{[\Delta S_n]^2}{\Delta^2 S_n}, \quad n = 0, 1, \ldots, \tag{0.1}$$

where $\{S_n\}$ is the sequence to be accelerated and Δ denotes the forward difference operator, $\Delta S_n = S_{n+1} - S_n$ and $\Delta^2 S_n = \Delta S_{n+1} - \Delta S_n = S_{n+2} - 2S_{n+1} + S_n$, the method is called Aitken's Δ^2 process and was obtained in 1926 [1].

In 1927, L. F. Richardson [98] proposed a technique for solving a 6th order differential eigenvalue problem, this extrapolation method being given by

$$T_{k+1}^{(n)} = \frac{x_{n+k+1} T_k^{(n)} - x_n T_k^{(n+1)}}{x_{n+k+1} - x_n}, \quad \kappa, n = 0, 1, \ldots$$

with $T_0^{(0)} = S(x_n)$ for $n = 0, 1, \ldots$.

Extensions and applications of the Richardson extrapolation process can be found in [52], [71], [131].

In 1955, D. Shanks [108], considered the following transformation that bears his name, Shanks transformation, given by

$$T_n = e_k(S_n) = \frac{\begin{vmatrix} S_n & S_{n+1} & \ldots & S_{n+k} \\ S_{n+1} & S_{n+2} & \ldots & S_{n+k+1} \\ \vdots & \vdots & & \vdots \\ S_{n+k} & S_{n+k+1} & \ldots & S_{n+2k} \end{vmatrix}}{\begin{vmatrix} \Delta^2 S_n & \ldots & \Delta^2 S_{n+k-1} \\ \vdots & & \vdots \\ \Delta^2 S_{n+k-1} & \ldots & \Delta^2 S_{n+2k-2} \end{vmatrix}}.$$

When $k = 1$, Shanks transformation reduces to Aitken's Δ^2 process.

In the same year, 1955, W. Romberg [102], suggested a new transformation, called Romberg transformation, given by

$$T_{k+1}^{(n)} = \frac{4^{k+1}T_k^{(n+1)} - T_k^{(n)}}{4^{k+1} - 1},$$

where $T_0^{(n)}$ is obtained by trapezoidal rule with the stepzise $h_0/2^n$. It was proved by P.J. Laurent, in 1963 [74] that the Romberg process is the same as the Richardson extrapolation process when taking $x_n = h_n^2$ and $h_n = h_0/2^n$.

The determinants used in Shanks transformation are difficult to compute and frequently give a wrong result because are needed too many arithmetical operations and because of the rounding errors due to the computer. Consequently a recursive scheme, for computing the $e_k(S_n)$'s without computing the determinants involved in their definition was needed. This algorithm was introduced in 1956, by Wynn [126], being called the ϵ-algorithm. It begins with

$$\epsilon_{-1}^{(n)} = 0, \quad \epsilon_0^{(n)} = S_n$$

and then computes

$$\epsilon_{k+1}^{(n)} = \epsilon_{k-1}^{(n+1)} + \frac{1}{\epsilon_k^{(n+1)} - \epsilon_k^{(n)}}.$$

Also in the same year, 1956, Wynn [127] proposed the following algorithm, called ρ-algorithm

$$\begin{aligned} &\rho_{-1}^{(n)} = 0, \quad \rho_0^{(n)} = S_n \\ &\rho_{k+1}^{(n)} = \rho_{k-1}^{(n)} + \frac{k+1}{\rho_k^{(n+1)} - \rho_k^{(n)}}, \qquad k, n \in \mathbb{N}_0. \end{aligned}$$

In 1971 Brezinski [21] proved some limitations of Wynn's ϵ-algorithm and improved it with the so called θ-algorithm, which is given by

$$\theta_{2k+1}^{(n)} = \theta_{2k-1}^{(n+1)} + \frac{1}{\Delta\theta_{2k}^{(n)}}, \quad \theta_{2k+2}^{(n)} = \theta_{2k}^{(n+1)} + \frac{\Delta\theta_{2k}^{(n+1)}\Delta\theta_{2k+1}^{(n+1)}}{\Delta^2\theta_{2k+1}^{(n)}},$$

where $\theta_{-1}^{(n)} = 0$, $\theta_0^{(n)} = S_n$, $\Delta\theta_k^{(n)} = \theta_k^{(n+1)} - \theta_k^{(n)}$, $\Delta^2\theta_k^{(n)} = \Delta\theta_k^{(n+1)} - \Delta\theta_k^{(n)}$ and $k, n \in \mathbb{N}$.

In 1973 D. Levin [75] introduced a new acceleration algorithm. This algorithm is given by

$$l(n,k) = \frac{\sum\limits_{i=0}^{k}(-1)^i \binom{k}{i} (n+i)^{k-2}(\Delta_{n+i+1})(\Delta S_{n+i})^{-1}}{\sum\limits_{i=0}^{k}(-1)^i \binom{k}{i} (n+i)^{k-2}(\Delta S_{n+i})^{-1}},$$

where $S_n = \sum\limits_{i=0}^{n} c_i$ and $n, k \in \mathbb{N}$.

For all the presented iterative methods the difference operator, Δ, acts only upon the superscript n and not upon the subscript k.

For a general historical development of sequence transformation see [1], [19], [21], [30], [34], [52], [71], [75], [98], [102], [108], [121], [122], [126], [127].

Motivated by the efficiency, the performance and the simplicity of Aitken's Δ^2 process, (0.1), in this thesis we proposed a new acceleration method, for a sequence that converges linearly, that was inspired from Aitken's Δ^2 process by a suitable manipulation of the algorithm.

The thesis is structured in three chapters as follows:

The first chapter, **Theoretical Background**, is divided in eight paragraphs in which we provide the mathematical symbols and the basic concepts that will be used throughout this thesis. For a better understanding and reading of the results presented here, in the first paragraph, we have introduced the special mathematical symbols used in the thesis. The second paragraph is dedicated to the terminology and to the notions from the fixed point theory, we also introduced here the well known fixed point iterative procedures: Picard, Krasnoselskij, Mann and Ishikawa for which we present some convergence results, in different classes of operators, the theorems are given without the proofs, for an extensive treatment we refer the readers to consult the articles and the monographs: [3], [4], [5], [12], [13], [14], [132]. As well in this paragraph we present Definition 1.10 given by Berinde in [12] that will help us to compare the convergence speed of two sequences, then we continue with the **Contraction mapping principle** or the **Banach theorem** or the **Picard-Banach-Caccioppoli theorem** [8], [12], which is the only theorem presented with proof in this chapter and we end this paragraph by presenting how a nonlinear equation can be rewritten as the equivalent fixed point problem, as an application we give Example 1.4, which is the author personal contribution in this paragraph.

The Paragraph 1.3 contains a short presentation for the scalar sequences, then in Paragraph 1.4 we introduced the types of sequences that occur in practical applications, as well we present Definition 1.13 [12], [19] which will help us to compare the convergence speed of to sequences, both convergent to the same limit. The next two paragraphs, Paragraph 1.5 and Paragraph 1.6 contain concepts that will help us get familiar with the notions of sequence transformations and iterated sequence transformations, following that in Paragraph 1.7 to present the most important sequence acceleration processes from the last century, that are: Richardson's process [98], Romberg process [102], Shanks transformation [108], Aitken's Δ^2 process [1], Wynn's ϵ [126] and ρ [127] algorithms, Brezinski θ-algorithm [21] and Levin transformation [75], the above nonlinear sequence transformations are shortly presented in this thesis and we here give only a few of their properties, for their extensive treatment we refer the readers to see the articles and the monographs in the reference list.

The last paragraph of this chapter, Paragraph 1.8, contains the presentation of two orders of convergence for the iterative methods, q-order and r-order [83], that can be applied for every convergent sequence from $\mathbb{R}^n$.

In [16], Biazar and Amirteimoori proposed a Padé-type acceleration technique to accelerate Picard iteration method for solving three scalar equations of the form $f(x) = 0$. Starting from this article our aim in the second chapter, entitled **A Padé-type acceleration technique**, is to apply the Padé-type acceleration technique to some nonlinear equations, for different values of k, in order to see which value gives the best convergence speed, then to apply the Padé-type acceleration technique to some initial value problems and in the last part of the chapter to make an empirical study of the possibilities of accelerating the Krasnoselskij, Mann and Ishikawa iterations for some operators taken from the literature.

In Paragraph 2.1 is presented Biazar and Amirteimoori's Padé-type acceleration technique for accelerating Picard iteration, authors personal contribution are: Theorem 2.1, Remark 2.1 and the Examples 2.1, 2.2, 2.3.

The Paragraph 2.2 contains a Padé-type acceleration technique, given in two ways, for solving initial value problems, all paragraph is author original result. The method is given only in a empiric manner for both cases, being applied to one initial value problem presented in the Example 2.4.

As well Paragraph 2.3, Paragraph 2.4 and Paragraph 2.5 are original author results, in Paragraph 2.3 we apply the Padé-type acceleration technique, in two ways, to some sequences arising when applying the Krasnoselskij iteration, in Paragraph 2.4 we apply the Padé-type acceleration technique to some sequences arising when applying the Mann iteration and in Paragraph 2.5 we apply the Padé-type acceleration technique to some sequences arising when applying the Ishikawa iteration at the end of each paragraph is presented one example to see the efficiency of the new proposed method.

The last chapter, **New acceleration methods for sequences that converge linearly**, is author original result and it represents the main result of this thesis. Starting from one of the most famous and well known sequence acceleration processes, Aitken's Δ^2 process, given by the relation (0.1) we were able to develop a new acceleration method, that we called the B-algorithm, being given by the following formula

$$B_1^{(n)} = S_{n+3} - \frac{[\overline{\Delta} S_{n+1}][\Delta S_{n+2}]}{\overline{\Delta} S_{n+1} - \overline{\Delta} S_n}, \quad n = 0, 1, \ldots \tag{0.2}$$

where $\{S_n\}$ is the sequence to be accelerated, Δ denotes the forward difference operator, defined by $\Delta S_n = S_{n+1} - S_n$ and we denote $\overline{\Delta}$, the forward difference operator with two steps, defined by $\overline{\Delta} S_n = S_{n+2} - S_n$.

This chapter is divided in two big paragraphs, in the first one, Paragraph 3.1, we have introduced the B-algorithm for which we gave some convergence results represented by the Theorems 3.1 which shows that the B-algorithm applied to some sequence has a better converge speed than the sequence to be accelerated and the same rate of converge as Aitken's Δ^2 process, Theorem 3.3 in which is demonstrated the convergence order for the B-algorithm, the following to theorems 3.4, 3.5 show that the B-algorithm is exact for some sequences, Theorem 3.7 shows that B-algorithm is quasi-linear, and theorems 3.6 and 3.8 show that the B-algorithm converges to the limit of some sequences that fulfill some relations, we also give the Remark 3.3 and the Examples 3.2-3.12.

In Paragraph 3.1.1 we suggested some other representations for the B-algorithm, obtained by a suitable manipulation of the relation (0.2), for the new proposed algorithms we gave some convergence results, similar to that for the B-algorithm represented, for all methods, by Theorem 3.9 which shows that the other representations for the B-algorithm applied to some sequence have a better convergence speed than the sequence to be accelerated and the same rate of convergence as the B-algorithm. The paragraph ends with Example 3.13 which shows the practical implication of the new proposed algorithms.

In Paragraph 3.1.2 we gave the iterated form for the B-algorithm, obtaining a nonlinear recursive scheme with a better convergence speed. For the new method we obtained the following convergence results given by Theorem 3.10 in which we give conditions on $B_i^{(n)}$ which ensure that $\lim\limits_{n\to\infty} B_k^{(n)} = S$, Theorems 3.11 , 3.12, 3.13, 3.14 in which is compared the convergence speed of the $B_k^{(n)}$ with different superscripts and subscripts, we end the paragraph with some application in which we apply the iterated B-algorithm to some linear sequences in the Examples 3.14-3.19 and then we apply the algorithm to the sequence of successive approximations obtained in solving a nonlinear equation by means of fixed point theory in Example 3.21.

Motivated by the efficiency and the performance obtained for the iterated B-algorithm, in Paragraph 3.1.3 we proposed the iterated forms for the other representation of the technique, giving then some similar theoretical results in Theorem 3.15 we give conditions on $B_i^{(n)}$ which ensure that $\lim_{n\to\infty} B_k^{(n)} = S$ and in Theorem 3.16 we compare the convergence speed of the $B_k^{(n)}$ with different superscripts and subscripts and a numerical result in Example 3.22, then in the ending of the first part of the chapter, Paragraph 3.1.4, we gave some extensions for the iterated B-algorithm, author theoretical results are Theorem 3.17, Theorem 3.18 in which are presented the same results as in the previous paragraph and Remark 3.4, the paragraph ends with the Example 3.23.

The second part of the chapter is dedicated to an empirical study for accelerating the Krasnoselskij, Mann and Ishikawa iterations with the help of the B-algorithm, Paragraph 3.2.1, and its iterated form, Paragraph 3.2.2, with the other representations for the B-algorithm, Paragraph 3.2.3, and their iterated forms, Paragraph 3.2.4, and for the extensions of the iterated B-algorithm, Paragraph 3.2.5, at the end of each paragraph we applied the new acceleration techniques to some operators taken from the literature in order to see the practical implication and the performance of the new methods.

Because in the literature very little seems to be known about the theoretical properties of iterated Aitken's Δ^2 process, as far it is known, only one article written by Hillion [64], our next goal is to create a model sequence for which iterated Aitken's Δ^2 process is exact and then to extend that theory to the iterated B-algorithm.

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1 Theoretical Background

1.1 Special mathematical symbols and notions

In this thesis we will use essentially standard mathematical terminology:
$\mathbb{N}$-stands for the set of positive integers $n = 1, 2, 3, \ldots$;
$\mathbb{N}_0$- stands for the set of nonnegative integers $n = 0, 1, 2, \ldots$;
$\mathbb{R}$-denotes the set of real numbers;
$\{S_n\}$-denotes the set of elements S_n with $n \in \mathbb{N}_0$.

Let X be a nonempty set and $T : X \to X$ a selfmap. We note by:
1_x-the identity operator; $F_T := \{x \in X | Tx = x\}$-the fixed point set of T;
$T^0 := 1_x,\ T^1 := T,\ \ldots, T^n := T \circ T^{n-1}$-the iterates of T.

Let f be a function defined on the set of integers $\mathbb{N}_0$. We denote by
$\Delta f(n) = f(n+1) - f(n),\ n \in \mathbb{N}_0$-the forward difference operator;
$\Delta^k f(n) = \Delta[\Delta^{k-1} f(n)]$, with $\Delta^0 f(n) = f(n),\ k \in \mathbb{N}$-the powers of the difference operator Δ.

1.2 The basic concepts of fixed point theory

In this paragraph we present the terminology, basic concepts and notations from the fixed point theory that we will use throughout the thesis. The results are presented without the proofs, they can be found in the monographs that are presented in the list of references, exception is one result, The Contraction Mapping Principle. Also, at the end of the paragraph we will illustrate how to obtain a solution for a certain functional equation by means of iterative processes.

Let X be a nonempty set and $T : X \to X$ a selfmap. We say that $x \in X$ is a fixed point of T if

$$T(x) = x$$

and denote by F_T or $Fix(T)$ the set of all fixed points of T.

Example 1.1 *[12]*
1. If $X = \mathbb{R}$ and $T(x) = x^2 - 2$ then $F_T = \{0, 2\}$.
2. If $X = \left[\frac{1}{2}, 2\right]$ and $T(x) = 1/x$ then $F_T = \{1\}$.

Let X be any set and $T : X \to X$ a selfmap. For any given $x \in X$ we define $T^n(x)$ inductively by $T^0(x) = x$ and $T^{n+1}(x) = T(T^n(x))$ we say that $T^n(x)$ is the n^{th} iterate of x under T. To simplify the notations we will use Tx instead of $T(x)$.

The mapping $T^n,\ (n \geq 1)$ is called the n^{th} *iterate of* T. For any $x_0 \in X$, the sequence $\{x_n\}_{n\geq 0} \subset X$ given by

$$x_n = Tx_{n-1} = T^n x_0, \quad n = 1, 2, \ldots$$

is called the **sequence of successive approximations** with the initial value x_0 or the **Picard iteration** starting at x_0.

1.2.1 Metric spaces

Definition 1.1 *[12] Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}_+$ is called a **metric** or a **distance** on X if we have:*
(d1) $d(x, y) = 0 \Longleftrightarrow x = y$ ("separation axiom")
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$ ("symmetry")
(d3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ ("the triangle inequality").
*A set X endowed with a metric d is called **metric space** and it is denoted by (X, d).*

Example 1.2 *[12] $X = \mathbb{R}$, $d(x, y) = |x - y|$, $\forall x, y \in \mathbb{R}$, where $|\cdot|$ denotes the absolute value, is a metric (a distance) on $\mathbb{R}$.*

Definition 1.2 *[12] Let $\{x_n\}$ be a sequence in a metric space (X, d). We say that the sequence $\{x_n\}$*
*(a) is **convergent** to $a \in X$ if, for any $\epsilon > 0$, there is $n_0 = n_0(\epsilon)$ such that $d(x_n, a) < \epsilon$, for any $n \in \mathbb{N}$, $n \geq n_0$;*
*(b) is **fundamental** or **Cauchy sequence** if, for any $\epsilon > 0$, there is $n_0 = n_0(\epsilon)$ such that $d(x_n, x_{n+p}) < \epsilon$, for any $n \in \mathbb{N}$, $n \geq n_0$ and for any $p \in \mathbb{N}^*$.*

Remark 1.1 *[12] In a metric space, any convergent sequence is a Cauchy sequence, too, but the reserve is not generally true.*

Definition 1.3 *[12] A metric space (X, d) is called **complete** if any Cauchy sequence in X is convergent.*

Definition 1.4 *[12] Let (X, d) be a metric space. A mapping $T : X \to X$ is called:*
*(1) **Lipschitzian** or **L-Lipschitzian** if there exits $L > 0$ such that $d(Tx, Ty) \leq Ld(x, y)$, for all $x, y \in X$.*
*(2) **(strict) contraction** or a-**contraction** if T is a-Lipschitzian, with $a \in [0, 1)$.*
*(3) **nonexpansive** if T is 1-Lipschitzian.*

Example 1.3 *[12]*
1, $T : \mathbb{R} \to \mathbb{R}$, $Tx = \frac{x}{2} + 3$, $x \in \mathbb{R}$ is a strict contraction and $F_T = \{6\}$.
2, The function $T : \left[\frac{1}{2}, 2\right] \to \left[\frac{1}{2}, 2\right]$, $Tx = \frac{1}{x}$ is 4-Lipschitzian with $F_T = \{1\}$.

1.2.2 Normed spaces

Definition 1.5 *[12] Let E be a real (complex) vector space. A norm on E is a mapping $\|\cdot\| : E \times E \to \mathbb{R}_+$ having the following properties:*
(n1) $\|x\| = 0 \Longleftrightarrow x = 0$, the null element of E;
(n2) $\|\lambda x\| = |\lambda| \cdot \|x\|$, for any $x \in E$ and any scalar λ;
(n3) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in E$. ("the triangle inequality")
*The pair $(E, \|\cdot\|)$ is called **normed (linear) space**.*

Remark 1.2 *[12] If $\|\cdot\|$ is a norm on the (linear) vector space E, then $d : E \times E \to \mathbb{R}_+$ given by*

$$d(x, y) = \|x - y\|, \quad x, y \in E,$$

is a distance on E. This shows that any normed space can be always regarded as a metric space with respect to the distance introduced by the norm.

Remark 1.3 *[12] A **Banach space** is a normed space which is complete (as a metric space).*

From the Remark 1.2 we deduce that all the concepts related to the norm in a normed space could be adapted from the metric space. The nonexpansive condition can be rewritten in the following form $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in E$.

Definition 1.6 *[12] A Banach space $(E, \|\cdot\|)$ is called **uniformly convex** if, given any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for all $x, y \in E$ satisfying $\|x\| \leq 1,\ \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, we have*

$$\frac{1}{2}\|x + y\| < 1 - \delta(\epsilon).$$

Definition 1.7 *[12] A set C of a real vector space E is called **convex** if, for any pair of points x, y in C, the closed segment with the extremities x, y, that is, the set $\{\lambda x + (1 - \lambda)y | \lambda \in [0, 1]\}$ is contained in C. A set C of a real normed space is called **bounded** if there exists M such that $\|x\|\| \leq M$, for all $x \in C$.*

Definition 1.8 *[12] Let X be a linear normed space. A mapping $T : D(T) \subset X \to X$ where $D(T)$ is the domain of T, is called **strongly** (or **strictly**) **pseudo-contractive** if there exists $t > 1$ such that*

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|$$

for $\forall\ x, y \in D(T)\ r > 0$.

If $t = 1$ then T is called **pseudo-contractive**. An operator is strong pseudo-contractive if and only if $I - T$ is accretive, which is equivalent with

$$\|x - y\| \leq \|x - y + r[(I - T - kI)x - (I - T - kI)y]\|$$

holds for all $\forall\ x, y \in D(T),\ r > 0,\ k = \frac{t-1}{t}$.

1.2.3 Hilbert spaces

Definition 1.9 *[12] Let H be a real vector space. An **inner product** is a functional $< \cdot, \cdot >: H \times H \to \mathbb{R}$ satisfying:*
(p1) $< x, x > \geq 0$ for all $x \in H$ and $< x, x >= 0$ if and only if $x = 0$, the null vector in H;
(p2) $< x, y >=< y, x >$, for all $x, y \in H$;
(p3) $< ax + by, z >= a < x, z > +b < y, z >$, for each $x, y, z \in H$ and all $a, b \in \mathbb{R}$.

If $< \cdot, \cdot >$ is an inner product on H, then the function $x \to < x, x, >^{1/2}$ defines a norm on H, called the **norm induced by the inner product**. The pair $(H, < \cdot, \cdot >)$ is called a **prehilbertian space**.
A prehilbertian space that is complete is called **Hilbert space**.

Remark 1.4 *[12] Any Hilbert space is a uniformly convex Banach space.*

1.2.4 Fixed point iteration procedures

Let (X, d) be a metric space, $D \subset X$ a closed subset of X (we often have $D = X$) and $T : D \to D$ a selfmap that has at least one fixed point $p \in F_T$. For a initial value $x_0 \in X$ we consider the sequence $\{x_n\}$ defined by

$$x_n = T(x_{n-1}) = T^n(x_0), \quad n = 1, 2, \ldots. \tag{1.1}$$

The sequence defined by the relation (1.1) is called **the sequence of successive approximations** or the **Picard iteration**.

In the case when the contractive conditions are slightly weaker, the Picard iteration will not converge to the fixed point of T, consequently other iteration procedures must be applied.

The following fixed point iteration procedures will be introduced in a real normed space $(E, \|\cdot\|)$. Let $T : E \to E$ be a selfmap, $x_0 \in E$ and $\lambda \in [0, 1]$. The sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, n \geq 0 \tag{1.2}$$

is called **the Krasnoselskij iteration procedure** or **Krasnoselskij iteration**.

The Krasnoselskij iteration $\{x_n\}$ defined by (1.2) is exactly the Picard iteration corresponding to the averaged operator

$$T_\lambda = (1 - \lambda)I + \lambda T \tag{1.3}$$

where I is the identity operator and for $\lambda = 1$ the Krasnoselskij iteration reduces to Picard iteration, so we have $Fix(T) = Fix(T_\lambda)$, for all $\lambda \in (0, 1]$.

The sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, n \geq 0 \tag{1.4}$$

where $\alpha_n \subset [0, 1]$ satisfies certain appropriate conditions is called **the Mann iteration procedure** or **Mann iteration**.

If we take

$$T_n = (1 - \alpha_n)I + \alpha_n T, \tag{1.5}$$

then $Fix(T) = Fix(T_{\alpha_n})$, for all $\alpha_n \in (0, 1]$.

If we consider $\alpha_n = \lambda$ (const.), then the Mann iteration reduces to the Krasnoselskij iteration and when $\alpha_n = 0$, then the Mann iteration reduces to Picard iteration.

The sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T[(1 - \beta_n) + \beta_n T x_n], \quad n = 0, 1, 2, \ldots, \tag{1.6}$$

where $\{\alpha_n\}, \{\beta_n\} \in [0, 1]$ are two sequences of real numbers satisfying certain appropriate conditions, is called **Ishikawa iteration procedure** or **Ishikawa iteration**.

The relation (1.6) can be rewritten in a system form as follows

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T x_n, \ n \geq 0 \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \ n \geq 0. \end{cases} \tag{1.7}$$

If we take $\beta_n = 0$ in (1.6) or (1.7) the Ishikawa iteration reduces to the Mann iteration, but there is not a general dependence between convergence results for Mann iteration and Ishikawa iteration. If we take $\alpha_n = 0$ the Ishikawa iteration reduces to Picard iteration.

1.2.5 Other important fixed point iteration procedures

Let E be a Banach space and $T : E \to E$ a mapping.

The Kirk's iteration procedure is defined by $x_0 \in E$ and

$$x_{n+1} = \lambda_0 x_n + \lambda_1 T x_n + \lambda_2 T^2 x_n + \ldots + \alpha_k T^k x_n,$$

where k is a fixed integer, $k \geq 1,\ \alpha_i \geq 0$, for $i = 0, 1, \ldots, k,\ \alpha_1 > 0$ and

$$\alpha_0 + \alpha_1 + \ldots + \alpha_k = 1.$$

For $k = 0$ is reduces to Picard iteration and for $k = 1$ is reduces to the Krasnoselskij iteration.

The Kirk, Krasnoselskij, Mann and Ishikawa iteration procedures are used to generate successive approximations for the fixed points for various classes of mappings in normed linear spaces, for which the Picard iteration does not converge.

Let H be a Hilbert space and C be a closed, bounded and convex subset of H containing 0. The sequence $\{x_n\}$ defined by $x_0 \in C$ and

$$x_n = T_n^{n^2} x_{n-1}, \quad n = 1, 2, \ldots$$

where $T_n x = \frac{n}{n+1} T x,\ n \geq 1$ is called the **Figueiredo iteration procedure**.

Let T be a selfmap of a Hilbert space H, and $\alpha = \{\alpha\}$ be a sequence in $[0, 1]$. The sequence $\{A_n^\alpha\}$ defined by $A_0^\alpha x = x$ and

$$A_{n+1}^\alpha x = \alpha_{n+1} x + (1 - \alpha_{n+1}) T A_n^\alpha x,\ n = 0, 1, \ldots,$$

will be called **the Halpern iteration procedure**.

If T is positively homogeneous ($T(tx) = tT(x)$, for $t \geq 0$ and $x \in H$) and $\alpha_n = \frac{1}{n+1},\ n \geq 0$, we have

$$A_n^\alpha = \frac{1}{n+1} S_n x,$$

where $S_0 x = x, S_{n+1} = x + T(S_n, x)$ is a nonlinear generalization of the Cesaro averages.

Another iteration scheme is called **Wittmann iteration** and is given by

$$A_0^\alpha = x,\ A_{n+1}^\alpha x = \alpha_{n+1} x + T((1 - \alpha_{n+1}) A_n^\alpha x),$$

which reduces to the Halpern if T is positively homogeneous.

1.2.6 Some convergence results for the Picard, Krasnoselskij, Mann and Ishikawa iterations

In the last twenty years many authors have been studied the convergence of the Picard, Krasnoselskij, Mann and Ishikawa iterations, defined from T to a fixed point of

T, under several contraction conditions. So, it is of theoretical and practical importance to compare these iterative methods in order to see which one converges faster. In what follows we will present some convergence results for the: Picard, Krasnoselskij, Mann and Ishikawa iterations in different classes of operators.

We begin this paragraph with some results taken from [12] that will help us to compare two iteration procedures.

In order to compare two fixed point iterative procedures $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ that converge to a fixed point p of a operator T, Rhoads [100] considered that $\{u_n\}$ is "better" that $\{v_n\}$ if

$$\|u_n - p\| \leq \|v_n - p\|, \quad \forall n.$$

In the sequel we will use the following concepts introduced by Berinde in [12].

Definition 1.10 *[12] Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of positive numbers that converge to a, respectively b. Assume there exists*

$$l = \lim_{n\to\infty} \left| \frac{a_n - a}{b_n - b} \right|$$

*(i) If $l = 0$, then it is said that the sequence $\{a_n\}_{n=0}^{\infty}$ converges to a **faster** than the sequence $\{b_n\}_{n=0}^{\infty}$ to b.*
*(ii) If $0 < l < \infty$, then we say that the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n0}^{\infty}$ have the **same rate of convergence**.*

Remark 1.5 *[12]*
(a) In the case (i) we use the notation $a_n - a = o(b_n - b)$
(b) If $l = \infty$, then the sequence $\{b_n\}_{n=0}^{\infty}$ converges faster than the $\{a_n\}_{n=0}^{\infty}$, that is $b_n - b = o(a_n - a)$.

Assume that for two fixed point procedures $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ both converging to the same fixed point p, the following a priori error de estimates:

$$\|u_n - p\| \leq a_n, \; n = 0, 1, 2, \ldots \tag{1.8}$$

$$\|v_n - p\| \leq b_n, \; n = 0, 1, 2, \ldots \tag{1.9}$$

are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive real numbers (converging to zero).

Then in view of the Definition 1.10 we will have the following concept.

Definition 1.11 *[12] Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two fixed point iterative procedures converging to the same fixed point p and satisfying the relations (1.8) and (1.9). If $\{a_n\}$ converges faster than $\{b_n\}$ then we shall say that $\{u_n\}$ converges faster to p than $\{v_n\}$.*

In the sequel we present some convergence results for the Picard, Krasnoselskij, Mann and Ishikawa iterations in class of Zamfirescu operators and in the class of Lipschitzian pseudo-contractive mappings. (we here give the proof only for the *Contraction mapping principle* the other results are presented without the proofs, for an extensive treatment see [3], [4], [5], [12], [13], [14], [132])

Theorem 1.1 ***Contraction mapping principle****(or Banach theorem or Picard-Banach-Caccioppoli theorem)[8], [12]*
Let (X, d) be a complete metric space and $T : X \to X$ be a-contraction, that is an operator satisfying

$$d(Tx, Ty) \leq ad(x, y), \quad \forall x, y \in X \tag{1.10}$$

with $a \in [0, 1)$ fixed. Then
(i) T has a unique fixed point, that is, $F_T = \{x^*\}$;
(ii) The Picard iteration associated to T, i.e., the sequence $\{x_n\}_{n\geq 0}$ defined by

$$x_n = T(x_{n-1}) = T^n(x_0), \quad n = 1, 2, \dots \tag{1.11}$$

converges to x^*, for any initial guess $x_0 \in X$;
(iii) The following a priori and a posteriori error estimates hold:

$$d(x_n, x^*) \leq \frac{a^n}{1-a} d(x_0, x_1), n = 0, 1, 2, \dots \tag{1.12}$$

$$d(x_n, x^*) \leq \frac{a}{1-a} d(x_{n-1}, x_n), n = 0, 1, 2, \dots \tag{1.13}$$

(iv) The rate of convergence is given by

$$d(x_n, x^*) \leq ad(x_{n-1}, x^*) \leq a^n d(x_0, x_1), n = 1, 2, \dots \tag{1.14}$$

Proof[12] There is at most one fixed point $cardF_T \leq 1$. Indeed assuming $x^*, y^* \in F_T$, $x^* \neq y^*$, since $0 \leq a < 1$, we get the contradiction

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq ad(x^*, y^*) < d(x^*, y^*).$$

To prove the existence of the fixed point, we will show that, for any given $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

Notice that, by (1.10), we have

$$d(x_2, x_1) = d(Tx_1, Tx_0) \leq ad(x_1, x_0),$$

and by induction

$$d(x_{n+1}, x_n) \leq a^n d(x_1, x_0), \quad n = 0, 1, 2, \dots .$$

Thus, for any numbers $n, p \in \mathbb{N}, p > 0$, we have

$$d(x_{n+p}, x_n) \leq \sum_{k=n}^{n+p-1} d(x_{k+1}, x_k) \leq \sum_{k=n}^{n+p-1} a^k d(x_1, x_0) \leq \frac{a^n}{1-a} d(x_1, x_0). \tag{1.15}$$

Since $0 \leq a < 1$, it results that $a^n \to n$ (as $n \to \infty$), which together with (1.15) shows that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. But (X, d) is a complete metric space, therefore $\{x_n\}_{n=0}^{\infty}$ converges to some $x^* \in X$. On the other hand, any Lipschitzian mapping is continuous. So, denoting

$$\lim_{n\to\infty} x_n = x^*,$$

we find

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T\left(\lim_{n\to\infty} x_n\right) = Tx^*,$$

which gives $x^* = Tx^*$, i.e., x^* is a fixed point of T.

This shows that for any $x_0 \in X$, the Picard iteration converges in X and its limit is a fixed point of T. Since T has at most one fixed point, we deduce that for every choice of $x \in X$, the Picard iteration converges to some value x^*, that is, the unique fixed point of T. So, we proved (i) and (ii).

To prove (iii) we use (1.15),

$$d(x_{n+p}, x_n) \leq \frac{a^n}{1-a} d(x_0, x_1), \quad \forall p \in \mathbb{N}^*,$$

and the continuity of the metric and so, by letting $p \to \infty$, we find

$$d(x_n, x^*) = d(x^*, x_n) = \lim_{p \to \infty} d(x_{n+p}, x_n) \leq \frac{a^n}{1-a} d(x_0, x_1), \; n \geq 0$$

and so (iii) is proved.

To obtain the a posteriori estimation (3.7), let us notice that by (1.10) we have

$$d(x_{n+1}, x_n) \leq a d(x_n, x_{n-1})$$

and, by induction

$$d(x_{n+k}, x_{n+k-1}) \leq a^k d(x_n, x_{n-1}), \; k \in \mathbb{N}^*,$$

so

$$d(x_{n+p}, x_n) \leq (a + a^2 + \ldots + a^p) d(x_n, x_{n-1}) \leq \frac{a}{1-a} d(x_n, x_{n-1}).$$

By letting $p \to \infty$ in the last inequality we get exact (iv).

In 1972 [133], Zamfirescu obtained the following contractive definition in the class of quasi-nonexpansive mappings.

Definition 1.12 *[12], [133] An operator $T : X \to X$ satisfies Zamfirescu's conditions if and only if, there exists the real numbers a, b, c satisfying $0 < a < 1, 0 < b, c < \frac{1}{2}$ such that for each pair x, y from X, at least one of the following conditions is true:*
(z1) $d(Tx, Ty) \leq a d(x, y)$
(z2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$
(z3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

It is known that every Zamfirescu operator, T, satisfies the following inequality:

$$d(Tx, Ty) \leq \delta d(x, y) + 2\delta d(x, Tx) \tag{1.16}$$

where $x, y \in D$, $\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$, with $0 < \delta < 1$.

Also, in 1972 [133], Zamfirescu obtained a fixed point theorem for the Zamfirescu operator, his theorem is a generalization of Banach's, Kannan's and Chatterjea's fixed point theorems.

Theorem 1.2 *[12], [133] Let (X, d) be a complete metric space and $T : X \to X$ a Zamfirescu operator. Then T has a unique fixed point p, and the Picard iteration converges to p.*

Later Berinde has improved and extended the above result with the next one.

Theorem 1.3 *[12], [13], [14] Let E be an arbitrary Banach space, D be a closed convex subset of E, and $T : D \to D$ a operator satisfying $(z1) - (z3)$. Let $\{x_n\}_{n=0}^{\infty}$ be the Mann iteration defined by* (1.5) *and $x_0 \in D$, with $\{\alpha_n\} \subset (0, 1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.*

In the sequel we present some convergence theorems for the Picard, Krasnoselskij, Mann and Ishikawa iterations in the class of Zamfirescu operators.

Theorem 1.4 *[12], [13] Let E an arbitrary Banach space, D be a closed convex subset of E, and $T : D \to D$ a operator satisfying $(z1) - (z3)$. Let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration defined by* (1.7) *for $x_0 \in D$, with $\{\alpha_n\}$ and $\{\beta_n\}$ sequences of positive numbers in $[0, 1]$ with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.*

The following theorem compares the convergence speed between Picard and Mann iterations.

Theorem 1.5 *[132] Let E an arbitrary real Banach space, D be a closed convex subset of E, and $T : D \to D$ a operator satisfying $(z1) - (z3)$. Let $\{x_n\}_{n=0}^{\infty}$ be defined by* (1.1) *for $x_0 \in D$, and $\{y_n\}_{n=0}^{\infty}$ be defined by* (1.5) *for $y_0 \in D$, with $\{\alpha_n\}$ in $\left[0, \frac{1}{1+\delta}\right)$ satisfying:*
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\alpha_n \to 0$ as $n \to \infty$.
Then the Picard iteration, (1.1)*, converges faster than the Mann iteration,* (1.5)*, to the fixed point of T.*

The following two theorems refer to the Krasnoselskij, Mann and Ishikawa iterations in arbitrary real Banach spaces.

Theorem 1.6 *[132] Let E an arbitrary real Banach space, D be a closed convex subset of E, and $T : D \to D$ a operator satisfying $(z1) - (z3)$. Let $\{x_n\}_{n=0}^{\infty}$ be defined by* (1.2) *for $x_0 \in D$, and $\{y_n\}_{n=0}^{\infty}$ be defined by* (1.7) *for $y_0 \in D$, with $\{\alpha_n\}$ and $\{\beta_n\}$ in $\left[0, \frac{1}{1+\delta}\right)$ satisfying:*
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\alpha_n, \beta_n \to 0$ as $n \to \infty$.
Let p be the fixed point of T in D. Then the Krasnoselskij iteration, (1.2)*, converges faster than the Ishikawa iteration,* (1.7)*, to the fixed point p of T.*

Theorem 1.7 *[4], [5] Let E an arbitrary real Banach space, D be a arbitrary closed convex subset of E, and $T : D \to D$ a operator satisfying $(z1) - (z3)$. Let $\{x_n\}_{n=0}^{\infty}$ be defined by* (1.5) *for $x_0 \in D$, and $\{y_n\}_{n=0}^{\infty}$ be defined by* (1.7) *for $y_0 \in D$, with $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying:*
(i) $0 \le \alpha_n, \beta_n < 1$ (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.
Then $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ converge strongly to the fixed point of T and moreover, the Mann iteration, (1.5)*, converges faster than the Ishikawa iteration,* (1.7)*, to the fixed point of T.*

The next Corollary shows that if we increase the number of unknowns the convergence order will decrease.

Corollary 1.1 *[4] Under the hypothesis of Theorem 1.7, the Picard iteration defined by* (1.1) *converges faster than the Ishikawa iteration defined by* (1.7)*, to the fixed point of Zamfirescu operator.*

The following two results refer to Mann and Picard iterations in uniformly convex Banach spaces.

Theorem 1.8 *[12], [14] Let E be a uniformly convex Banach space, D a closed convex subset of E, and $T : D \to D$ a operator satisfying $(z1) - (z3)$. Then the Mann iteration $\{x_n\}_{n=0}^{\infty}$ defined by* (1.5) *and $x_0 \in D$, with $\{\alpha_n\} \subset (0, 1]$ satisfying the conditions:*
(i) $\alpha_1 = 1$;
(ii) $0 \leq \alpha_n < 1$ *for* $n > 1$;
(iii) $\sum\limits_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$
converges to the fixed point of T.

Theorem 1.9 *[12], [14] Let E be a uniformly convex Banach space, K be a closed convex subset of E and $T : K \to K$ a Zamfirescu operator, that is a operator that satisfies $(z1) - (z3)$. Let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration associated with T, starting from $x_0 \in K$, given by* (1.1)*, and $\{y_n\}_{n=0}^{\infty}$ the Mann iteration given by* (1.5)*, where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence satisfying:*
(i) $\alpha_1 = 1$;
(ii) $0 \leq \alpha_n < 1$, *for* $n > 1$;
(iii) $\sum\limits_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$.
Then
1, T has a unique fixed point in E, that is, $F_T = \{p\}$.
2, The Picard iteration $\{x_n\}_{n=0}^{\infty}$ converges to p for any $x_0 \in K$.
3, The Mann iteration $\{y_n\}_{n=0}^{\infty}$ converges to p for any $y_0 \in K$ and $\{\alpha_n\}_{n=0}^{\infty}$ satisfying (i),(ii),(iii).
4, The Picard iteration is faster than the Mann iteration.

An analogous result, but for arbitrary Banach spaces, is given by the following theorem.

Theorem 1.10 *[12], [14] Let E be a arbitrary Banach space, K be a closed convex subset of E and $T : K \to K$ a Zamfirescu operator, that is a operator that satisfies $(z1) - (z3)$. Let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration associated with T, starting from $x_0 \in K$, given by* (1.1)*, and $\{y_n\}_{n=0}^{\infty}$ the Mann iteration given by* (1.5)*, where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence satisfying:*
(iv) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$.
Then $\{y_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T and, moreover, the Picard iteration, (1.1)*, converges faster than the Mann iteration,* (1.5).

In what follows we present some convergence theorems for Picard, Krasnoselskij, Mann and Ishikawa iterations in class of Lipschitzian operators.

The first two theorems refer to the Mann iteration.

Theorem 1.11 *[12], [15] Let X be a Banach space and K a nonempty closed convex bounded subset of X. Let $T : K \to K$ be a Lipschitzian strong pseudo-contractive mapping. If the fixed point set of T, $F(T)$ is nonempty, then the Mann iteration $\{x_n\} \subset K$ defined by* (1.5) *with $x_0 \in K$ and with the sequence $\{\alpha_n\} \subset (0,1]$, satisfying*

$$\lim_{n\to\infty} \alpha_n = 0 \quad \textit{and} \quad \sum_{n=0}^{\infty} \alpha_n = \infty$$

converges strongly to $p \in F(T)$ and $F(T)$ is a single set.

Theorem 1.12 *[12], [15] Let X be a Banach space and K a nonempty closed convex subset of X. If $T : K \to K$ is a Lipschitzian (with constant L) and strongly pseudo-contractive operator (with constant k) such that the fixed point set of T, $F(T)$ is nonempty, then the Mann iteration $\{x_n\} \subset K$ generated by* (1.5) *with $x_0 \in K$ and the sequence $\{\alpha_n\} \subset (0,1]$ with $\{\alpha_n\}$ satisfying*

$$\sum_{n=0}^{\infty} \alpha_n = \infty \quad \textit{and} \quad \alpha_n \leq \frac{k-\eta}{(L+1)(L+2-k)}$$

for some $\eta \in (0,k)$ converges strongly to the unique fixed point of T. Moreover, there exists $\{\beta_n\}_{n=0}^{\infty}$ a sequence from $(0,1)$ with $\beta_n \geq \frac{\eta}{1+k}\alpha_n$, such that for all $n \in \mathbb{N}$, the following estimate holds

$$\|x_{n+1} - p\| \leq \prod_{j=1}^{n} (1-\beta_j) \|x_1 - p\| .$$

The next two results show the convergence speed of the Krasnoselskij iteration in class of Lipschitzian and strongly pseudo-contractive operators.

Corollary 1.2 *[12], [15] Let X, K, T, L, k, p be as in Theorem* 1.12*. Then the Krasnoselskij iteration $\{x_n\} \subset K$ generated by* (1.2) *and $x_1 \in K$ with $\lambda \in (0,a)$ where*

$$a = \frac{k}{(L+1)(L+2-k)}$$

converges strongly to the (unique) fixed point p of T. Moreover, the following estimate holds

$$\|x_{n+1} - p\| \leq q^n \|x_1 - p\|$$

where

$$q = \frac{1 + (1-k)\lambda + (L+1)(L+2-k)\lambda^2}{1+\lambda}.$$

Theorem 1.13 *[12], [15] Let X be a Banach space and K a nonempty closed convex subset of X. If $T : K \to K$ is a Lipschitzian (with constant L) and strongly pseudo-contractive operator (with the constant k) such that the fixed point set of T, $F(T)$ is nonempty, then the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ generated by $x_0 \in K$ and* (1.2)*, with $\lambda \in (0,a)$, where*

$$a = \frac{k}{(L+1)(L+2-k)} \tag{1.17}$$

converges strongly to the (unique) fixed point p of T. Moreover, among all above Krasnoselskij iterations, there exists one which is the fastest one. It is obtain for

$$\lambda_0 = -1 + \sqrt{1+a}$$

where a is given by (1.17)*.*

We end this paragraph by two theorems in which is compared the convergence speed between Krasnoselskij and Mann iterations and between Mann and Ishikawa iterations in the class of Lipschitzian and pseudo-contractive mappings.

Theorem 1.14 *[3] If T is a Lipschitzian, pseudo-contractive map with the constants $L \geq 1$ and $k \in (0,1)$, then the following hold:*
(a) for any $x_0 \in K$ and $\epsilon \in (0, k/2M) \cap (0,1)$, the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ defined by (1.2) converges strongly to the fixed point p of T, where $M = 1 + (2-k+L)(L+1)$;
(b) for any $x_0 \in K$, the Mann iteration $\{x_n\}_{n=0}^{\infty}$ defined by (1.5) with $\{\alpha_n\} \subset [0,1)$ satisfying:
(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;
(ii) $\sum\limits_{n=0}^{\infty} = \infty$,
converges strongly to the fixed point p of T;
(c) for any $x_0 \in K$ and for any Mann iteration $\{x_n\}_{n=0}^{\infty}$ defined by (1.5) with $\{\alpha_n\} \subset [0,1)$ satisfying (i) and (ii) of (b), converging to the fixed point p of T, there is an $\epsilon \in (0,1)$ such that the Krasnoselskij iteration (1.2) converges faster to the fixed point p of T. Moreover p is unique.

Theorem 1.15 *[3] Let E, K and T be as in Theorem 1.14. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1)$ such that $\sum\limits_{n=0}^{\infty} \alpha_n = 0$ and $\lim\limits_{n\infty} \alpha_n = \lim\limits_{n\to\infty} \beta_n = 0$. Then:*
(a) for any $x_0 \in K$, the Ishikawa iteration $\{x_n\}_{n=0}^{\infty}$ defined by (1.7) converges strongly to the fixed point p of T;
(b) the Mann iteration (1.5) converges faster then the Ishikawa iteration (1.7) to the fixed point p of T.

1.2.7 Single nonlinear equations

Fixed point iterative methods are very important in obtaining the approximate solution of the following nonlinear equation

$$f(x) = 0, \tag{1.18}$$

where $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is a given operator, when it is not possible to use usual algebraic processes. Usually we rewrite (1.18) as the equivalent fixed point problem

$$g(x) = x, \tag{1.19}$$

where g is a certain function associated to f and by considering a fixed point iterative process, usually the Picard iteration, we obtain a sequence that converges to the solution of (1.18). The operator g is called the iteration function.

Example 1.4 *[16] Consider the polynomial equation*

$$x^3 + 4x^2 - 10 = 0,$$

that has a unique solution in the interval $[1,2]$. The above equation can be rewritten in the form (1.19) in many different ways. Some of them are given below:
(i) $x = \frac{1}{2}\sqrt{10 - x^3}$
(ii) $x = \sqrt[3]{10 - 4x^2}$

(iii) $x = \sqrt{\frac{10}{x+4}}$
(iv) $x = \frac{10}{x^2+4x}$.
We denote by:
$g_1(x) = \frac{1}{2}\sqrt{10 - x^3}$
$g_2(x) = \sqrt[3]{10 - 4x^2}$
$g_3(x) = \sqrt{\frac{10}{x+4}}$
$g_4(x) = \frac{10}{x^2+4x}$.

An important observation is that for an initial value $x_0 \in [1, 2]$ the Picard iteration associated to g does not always converge to the fixed point of g. For example: the Picard iteration associated to g_1 and g_3 does converge, while in the case of g_2 and g_4 does not. We end this paragraph by presenting the first five iterations for the four iterative processes defined by the iteration functions g_1, g_2, g_3 and g_4 and for the initial guesses $x_0 = 1.2$ and $x_0 = 1.5$.

$x_{n+1} = g_1(x_n)$	$x_{n+1} = g_2(x_n)$	$x_{n+1} = g_3(x_n)$	$x_{n+1} = g_4(x_n)$
$x_0 = 1.2$	$x_0 = 1.2$	$x_0 = 1.2$	$x_0 = 1.2$
$x_1 = 1.43806$	$x_1 = 1.61853$	$x_1 = 1.38675$	$x_1 = 1.60256$
$x_2 = 1.32534$	$x_2 = 0.39110 + 0.67741i$	$x_2 = 1.36250$	$x_2 = 1.11378$
$x_3 = 1.38492$	$x_3 = 2.24771 - 0.14002i$	$x_3 = 1.36558$	$x_3 = 1.75573$
$x_4 = 1.35496$	$x_4 = 1.24266 + 1.79781i$	$x_4 = 1.36518$	$x_4 = 0.98956$
$x_5 = 1.37044$	$x_5 = 2.79700 - 0.78188i$	$x_5 = 1.36524$	$x_5 = 2.02533$

$x_{n+1} = g_1(x_n)$	$x_{n+1} = g_2(x_n)$	$x_{n+1} = g_3(x_n)$	$x_{n+1} = g_4(x_n)$
$x_0 = 1.5$	$x_0 = 1.5$	$x_0 = 1.5$	$x_0 = 1.5$
$x_1 = 1.28696$	$x_1 = 1$	$x_1 = 1.34840$	$x_1 = 1.21212$
$x_2 = 1.40254$	$x_2 = 1.81712$	$x_2 = 1.36738$	$x_2 = 1.58285$
$x_3 = 1.34546$	$x_3 = 0.73739 + 1.22721i$	$x_3 = 1.36496$	$x_3 = 1.13163$
$x_4 = 1.37517$	$x_4 = 2.49790 - 0.40609i$	$x_4 = 1.36527$	$x_4 = 1.72203$
$x_5 = 1.36010$	$x_5 = 1.62966 + 1.95189i$	$x_5 = 1.36523$	$x_5 = 1.01487$

1.3 Scalar sequences

Let us assume that the sequence $\{S_n\}$ converges to some limit S or if it is divergent it can be summed by an appropriate summation process to give S, in the second case it is called antilimit. Then the sequence $\{S_n\}$ satisfies the relation

$$S_n = S + R_n,$$

where R_n is the remainder and $n \in \mathbb{N}_0$. If $\{S_n\}$ is the partial sum of a series

$$S_n = \sum_{k=0}^{n} a_k,$$

the remainder R_n satisfies

$$R_n = -\sum_{k=n+1}^{\infty} a_k.$$

1.4 Types of sequences

It is not possible to set up a complete classification that covers all types of convergence, however, in practical applications only a few types of convergence occur.

Let $\{S_n\}$ be a scalar sequence converging to a limit S, that satisfies

$$\lim_{n\to\infty} \frac{S_{n+1} - S}{S_n - S} = \lim_{n\to\infty} \frac{R_{n+1}}{R_n} = \lambda.$$

If $0 < |\lambda| < 1$ holds we can say that the sequence $\{S_n\}$ converges linearly, if $\lambda = 1$ holds, we say that $\{S_n\}$ converges logarithmically, if $\lambda = 0$ holds, we say that $\{S_n\}$ converges hyperlinearly and if $|\lambda| > 1$ holds, the sequence $\{S_n\}$ is divergent.

Because in this thesis we will use only linear convergence and some examples of logarithmically convergence we shall give an example only for the linear convergence. The standard example for linear convergence is the sequence of partial sums of the geometric series

$$S_n(z) = \sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}, \quad 0 < |z| < 1, \quad n \in \mathbb{N}_0.$$

We end this paragraph with a result that will help us to compare the convergence speed of two scalar sequences.

Definition 1.13 *[12], [19] Let us assume that $\{S_n\}$ and $\{S'_n\}$ are two sequences converging to the same limit S. We shall say that the sequence $\{S'_n\}$ converges more rapidly than $\{S_n\}$ if*

$$\lim_{n\to\infty} \frac{S'_n - S}{S_n - S} = 0. \tag{1.20}$$

In the convergence acceleration process, this definition has the inconvenient that requires the knowledge of the limit S which is usually not known. Because of that it would be better to replace the equation (1.20) by the alternative condition

$$\lim_{n\to\infty} \frac{S'_{n+1} - S'_n}{S_{n+1} - S_n} = \lim_{n\to\infty} \frac{\Delta S'_n}{\Delta S_n} = 0. \tag{1.21}$$

Unfortunately, it is not possible to prove the equivalence between the equations (1.20) and (1.21) without making some explicit assumptions about how fast the sequences $\{S_n\}$ and $\{S'_n\}$ approach their common limit S.

In the case when $\{S_n\}$ converges linearly, the transformed sequence $\{S'_n\}$ can only converge more rapidly than $\{S_n\}$ if it converges at least linearly or even faster. Consequently, the equivalence between the conditions (1.20) and (1.21) follows at once from the relationship

$$\frac{S'_{n+1} - S'_n}{S_{n+1} - S_n} = \frac{S'_n - S}{S_n - S} \, \frac{\frac{S'_{n+1} - S}{S'_n - S} - 1}{\frac{S'_{n+1} - S}{S'_n - S} - 1}.$$

1.5 Sequence transformations

A sequence transformation τ will allows us to transform a given sequence $\{S_n\}$ into a new one $\{S'_n\}$

$$\tau : \{S_n\} \to \{S'_n\}, \quad n \in \mathbb{N}_0.$$

Because a computational algorithm involves only a finite number of operations, by a sequence transformation τ we can associate to each new sequence element S'_n only finite subsets of a sequence $\{S_n\}$. It will be considered that the finite subset, which is to be transformed, will be made up of consecutive elements, these means that only subsets of the form $\{S_n, S_{n+1}, \ldots, S_{n+l}\}$ with $n, l \in \mathbb{N}_0$ will be taken into account. Because the subsets, that must be transformed, contain $l+1$ elements, l will be called the order of the transformation τ. So, if all elements of the sequence $\{S_n\}$ are real and if it is no necessary to use a sequence of interpolation points $\{x_n\}$ or a sequence of remainders estimates, called it, $\{w_n\}$, a sequence transformation τ of order l is a map of the following form

$$\tau : \mathbb{R}^{l+1} \to \mathbb{R}.$$

A sequence transformation can be represented by an infinite set of doubly indexed quantities $T_k^{(n)}$ with $k, n \in \mathbb{N}_0$. The superscript n indicates the minimal index that appears in the finite subset of the sequence elements, which are used for the computation of the transformation $T_k^{(n)}$ and the subscript k is a measure for the complexity of such a $T_k^{(n)}$.

For each sequence transformation, $T_k^{(n)}$, we will consider that $T_0^{(n)}$ corresponds to a untransformed sequence element

$$T_0^{(n)} = S_n, \quad n \in \mathbb{N}_0.$$

If we increase the values of k then the order l of the transformation, $T_k^{(n)}$, will increase also. That is, that for every $k, n \in \mathbb{N}_0$ the sequence transformation τ will give a new sequence transformation of the form

$$T_k^{(n)} = \tau(S_n, S_{n+1}, \ldots, S_{n+l}).$$

Here, the order l is a function of k. In this thesis we shall find relationships such as $l = k$, $l = k+1$, $l = 2k$ and $l = 3k$.

The transform $T_k^{(n)}$ with $k, n \in \mathbb{N}_0$ can be displayed in a 2-dimensional array called the table of the transformation τ. In this thesis, the transforms $T_k^{(n)}$ will be ordered in a rectangular scheme, where the superscript n indicates the row and the subscript k indicates the column of the array. Thus, the table of the transformation τ will be displayed in the following scheme

$$
\begin{array}{cccccc}
T_0^{(0)} & T_1^{(0)} & T_2^{(0)} & \ldots & T_n^{(0)} & \ldots \\
T_0^{(1)} & T_1^{(1)} & T_2^{(1)} & \ldots & T_n^{(1)} & \ldots \\
T_0^{(2)} & T_1^{(2)} & T_2^{(2)} & \ldots & T_n^{(2)} & \ldots \\
T_0^{(3)} & T_1^{(3)} & T_2^{(3)} & \ldots & T_n^{(3)} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
T_0^{(n)} & T_1^{(n)} & T_2^{(n)} & \ldots & T_n^{(n)} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}
$$

In case we want to obtain convergence up to a certain accuracy for a sequence process or a summation process we should compute only those elements of the table of the sequence transformation τ which are necessary in obtaining that accuracy, this implies that we must known in advance which transformations $T_k^{(n)}$ we will use to achieve our goal.

1.6 Iterated sequence transformations

Let $\{S_n\}$ be a sequence to be transformed by a sequence transformation $T_k^{(n)}$ with $k, n \in \mathbb{N}_0$ and for some $\kappa \in \mathbb{N}$, which is usually a relatively small number, a transformation $T_k^{(n)}$ can be expressed explicitly in terms of the sequence elements $S_n, S_{n+1}, \ldots, S_{n+\alpha}$ as follows

$$
T_\kappa^{(n)} = F(S_n, S_{n+1}, \ldots, S_{n+\alpha}). \tag{1.22}
$$

Iterating the expression for $T_\kappa^{(n)}$ we can obtain a new sequence transformation $\Theta_k^{(n)}$, consequently the expression (1.22) is rewritten in the following form

$$
\Theta_1^{(n)} = F_0(\Theta_0^{(n)}, \Theta_0^{(n+1)}, \ldots, \Theta_0^{(n+\alpha)}), \quad n \in \mathbb{N}_0 \tag{1.23}
$$

where

$$
\Theta_0^{(n)} = S_n, \quad n \in \mathbb{N}_0.
$$

The relationship (1.23) can be used to construct a recursive scheme that will allow us to compute the transformations $\Theta_k^{(n)}$, with $k \geq 2$. The simplest way to obtain such a recursive scheme is to assume that the equation (1.23) corresponds to the special case $k = 0$ of the following, more general, recursive scheme

$$
\Theta_{k+1}^{(n)} = F_k(\Theta_k^{(n)}, \Theta_k^{(n+1)}, \ldots, \Theta_k^{(n+\alpha)}), \quad k, n \in \mathbb{N}_0.
$$

In this thesis we will present several very powerful sequence transformations which are derived by iterating explicit expressions of other sequence transformations. For example Aitken's iterated Δ^2 process which is obtained by iterating the explicit expression for Aitken's Δ^2 process. Strangely, sometimes the properties of the new sequence transformation differ significantly from the properties of the transformation from which it was derived.

1.7 Sequence acceleration processes

We begin this paragraph with the next important observation.

Remark 1.6 *[122] (i) A sequence transformation able to accelerate the convergence of all types of scalar sequences cannot exist. Consequently, for each particular class of sequences it is necessary to obtain different convergence acceleration methods.*
(ii) The study of the properties of these procedures lead to the appearance of some classes of sequences that can be accelerated with a certain algorithm.
(iii) The scalar sequence transformations have also been studied from a theoretical point of view.

In what follows we will present the most important and well known sequence acceleration processes that can be found in the literature.

1.7.1 Richardson's process

The first occurrence of a particular case of the so called Richardson extrapolation process is due C. Huygens. Then in 1903 [85], R. Milne applied Huygens idea for computing the value of π. Later in 1936 [72], Kommerell used the same scheme for approximating π, so he can be considered the real discoverer of Romberg's method.

Consider the procedures used for improving the accuracy of the trapezoidal rule for computing the approximations to a definite integral. When we have a sufficiently smooth function, the error of the method is given by the Euler-Maclaurin expansion. In 1742 [78], Maclaurin showed that its accuracy could be enhanced if linear combinations are formed from the results with different stepsizes. His procedure can be seen like a first version of Romberg's method.

In 1900 [109], W. F. Sheppard used an elimination method in the Euler-Maclaurin quadrature formula with $h_n = r_n h$ and $1 = r_0 < r_1 < r_2 < \ldots$ to produce a better approximation to the given integral.

In 1910 [97], Richardson obtained a new procedure, called *the difference approach to a limit* or *h^2-extrapolation*, this algorithm was realized by combining the results obtained with the stepsizes h and $2h$ and then by eliminating the first term in the discretization process using central differences. The transformed sequence $\{T_n\}$ is given by

$$T_n = \frac{h_{n+1}^2 S(h_n) - h_n^2 S(h_{n+1})}{h_{n+1}^2 - h_n}.$$

Richardson referred to a paper from 1926 [31] written by N. N. Bogolyubov and N. Mitrofanovich, where the deferred approach to the limit can already be found.

In 1927 [98] he used the same process for solving a 6th order differential eigenvalue problem. This technique was called *(h^2, h^4)-extrapolation.* Richardson's process consist of computing the value at 0, denoted by $T_k^{(n)}$, of the interpolation polynomial of degree at most k, which passes through the points $(x_n, S_n), \ldots, (x_{n+k}, S_{n+k})$.

If we use the Neville-Aitken scheme for these interpolation polynomials we obtain

$$T_{k+1}^{(n)} = \frac{x_{n+k+1} T_k^{(n)} - x_n T_k^{(n+1)}}{x_{n+k+1} - x_n},$$

where $T_0^{(n)} = S_n$.
For a detailed history of the Richardson's extrapolation process, its developments and its applications, see [52], [71], [121].

1.7.2 Romberg's process

In 1955 [102], W. Romberg had the idea to use repeatedly an elimination approach for improving the precision of the trapezoidal rule. The method is given by the next formula

$$T_{k+1}^{(n)} = \frac{4^{k+1}T_k^{(n+1)} - T_k^{(n)}}{4^{k+1} - 1},$$

where $T_0^{(n)}$ is the result obtained by the trapezoidal rule with the stepsize $h_0/2^n$.

We have to mention that Romberg refers to a paper from 1951 [46] written by L. Collatz.

Romberg's extrapolation process had became well known after the rigorous analysis given by F. L. Bauer in 1961 [9] and by the synthesis of E. L. Stiefel in 1961 [115]. Romberg's extrapolation process is mainly heuristic. Later in 1963 [74], P. J. Laurent proved that the process derives, in fact, from the Richardson process by choosing $x_n = h_n^2$ and $h_n = h_0/2^n$. Laurent other contributions were the conditions in choosing the sequence $\{x_n\}$ in order that the sequence $T_k^{(n)}$ tends to S either when k or n tends to infinity. Also, weaker conditions were given by M. Crouzeix and L. Mignot in [49].

Romberg's work on the extrapolation of the trapezoidal rule was continued by T. Håvie for less regular integrands [62].

1.7.3 Shanks transformation

In the paper [108] about nonlinear sequence transformations, D. Shanks considered the next model sequence

$$S_n = S + \sum_{j=0}^{k-1} c_j \Delta S_{n+j}, \quad n \in \mathbb{N}_0. \tag{1.24}$$

In the case when the sequence elements S_n are partial sums of an infinite series

$$S_n = \sum_{\nu=0}^{n} a_\nu,$$

the model sequence (1.24) can be rewritten in the following form

$$S_n = S + \sum_{j=0}^{k-1} c_j a_{n+j+1}, \quad n \in \mathbb{N}_0.$$

This means that the limit S of the infinite series is approximated by the partial sum S_n plus a weighted sum of the next k terms $a_{n+1}, a_{n+2}, \dots, a_{n+k}$. The model sequence S_n is formed by $k+1$ unknowns, they are: the limit or antilimit S and the k linear coefficients $c_0, c_1, \dots, c_{k-1}$, which all occur linearly.

In view of the Cramer's rule, Shanks sequence transformation $e_k(S_n)$, which is by construction exact for the model sequence (1.24), can be written as a ratio of two determinants, as follows

$$T_n = e_k(S_n) = \frac{\begin{vmatrix} S_n & S_{n+1} & \dots & S_{n+k} \\ \Delta S_n & \Delta S_{n+1} & \dots & \Delta S_{n+k} \\ \vdots & \vdots & & \vdots \\ \Delta S_{n+k-1} & \Delta S_{n+k} & \dots & \Delta S_{n+2k-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \dots & 1 \\ \Delta S_n & \Delta S_{n+1} & \dots & \Delta S_{n+k} \\ \vdots & & \vdots & \\ \Delta S_{n+k-1} & \Delta S_{n+k} \dots & \Delta S_{n+2k-1} & \end{vmatrix}}. \tag{1.25}$$

The computation of the transform $e_k(S_n)$ requires the sequence elements $S_n, \dots,$ S_{n+2k}, consequently, $e_k(S_n)$ is a transformation of order $l = 2k$.

If we take $k = 1$ in the relation (1.25), Shanks transformation reduces to Aitken's Δ^2 process, which will be defined in the next paragraph.

In 1941 [106], R. J. Schmidt used the same ratio of determinants for the iterative solution of linear systems.

The determinants involved in the definition of $e_k(S_n)$ have a very special structure. They were studied by H. Hankel in his thesis in 1861 [61], so they are called Hankel determinants. These determinants satisfy a five-term recurrence relationship. O'Beirne and Shanks used these relation to implement an algorithm that allows to calculate separately the numerators and the denominators of the $e_k(S_n)$'s. Working with determinants is difficult, because are need to many arithmetical operations and due the rounding errors that occur to the computer we are often lead to a completely wrong result. Consequently, it was necessary to find a recursive scheme that is able to compute the $e_k(S_n)$'s without computing the determinants that are involved in their definition. Fortunately, this algorithm was obtained in 1956 [126] by Wynn. The technique is called the ϵ-algorithm, which will be presented in this thesis.

Shanks showed that the transformation $e_k(S_n)$ has close connection to Padé approximants, continued fractions and formal orthogonal polynomials, see [28].

More informations about the work of Shanks can be found in [123].

1.7.4 Aitken's Δ^2 process

The most popular and well known sequence acceleration method for a slowly converging sequence is Aitken's Δ^2 process. The method was given by A.C. Aitken in 1926 [1], who used it to accelerate the convergence of Bernoulli's method for computing the dominant zero of a polynomial. The same method was obtained by Hans von Naegelsbach in 1876 in his study of Furstenan's method for solving nonlinear equations [86] and by Maxwell in 1873 in his Treatise on Electricity and Magnetism [82]. Neither Naegelsbach nor Maxwell used it to accelerate the convergence of a slowly converging sequence. The method was also found at one of the greatest Japanese mathematicians, Takakazu Seki, who used it to compute the value of π, the length of a chord and the volume of a sphere in his book that was written around 1680 but only published in 1712. Parts of it can be found in [65].

Let assume the following model sequence

$$S_n = S + c\lambda^n \quad c \neq 0,\ |\lambda| < 1,\ n \in \mathbb{N}_0. \tag{1.26}$$

Each sequence element S_n is obtained by three unknowns c, λ and the limit S. So, a sequence transformation will be needed at least three elements of the model sequence (1.26) for obtaining the limit S. In order to establish such a transformation we need to define the first and the second differences of S_n:

$$\Delta S_n = c\lambda^n(\lambda - 1),$$

$$\Delta^2 S_n = c\lambda^n(\lambda - 1)^2.$$

Making a short computation we will be lead to the conclusion that the following sequence transformation is exact, that is,

Definition 1.14 *[122] Let T_n be an acceleration method for the sequence S_n, for which $\lim_{n\to\infty} S_n = S$. We say that the transformation T_n is exact for the sequence S_n if $\forall n,\ T_n = S$.*

for the model sequence (1.26)

$$A_1^{(n)} = S_n - \frac{(S_{n+1} - S_n)^2}{S_{n+2} - 2S_{n+1} + S_n} = S_n - \frac{[\Delta S_n]^2}{\Delta^2 S_n}, \quad n \in \mathbb{N}_0. \tag{1.27}$$

The above sequence transformation is called Aitken's Δ^2 process. The structure of it explains why it bears this name.

Aitken's Δ^2 process is a special case of Shanks transformation (1.25) or Wynn's ϵ-algorithm (1.31):

$$A_1^{(n)} = e_1(S_n) = \epsilon_2^{(n)}.$$

In the literature are many other representations for Aitken's Δ^2 process establish by suitable manipulations of the equation (1.27). In what follows we will present them:

$$A_1^{(n)} = S_{n+1} - \frac{[\Delta S_n][\Delta S_{n+1}]}{\Delta^2 S_n}, \quad n = 0, 1, \dots,$$

$$A_1^{(n)} = S_{n+2} - \frac{[\Delta S_{n+1}]^2}{\Delta^2 S_n}, \quad n = 0, 1, \dots,$$

$$A_1^{(n)} = \frac{S_{n+2}S_n - [\Delta S_{n+1}]^2}{\Delta^2 S_n}, \quad n = 0, 1, \dots,$$

$$A_1^{(n)} = \frac{[\Delta S_{n+1}]S_{n+1} - [\Delta S_n]S_{n+2}}{\Delta^2 S_n}, \quad n = 0, 1, \dots,$$

$$A_1^{(n)} = \frac{[\Delta S_{n+1}]S_n - [\Delta S_n]S_{n+1}}{\Delta^2 S_n}, \quad n = 0, 1, \dots,$$

$$A_1^{(n)} = S_{n+1} + \frac{1}{\Delta(\frac{1}{\Delta S_n})}, \quad n = 0, 1, \dots,$$

$$A_1^{(n)} = \frac{\Delta[\frac{S_{n+1}}{\Delta S_n}]}{\Delta(\frac{1}{\Delta S_n})}, \quad n = 0, 1, \ldots,$$

where Δ denotes the forward difference operator, $\Delta S_n = S_{n+1} - S_n$ and $\Delta^2 S_n = \Delta S_{n+1} - \Delta S_n = S_{n+2} - 2S_{n+1} + S_n$.

Aitken's Δ^2 process was studied in papers by Shanks [108], Clark, Grey and Adams [45], Lubkin [77], Tucker [118], [119], Bell and Phillips [10], even a multidimenisional generalization to the vector case was obtained by MacLeod [79]. Discussions about the close connection between the Aitken's Δ^2 process and the Fibonacci numbers were done in papers by McCabe and Philips [80] and Arai et al. [2]. Modifications of it were proposed by Drummond [51] and by Bjørstad, Dahlquist and Grosse [17]. The properties of Aitken's Δ^2 process were studied in books by Baker and Graves-Morries [7], Brezinski [20], [22], Walz [120] and Wimp [124]. Those properties are:
(i) Aitken's Δ^2 process accelerates linear convergence.
(ii) Aitken's Δ^2 process is regular but not accelerative for logarithmically convergent sequences.

The above properties show that Aitken's Δ^2 process has similar properties as Wynn's ϵ-algorithm, but it is not so powerful as Wynn's ϵ-algorithm, because the transform $A_1^{(n)}$ is produced by only three sequences elements S_n, S_{n+1}, S_{n+2}, which implies that $A_1^{(n)}$ is a transformation of order $l = 2$.

Aitken's Δ^2 process is not always successful for the sequences converging linearly, but its power can be increased by iterating it. This means that Aitken's Δ^2 process will be applied to the transformed sequence $\{A_1^{(n)}\}$ yielding a new sequence transformation $\{A_2^{(n)}\}$ and so on.

For obtaining some heuristic motivation for this iteration, we will apply Aitken's Δ^2 process to the following model sequence, which is a generalization of the sequence (1.26)

$$S_n = S + ax^n + by^n, \quad 0 < |y| < |x| < 1, \;\; a, b \neq 0. \tag{1.28}$$

Performing some calculus we can show that Aitken's Δ^2 process eliminates the dominant term ax^n from the sequence model (1.28)

$$A_1^{(n)} = S + \frac{b[(x-y)/(x-1)]^2 y^n}{1 + (b/a)[(y-1)/(x-1)]^2 (y/x)^n}. \tag{1.29}$$

In the hypothesis $0 < |y| < |x| < 1$, the transformed sequence (1.29) converges faster than the sequence (1.28). As well, because $(y/x)^n$ disappears when $n \to \infty$, at least for large values of n the elements of the transformed sequence (1.29) have the same structure as the elements of the sequence (1.26).

In the case of Aitken's Δ^2 process we can use the numerous representations for $A_1^{(n)}$ because they are mathematically equivalent, but it is important to say that these representations for $A_1^{(n)}$ differ considerably in their numerical stability. In their book [56] Flannery, Teukolsky and Vetterling pointed out that Aitken's Δ^2 process should be used in the form (1.27) because the other equivalent representations are numerically less reliable. If we determine the sequence elements S_n with the initial guess $A_0^{(n)}$ of the recursion we can obtain the following nonlinear recursive scheme:

$$A_{k+1}^{(n)} = A_k^{(n)} - \frac{[\Delta A_{k+1}^{(n)}]^2}{\Delta^2 A_k^{(n)}}, \quad A_0^{(n)} = S_n \quad n, k \in \mathbb{N}_0, \tag{1.30}$$

where $\Delta A_k^{(n)} = A_k^{(n+1)} - A_k^{(n)}, \Delta^2 A_k^{(n)} = \Delta A_k^{(n+1)} - \Delta A_k^{(n)} = A_k^{(n+2)} - 2A_k^{(n+1)} + A_k^{(n)}$. The forward difference operator Δ acts only upon the superscript n and not upon the subscript k. In the case of Aitken's iterated Δ^2 process every $A_k^{(n)}$ requires the sequence elements $S_n, S_{n+1}, \ldots, S_{n+2k}$, consequently, $A_k^{(n)}$ is a transformation of order $l = 2k$.
Regarding the fact that the $A_1^{(n)}$ has many different extensions in terms of S_n, S_{n+1}, S_{n+2}, these expressions can be iterated giving the following derivations for the Aitken's iterated Δ^2 process:

$$A_{k+1}^{(n)} = A_k^{(n+2)} - \frac{[\Delta A_k^{(n+1)}]^2}{\Delta^2 A_k^{(n)}}, \quad A_0^{(n)} = S_n \quad n, k \in \mathbb{N}_0,$$

$$A_{k+1}^{(n)} = A_k^{(n+1)} - \frac{[\Delta A_k^{(n)}][\Delta A_k^{(n+1)}]}{\Delta^2 A_k^{(n)}}, \; A_0^{(n)} = S_n \quad n, k \in \mathbb{N}_0.$$

Discussions about Aitken's iterated Δ^2 process were done by Smith and Ford [114], although about its theoretical properties very little seems to be known. As far as it is known only in an article by Hillion [64] the theoretical properties of Aitken's iterated Δ^2 process were studied. Hillion managed to create a model sequence for which the Aitken's iterated Δ^2 process was exact and he was able to give a determinantal representation for the transforms $A_k^{(n)}$. Because, in both cases, Hillion used explicitly the lower order transforms $A_0^{(n)}, \ldots, A_{k-1}^{(n)}, \ldots, A_0^{(n+k)}, \ldots, A_{k-1}^{(n+k)}$ it can not be said that these results provide a significant improvement to the properties of Aitken's iterated Δ^2 process.
In what follows we will arrange the elements $A_k^{(n)}$, of the Aitken table, in a rectangular scheme, where the superscript n indicates the row and the subscript k indicates the column of the 2-dimensional array:

$$\begin{array}{cccccc}
A_0^{(0)} & A_1^{(0)} & A_2^{(0)} & \ldots & A_n^{(0)} & \ldots \\
A_0^{(1)} & A_1^{(1)} & A_2^{(1)} & \ldots & A_n^{(1)} & \ldots \\
A_0^{(2)} & A_1^{(2)} & A_2^{(2)} & \ldots & A_n^{(2)} & \ldots \\
A_0^{(3)} & A_1^{(3)} & A_2^{(3)} & \ldots & A_n^{(3)} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
A_0^{(n)} & A_1^{(n)} & A_2^{(n)} & \ldots & A_n^{(n)} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

where the entries in the first column of the array are the starting values $A_0^{(n)} = S_n$ of the first part of the scheme (1.30) and the others elements of the Aitken table can be computed with the help of the second part of the relation (1.30).
Every four terms, which are connected with this nonlinear recursion can be arranged like the move of a knight on the chessboard, as follows

$$\begin{array}{ll}
A_k^{(n)} & A_{k+1}^{(n)} \\
A_k^{(n+1)} & \\
A_k^{(n+2)}. &
\end{array}$$

More details about the Aitken's Δ^2 process and its iterated forms can be found in [19], [30], [122].

1.7.5 Wynn's ϵ-algorithm and ρ-algorithm

As we mentioned in the Paragraph 1.7.3, the Shanks transformation $e_k(S_n)$ is not useful in case of practical examples, because its definition as a ratio of determinants. Luckily, just one year after the publication of Shanks transformation, Wynn [126] introduced a recursive scheme for the computation of $e_k(S_n)$'s, these new algorithm is called Wynn's ϵ-algorithm and it is given by the following scheme

$$\begin{aligned} \epsilon_{k+1}^{(n)} &= \epsilon_{k-1}^{(n+1)} + \frac{1}{\epsilon_k^{(n+1)} - \epsilon_k^{(n)}}, \quad k, n \in \mathbb{N}_0 \\ \epsilon_{-1}^{(n)} &= 0, \qquad \epsilon_0^{(n)} = S_n. \end{aligned} \tag{1.31}$$

In [126], Wynn showed that the elements of ϵ table with even subscript give the Shanks transformation

$$\epsilon_{2k}^{(n)} = e_k(S_n), \quad k, n \in \mathbb{N}_0 \tag{1.32}$$

when the elements of ϵ table with odd subscript are only auxiliary quantities, satisfying the relation

$$\epsilon_{2k+1}^{(n)} = \frac{1}{e_k(\Delta S_n)}, \quad k, n \in \mathbb{N}_0. \tag{1.33}$$

The ϵ's with an odd lower index can be eliminated from the algorithm, leading to the so called *cross rule* do to Wynn [130]. It follows from the recurrence formula that the computation of ϵ_{2k} requires the sequence elements $S_n, S_{n+1}, \ldots, S_{n+2k}$, consequently, it is a transformation of order $l = 2k$ like Aitken's iterated Δ^2 process.
When applying the ϵ-algorithm divisions by zero can occur, if these divisions by zero are for some adjacent values we can ignore them and continue the algorithm, in other case the algorithm must be stopped. When computations are done divisions by a number close to zero can occur and the algorithm becomes numerically unstable, because of the cancellation errors. In 1963 [129] Wynn proposed particular rules for the ϵ-algorithm who are more precise. They were extended by F. Cordellier in 1979 [47].
The convergence and acceleration properties, in case of ϵ-algorithm, were completely described only for the totally monotonic and totally oscillating sequences in [23], [26], [128].
In [128], Wynn studied the convergence properties of the ϵ-algorithm by applying it to several model sequences. One type of sequences which he has analysed, are those which have strictly alternating remainders r_n, given by the following relation

$$S_n \sim S + (-1)^n \sum_{j=0}^{\infty} c_j/(n+\beta)^{j+1}, \quad \beta \in \mathbb{R}_+, \ n \to \infty. \tag{1.34}$$

In the assumption that $c \neq 0$ for fixed k, Wynn was able to obtain an estimate which shows that the ϵ-algorithm accelerates convergence

$$\epsilon_{2k}^{(n)} \sim S + \frac{(-1)^n (k!)^2}{2^{2k}(n+\beta)^{2k+1}} c_0, \quad n \to \infty. \tag{1.35}$$

Another type of sequences which he has analysed, are those which generalize the model sequence

$$S_n = S + \sum_{j=0}^{k-1} c_j \lambda_j^n, \quad n \in \mathbb{N}_0, \tag{1.36}$$

being given by the next expression

$$S_n \sim S + \sum_{j=0}^{\infty} c_j \lambda_j^n, \quad 1 > \lambda_0 > \lambda_1 > \ldots > 0, \ n \to \infty. \tag{1.37}$$

For a fixed k he obtained the following estimate which shows that the ϵ-algorithm accelerates convergence

$$\epsilon_{2k}^{(n)} \sim S + c_k \frac{(\lambda_k - \lambda_0)(\lambda_k - \lambda_1)\ldots(\lambda_k - \lambda_{k-1})^2}{(1-\lambda_0)(1-\lambda_1)\ldots(1-\lambda_{k-1})^2} \lambda_k^n, \quad n \to \infty. \tag{1.38}$$

In the sequel we will arrange the elements $\epsilon_{2k}^{(n)}$ of the ϵ table in a rectangular scheme where the superscript n indicates the row and the subscript k indicates the column of the 2-dimensional array

$$\begin{array}{llllll}
\epsilon_0^{(0)} & \epsilon_1^{(0)} & \epsilon_2^{(0)} & \ldots & \epsilon_n^{(0)} & \ldots \\
\epsilon_0^{(1)} & \epsilon_1^{(1)} & \epsilon_2^{(1)} & \ldots & \epsilon_n^{(1)} & \ldots \\
\epsilon_0^{(2)} & \epsilon_1^{(2)} & \epsilon_2^{(2)} & \ldots & \epsilon_n^{(2)} & \ldots \\
\epsilon_0^{(3)} & \epsilon_1^{(3)} & \epsilon_2^{(3)} & \ldots & \epsilon_n^{(3)} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\epsilon_0^{(n)} & \epsilon_1^{(n)} & \epsilon_2^{(n)} & \ldots & \epsilon_n^{(n)} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array} \tag{1.39}$$

where the entries in the first column of the array are the starting values $\epsilon_0^{(n)} = S_n$ of the scheme (1.31), the second part and the others elements of the ϵ table can be computed with the help of the relation (1.31), first part.
Every four terms, which are connected with this nonlinear recursion can be arranged like the vertices of a rhombus, as follows

$$\begin{array}{ccc}
 & \epsilon_k^{(n)} & \epsilon_{k+1}^{(n)} \\
\epsilon_{k-1}^{(n+1)} & \epsilon_k^{(n+1)}. &
\end{array} \tag{1.40}$$

In the literature there is an enormous number of papers concerning Wynn's ϵ-algorithm, over 50 articles were published by Wynn alone, with a vector generalization in [125], at least 30 articles by Brezinski and many others.
Another important nonlinear recursive scheme obtained by Wynn is the ρ-algorithm, which is almost identical with the ϵ-algorithm, and it is given by the next relation

$$\rho_{k+1}^{(n)} = \rho_{k-1}^{(n+1)} + \frac{x_{n+k+1} - x_n}{\rho_k^{(n+1)} - \rho_k^{(n)}}, \quad k, n \in \mathbb{N}_0 \tag{1.41}$$

$$\rho_{-1}^{(n)} = 0 \quad \rho_0^{(n)} = S_n. \tag{1.42}$$

The difference between the ϵ-algorithm and the ρ-algorithm is that the ρ-algorithm also involves a sequence of interpolation points which has to fulfill the following conditions

$$0 < x_0 < x_1 < x_2 < \ldots < x_m < x_{m+1} < \ldots, \tag{1.43}$$

$$\lim_{n \to \infty} x_n = \infty. \tag{1.44}$$

When applying the ρ-algorithm only the elements, $\rho_{2k}^{(n)}$, with even subscripts give approximations to the limit, while the elements, $\rho_{2k+1}^{(n)}$, with odd subscripts are only auxiliary quantities which diverge if the whole process converges.
Although the ϵ-algorithm and the ρ-algorithm are almost the same, they differ significantly in their ability of accelerating convergence. As shown in the first part of the paragraph the ϵ-algorithm is exact for the model sequence (1.36) and it is a very good accelerator for the sequences converging linearly. It can even sum divergent series, but it is not a good accelerator for the sequences converging logarithmic.
Wynn's ϵ-algorithm and ρ-algorithm are, in some sense, opposite. For instance, the ρ-algorithm is not a good accelerator for a sequence that converges linearly and it is not able to sum divergent series, but it is very powerful for some logarithmically convergent sequences.
Discussions about the properties of ρ-algorithm were done by Brezinski [20], [22] and by Wimp in [124].
Taking the most evident interpolations points, for the sequence $\{x_n\}$, which are $x_n = n + \beta$ with $\beta > 0$, the ρ-algorithm reaches his standard form

$$\rho_{-1}^{(n)} = 0, \quad \rho_0^{(n)} = S_n \tag{1.45}$$

$$\rho_{k+1}^{(n)} = \rho_{k-1}^{(n)} + \frac{k+1}{\rho_k^{(n+1)} - \rho_k^{(n)}}, \quad k, n \in \mathbb{N}_0. \tag{1.46}$$

The elements of the ρ table can be arrange in the same way as the elements of the ϵ table in (1.39). Like in case of ϵ-algorithm every four elements which are connected by the ρ-algorithm can be arranged like the vertices of a rhombus

$$\begin{array}{ccc} & \rho_k^{(n)} & \rho_{k+1}^{(n)} \\ \rho_{k-1}^{(n+1)} & \rho_k^{(n+1)} & \end{array}. \tag{1.47}$$

Because Wynn's ϵ-algorithm (1.31) and Wynn's ρ-algorithm are by construction almost identical, one can construct a new sequence transformation proceeding as in case of Aitken's iterated Δ^2 process. These means that first the transform $\rho_2^{(n)}$ will be expressed by some sequence elements S_n and by some interpolation points x_n and then the new expression for $\rho_2^{(n)}$. Consequently, from (1.41) we obtain the next formulation for the ρ-algorithm analogue of Aitken's Δ^2-process

$$\rho_2^{(n)} = S_{n+1} + \frac{(x_{n+2} - x_n)\Delta S_{n+1} \Delta S_n}{\Delta x_{n+1} \Delta S_n - \Delta x_n \Delta S_{n+1}} \quad n \in \mathbb{N}_0. \tag{1.48}$$

The above relation can be iterated in many different ways. The difficult task is to choose the indices of the interpolation points x_n, because they are not define in a unique way. If we take into account the fact that when increasing k in Wynn's ρ-algorithm (1.31) the differences of the indices of the interpolation points x_n will increase too, we will see that the following nonlinear recursive scheme should be the most appropriate iteration of the transform (1.48)

$$W_{k+1}^{(n)} = W_k^{(n+1)} + \frac{(x_{n+2k+2} - x_n)\Delta W_k^{(n+1)} \Delta W_k^{(n)}}{(x_{n+2k+2} - x_{n+1})\Delta W_k^{(n)} - (x_{n+2k+1} - x_n)\Delta W_k^{(n+1)}},$$
$$W_0^{(n)} = S_n, \quad k, n \in \mathbb{N}_0. \tag{1.49}$$

As in case of the other iterated processes the forward difference operator, Δ, will act only upon the superscript n and not upon the subscript k. Taking the most suitable interpolation points, which are the same as in Wynn's ρ-algorithm, $x_n = n + \beta$ with $\beta > 0$, the iterated ρ_2 transformation achieves its standard form

$$W_{k+1}^{(n)} = W_k^{(n+1)} - \frac{(2k+2)\Delta W_k^{(n+1)} \Delta W_k^{(n)}}{(2k+1)\Delta^2 W_k^{(n)}},$$
$$W_0^{(n)} = S_n, \quad k, n \in \mathbb{N}_0. \tag{1.50}$$

The table of the transform can be arrange in the same rectangular scheme as the Aitken table (1.31). The fact that, the recursive formulas for Aitken's iterated Δ^2 process (1.30) and the recurrence formulas for the transforms $W_k^{(n)}$ are structurally identical, implies that every four elements, which are connected by the nonlinear recurrence formulas (1.49) or (1.50) can be arranged like the move of a knight on the chessboard, as follows

$$\begin{array}{ll} W_k^{(n)} & W_{k+1}^{(n)} \\ W_k^{(n+1)} & \\ W_k^{(n+2)}. & \end{array} \tag{1.51}$$

1.7.6 Brezinski's θ-algorithm and its iterated form

As we said in the previews section Wynn's ϵ-algorithm accelerates linear convergence quite good and it is also able to sum some divergent series, but Wynn's ρ-algorithm is one of the best accelerators for the logarithmic convergence, but fails to accelerate linear convergence and to sum divergent series. So, it is desirable to give a new acceleration method that combines the advantageous of these two algorithms. For this purpose, let us consider the following recurrence formula

$$T_{k+1}^{(n)} = T_{k-1}^{(n+1)} + w_k D_k^{(n)}, \tag{1.52}$$
$$T_{-1}^{(n)} = 0, \quad T_0^{(n)} = S_n, \quad k, n \in \mathbb{N}_0, \tag{1.53}$$

where $D_k^{(n)}$ is a known quantity which depends upon one or several other elements $T_k^{(\nu)}$ of the table of the transform and the quantity w_k is for the moment unspecified. Later, we will derive an expression for w_k that will guarantee that the above recursive scheme will lead to an acceleration of convergence.

The recursive scheme (1.52) is as particular case of the ϵ-algorithm and of the ρ-algorithm. If we choose $w_k = 1$ together with

$$D_k^{(n)} = \frac{1}{T_k^{(n+1)} - T_k^{(n)}}, \quad k, n \in \mathbb{N}_0, \tag{1.54}$$

we will obtain Wynn's ϵ-algorithm (1.31). If we choose $w_k = 1$ together with

$$D_k^{(n)} = \frac{x_{n+k+1} - x_n}{T_k^{(n+1)} - T_k^{(n)}}, \quad k, n \in \mathbb{N}_0, \tag{1.55}$$

we will obtain Wynn's ρ-algorithm (1.41). In what follows we will analyse how the quantity w_k has to be chosen in order to guarantee that the sequence transformation $T_k^{(n)}$ will lead to an acceleration of the convergence.
In the case of Wynn's ϵ-algorithm or Wynn's ρ-algorithm only the transforms with even superscripts are used as approximations to the limit, while the transforms with odd subscripts are only auxiliary quantities which diverge if the whole processes converges. Because, these processes are the starting points for the construction of the new sequence transformation, it will be considered that $T_k^{(n)}$ behaves in the same way.
In [21], Brezinski showed that the quantities with odd subscript do not really matter as long as they diverge if the whole process converges. Consequently, the most convenient choice for w_{2k} in (1.52) would be to proceed as in Wynn's ϵ-algorithm (1.31) or Wynn's ρ-algorithm (1.41)

$$w_{2k} = 1, \quad k \in \mathbb{N}_0. \tag{1.56}$$

The parameters w_{2k+1} can be determined by requiring that for fixed $k \in \mathbb{N}_0$ the sequence $\{T_{2k+2}^{(n)}\}$ should converge more rapidly than the sequence $\{T_{2k}^{(n+1)}\}$ in the following sense:

$$\lim_{n\to\infty} \frac{\Delta T_{2k+2}^{(n)}}{\Delta T_{2k}^{(n+1)}} = 0, \quad k, n \in \mathbb{N}_0. \tag{1.57}$$

If we form in (1.52) the first difference with respect to n, we see that the condition (1.57) is automatically fulfilled if we choose

$$w_{2k+1} = -\lim_{n\to\infty} \frac{\Delta T_{2k}^{(n+1)}}{\Delta D_{2k+1}^{(n)}}, \quad k, n \in \mathbb{N}_0. \tag{1.58}$$

Unfortunately, in practical situations it will be not possible to compute the limit when $n \to \infty$. In [21], Brezinski suggested the following alternative

$$w_{2k+1}^{(n)} = -\frac{\Delta T_{2k}^{(n+1)}}{\Delta D_{2k+1}^{(n)}}, \quad k, n \in \mathbb{N}_0. \tag{1.59}$$

This choice together with the relation (1.56) leads to the following recursive scheme for the sequence transformation $T_k^{(n)}$

$$T_{-1}^{(n)} = 0, \quad T_0^{(n)} = S_n \tag{1.60}$$

$$T_{2k+1}^{(n)} = T_{2k-1}^{(n+1)} + D_{2k}^{(n)} \tag{1.61}$$

$$T_{2k+2}^{(n)} = T_{2k}^{(n+1)} - \frac{\Delta T_{2k}^{(n+1)}}{\Delta D_{2k+1}^{(n)}} D_{2k+1}^{(n)}, \quad k, n \in \mathbb{N}_0. \tag{1.62}$$

In the case we choose in the above recursive scheme $D_k^{(n)}$ according to the expression (1.54) we obtain Brezinski's θ-algorithm, given by

$$\begin{aligned} \vartheta_{-1}^{(n)} &= 0, \quad \vartheta_0^{(n)} = S_n \\ \vartheta_{2k+1}^{(n)} &= \vartheta_{2k-1}^{(n+1)} + \frac{1}{\vartheta_{2k}^{(n)}} \\ \vartheta_{2k+2}^{(n)} &= \vartheta_{2k}^{(n+1)} + \frac{\Delta \vartheta_{2k}^{(n+1)} \Delta \vartheta_{2k+1}^{(n+1)}}{\Delta^2 \vartheta_{2k+1}^{(n)}}, \quad k, n \in \mathbb{N}_0. \end{aligned} \tag{1.63}$$

As in the other cases, the forward difference operator, Δ, acts only upon the superscript n and not upon the subscript k.

In [113], [114] Smith and Ford demonstrated that Brezinski's ϑ-algorithm is a very powerful and a very versatile sequence transformation because, it is able to accelerate both linear and logarithmic convergence and to sum even wildly divergent series.

Brezinski's ϑ-algorithm it is not derived via a model sequence and because, the recursive scheme (1.63) is significantly more complicated than the recursive schemes of the most other sequences transformations very little seems to be known about its theoretical properties.

In [113], Smith and Ford proved that $\vartheta_2^{(n)}$ accelerates linear convergence. Short discussions about the properties of the ϑ-algorithm can be found in books by Brezinski [20], [22] and Wimp [124].

In [29], Brezinski suggested a new derivation of the ϑ-algorithm, which is based upon Wynn's ρ-algorithm (1.41)

$$\begin{aligned} \theta_{-1}^{(n)} &= 0, \quad \theta_0^{(n)} = S_n \\ \theta_{2k+1}^{(n)} &= \theta_{2k-1}^{(n+1)} + \frac{x_{n+2k+1} - x_n}{\Delta \theta_{2k}^{(n)}} \\ \theta_{2k+2}^{(n)} &= \theta_{2k}^{(n+1)} - \frac{(x_{n+2k+2} - x_n) \Delta \theta_{2k}^{(n+1)} \Delta \theta_{2k+1}^{(n+1)}}{(x_{n+2k+2} - x_{n+1}) \Delta \theta_{2k+1}^{(n)} - (x_{n+2k+1} - x_n) \Delta \theta_{2k+1}^{(n+1)}}, \quad k, n \in \mathbb{N}_0. \end{aligned} \tag{1.64}$$

Numerical tests showed that these sequence transformation $\theta_k^{(n)}$ is more versatile than Wynn's ρ-algorithm (1.41), since it is able to accelerate linear convergence and to sum some divergent series. But it is much less efficient than Wynn's ρ-algorithm (1.41) in the case of logarithmic convergence and it is not particularly powerful in the case of linear convergence or divergence.

In what follows we arrange the elements of the ϑ table in a rectangular scheme in such a way that the superscript n indicates the row and the subscript k indicates the column of the 2-dimensional array:

$$
\begin{array}{cccccc}
\vartheta_0^{(0)} & \vartheta_1^{(0)} & \vartheta_2^{(0)} & \dots & \vartheta_n^{(0)} & \dots \\
\vartheta_0^{(1)} & \vartheta_1^{(1)} & \vartheta_2^{(1)} & \dots & \vartheta_n^{(1)} & \dots \\
\vartheta_0^{(2)} & \vartheta_1^{(2)} & \vartheta_2^{(2)} & \dots & \vartheta_n^{(2)} & \dots \\
\vartheta_0^{(3)} & \vartheta_1^{(3)} & \vartheta_2^{(3)} & \dots & \vartheta_n^{(3)} & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots \\
\vartheta_0^{(n)} & \vartheta_1^{(n)} & \vartheta_2^{(n)} & \dots & \vartheta_n^{(n)} & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots
\end{array}
\tag{1.65}
$$

The entries in the first column of the array are the starting values $\vartheta_0^{(n)} = S_n$ of the recursion and the remaining elements of the ϑ table can be computed with the recurrence formulas in (1.63). The sequence transformation $\vartheta_k^{(n)}$ requires the sequence elements $S_n, S_{n+1}, \dots, S_{n+3k}$ which implies that it is a transformation of order $l = 3k$. The four elements that are connected by the recursive scheme (1.63), second part, form the same pattern in the ϑ table as the four elements of the ϵ table (1.40)

$$
\begin{array}{ccc}
 & \vartheta_k^{(n)} & \vartheta_{k+1}^{(n)} \\
\vartheta_{k-1}^{(n+1)} & \vartheta_k^{(n+1)}, &
\end{array}
\tag{1.66}
$$

and the six elements, which are connected by the nonlinear recursion (1.63), third part, form the following pattern in the ϑ table

$$
\begin{array}{ccc}
 & \vartheta_{2k+1}^{(n)} & \vartheta_{2k+2}^{(n)} \\
\vartheta_{2k}^{(n+1)} & \vartheta_{2k+1}^{(n+1)} & \\
\vartheta_{2k}^{(n+2)} & \vartheta_{2k+1}^{(n+2)}. &
\end{array}
\tag{1.67}
$$

In Paragraph 1.7.4 it was shown that Aitken's Δ^2 process is identical with $\epsilon_2^{(n)}$ and it can be iterated to give the sequence transformation $A_k^{(n)}$. In the same way, Paragraph 1.7.5, it was shown that the $\rho_2^{(n)}$ can be iterated to give the sequence transformation $W_k^{(n)}$. In the sequel we show how $\vartheta_2^{(n)}$ can be iterated. From the recursion (1.63) we obtain the following expression

$$
\vartheta_2^{(n)} = S_{n+1} - \frac{\Delta S_n \Delta S_{n+1} \Delta^2 S_{n+1}}{\Delta S_{n+2} \Delta^2 S_n - \Delta S_n \Delta^2 S_{n+1}}, \quad n \in \mathbb{N}_0. \tag{1.68}
$$

By suitable manipulations of the above formula we can derive many other representations for the $\vartheta_2^{(n)}$, as follows

$$
\vartheta_2^{(n)} = \frac{S_{n+1} \Delta S_{n+2} \Delta^2 S_n - S_{n+2} \Delta S_n \Delta^2 S_{n+1}}{\Delta S_{n+2} \Delta^2 S_n - \Delta S_n \Delta^2 S_{n+1}} \tag{1.69}
$$

$$
= \frac{\Delta^2 [S_{n+1}/\Delta S_n]}{\Delta^2 [1/\Delta S_n]}. \tag{1.70}
$$

Iterating the expression (1.68) we obtain the following recursive scheme

$$
\begin{aligned}
I_0^{(n)} &= S_n \\
I_{k+1}^{(n)} &= I_k^{n+1} - \frac{\Delta I_k^{(n)} \Delta I_k^{(n+1)} \Delta^2 I_k^{(n+1)}}{\Delta I_k^{(n+2)} \Delta^2 I_k^{(n)} - \Delta I_k^{(n)} \Delta^2 I_k^{(n+1)}}, \quad n \in \mathbb{N}_0.
\end{aligned}
\tag{1.71}
$$

As in the previous cases, the forward difference operator, Δ, acts only upon the superscript n and not upon the subscript k. For the computation of $I_k^{(n)}$ we have to known the sequence elements $S_n, S_{n+1}, \ldots, S_{n+3k}$, this implies that $I_k^{(n)}$ is a transformation of order $l = 3k$.

$I_k^{(n)}$ is a powerful sequence transformation, which has similar properties as $\vartheta_{2k}^{(n)}$ and it is able to accelerate linear and logarithmic convergence and to sum even wildly divergent series.

If we would replace the recursion (1.63) by the recursion

$$\vartheta_{2k+1}^{(n)} = 1/\Delta_{2k}^{(n)}, \quad k, n \in \mathbb{N}_0 \tag{1.72}$$

then we would obtain

$$\vartheta_{2k}^{(n)} = I_k^{(n)}. \tag{1.73}$$

In the sequel we will arrange the elements $I_k^{(n)}$ in a rectangular scheme in such a way that the superscript n indicates the row and the superscript k indicates the column of the 2-dimensional array

$$\begin{array}{cccccc}
I_0^{(0)} & I_1^{(0)} & I_2^{(0)} & \ldots & I_n^{(0)} & \ldots \\
I_0^{(1)} & I_1^{(1)} & I_2^{(1)} & \ldots & I_n^{(1)} & \ldots \\
I_0^{(2)} & I_1^{(2)} & I_2^{(2)} & \ldots & I_n^{(2)} & \ldots \\
I_0^{(3)} & I_1^{(3)} & I_2^{(3)} & \ldots & I_n^{(3)} & \ldots \\
\multicolumn{6}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\
I_0^{(n)} & I_1^{(n)} & I_2^{(n)} & \ldots & I_n^{(n)} & \ldots \\
\multicolumn{6}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}
\end{array} \tag{1.74}$$

the entries in the first column of the array are the starting values $I_k^{(n)} = S_n$ and the remaining elements of the table can be computed with the help of the recursion (1.71), second part. Every five elements, which are connected with the nonlinear recursion (1.71) form the following pattern

$$\begin{array}{ll}
I_k^{(n)} & I_{k+1}^{(n)} \\
I_k^{(n+1)} & \\
I_k^{(n+2)} & \\
I_k^{(n+3)}. &
\end{array} \tag{1.75}$$

1.7.7 Levin's transformation

In [75], Levin proposed a sequence transformation that is exact for the following type of sequences

$$S_n = S + w_n \sum_{j=0}^{k-1} c_j/(n+\beta)^j, \quad k, n \in \mathbb{N}_0 \tag{1.76}$$

where the remainder estimates w_n, is a arbitrary function depending on n, different from zero for all finite values of n. It is important to say that, it would not make much sense to consider in convergence acceleration and summation processes remainder estimates which are constants. So, we will consider the sequence elements $\{w_n\}$ distinct for all finite values of n. According to the behaviour of the remainder estimates $\{w_n\}$ when $n \to \infty$, the sequence $\{S_n\}$ may either converge or diverge. In the equation (1.76) the term $n+\beta$ must be taken different from zero, this implies that the parameter β must be different from zero or different from a negative integer. Anyway, the elements of the model sequence (1.76) will serve as finite approximations to Poincaré-type asymptotic expansions of the following type

$$S_n \sim S + w_n \sum_{j=0}^{n} c_j/(n+\beta)^j, \quad n \to \infty. \tag{1.77}$$

When having expansions of these kind, negative values of β will lead to different signs of the terms when either $n+\beta < 0$ or $n+\beta > 0$ holds. Let be the model sequence, of the type (1.76), be used as approximations of the above asymptotic expansions and because, the approximations should be uniformly valid over a wide range of admissible values of n it is necessary that the sign pattern of the terms of the sum in (1.76) must not depend upon n. This implies that β must be taken positive, but otherwise β is completely arbitrary. In the literature only the case $\beta = 1$ has been considered so far.

In the equation (1.76) occur $k+1$ unknowns, these are: the limit or antilimit S and the k linear coefficients $c_0, c_1, \ldots, c_{k-1}$. So, for the computation of S are needed the $k+1$ sequences elements $S_n, \ldots, S_{n+k}$, this implies from the Cramer's rule, that the general Levin transformation $L_k^{(n)}(\beta, S_n, w_n)$ can be defined as a ratio of two determinants of the following form

$$L_k^{(n)}(\beta, S_n, w_n) = \frac{\begin{vmatrix} S_n & \ldots & S_{n+k} \\ w_n & \ldots & w_{n+k} \\ \vdots & \vdots & \vdots \\ w_n/(n+\beta)^{k-1} & \ldots & w_{n+k}/(n+\beta+k)^{k-1} \end{vmatrix}}{\begin{vmatrix} 1 & \ldots & 1 \\ w_n & \ldots & w_{n+k} \\ \vdots & \vdots & \vdots \\ w_n/(n+\beta)^{k-1} & \ldots & w_{n+k}/(n+\beta+k)^{k-1} \end{vmatrix}}. \tag{1.78}$$

If the sequence elements $S_n, S_{n+1}, \ldots, S_{n+k}$ satisfy the relation (1.76) the Levin's general sequence transformation is exact by construction

$$L_k^{(n)}(\beta, S_n, w_n) = S. \tag{1.79}$$

For practical applications Levin's transformation is not so efficient due the definition of a ratio of two determinants, so alternative expressions for it must be used, lucky they can be derived quite easily.

In [110], Sidi proposed a new approach for the Levin transformation, which can be easily extended to other sequence transformations, and for which he exploited the properties of the difference operator Δ. For that, the model sequence is rewritten in the following form

$$(n+\beta)^{k-1}[S_n - S]/w_n = \sum_{j=0}^{k-1} c_j (n+\beta)^{k-j-1}. \tag{1.80}$$

In the right side of the above formula the highest power of n is n^{k-1} if we utilize the fact that for polynomial of degree $k-1$ in n will be annihilated by the difference operator Δ^k, where Δ^k is linear from the equation (1.79) we obtain the following general Levin transformation

$$L_k^{(n)}(\beta, S_n, w_n) = \frac{\Delta^k\{(n+\beta)^{k-1} S_n/w_n\}}{\Delta^k\{(n+\beta)^{k-1} 1/w_n\}} \tag{1.81}$$

with the help of the next relation

$$\Delta^k f(n) = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(n+j), \quad k \in \mathbb{N}_0, \tag{1.82}$$

the relation (1.81) can be represented as a ratio of two finite sums

$$L_k^{(n)}(\beta, S_n, w_n) = \frac{\sum\limits_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(n+\beta+j)^{k-1}}{(n+\beta+k)^{k-1}} \frac{S_{n+j}}{w_{n+j}}}{\sum\limits_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(n+\beta+j)^{k-1}}{(n+\beta+k)^{k-1}} \frac{1}{w_{n+j}}}, \quad k, n \in \mathbb{N}_0. \tag{1.83}$$

The common factor $(n+\beta+k)^{k-1}$ in the equation (1.84) was introduced in order to increase the magnitude of the terms of the numerator and denominator sums, because, otherwise overflow may happen too easily for large values of k.

A simple extension for the general Levin transformation is the following transformation

$$L_k^{(n)}(\beta, S_n, w_n) = \frac{\sum\limits_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(n+\beta+j)^{k-l-1}}{(n+\beta+k)^{k-l-1}} \frac{S_{n+j}}{w_{n+j}}}{\sum\limits_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(n+\beta+j)^{k-l-1}}{(n+\beta+k)^{k-l-1}} \frac{1}{w_{n+j}}}, \quad k, n \in \mathbb{N}_0, \tag{1.84}$$

when $l = 0$ it is reduced to general Levin transformation. An other representation for this generalization of Levin transformation can be derived with the help of the relation (1.82)

$$L_{k,l}^{(n)}(\beta, S_n, w_n) = \frac{\Delta^k\{(n+\beta)^{k-l-1} S_n/w_n\}}{\Delta^k\{(n+\beta)^{k-l-1} 1/w_n\}} \tag{1.85}$$

the above transformation is exact for the following model sequence

$$S_n = S + (n+\beta)^l w_n \sum_{j=0}^{k-1} c_j/(n+\beta)^j, \quad k, n \in \mathbb{N}_0. \tag{1.86}$$

In [55], Fessler, Ford and Smith proposed a recursive scheme for the computation of the general Levin transformation. It is known that both numerator and denominator are the following form

$$P_k^{(n)}(\beta) = \Delta^k X_k^{(n)}(\beta), \quad k, n \in \mathbb{N}_0. \tag{1.87}$$

As in the other cases, the difference operator, Δ, acts only upon n and not upon k. Analogies with (1.81) show that $X_k^{(n)}(\beta)$ satisfies the next recursion in k

$$X_k^{(n)}(\beta) = (n+\beta) X_{k-1}^{(n)}(\beta), \quad k \geq 1,\ n \geq 0. \tag{1.88}$$

The following relation can be proved by complete induction with respect to

$$\Delta^k (n+\beta) - (n+\beta)\Delta^k = kE\Delta^{k-1}. \tag{1.89}$$

Combining the relation $Ef(n) = f(n+1)$ with (1.87), (1.88), (1.89) we obtain

$$\begin{aligned} P_k^{(n)}(\beta) &= \{kE + (n+\beta)\Delta\}\Delta^{k-1} X_{k-1}^{(n)}(\beta) \\ &= \{kE + (n+\beta)\Delta\} P_{k-1}^{(n)}(\beta) \\ &= (n+\beta+k) P_{k-1}^{(n+1)} - (n+\beta) P_{k-1}^{(n)}(\beta). \end{aligned} \tag{1.90}$$

With the help of the formula (1.90), third part, the numerator and denominator of the general Levin transformation $L_k^{(n)}(\beta, S_n, w_n)$ can be computed for $k \geq 1$. If we define

$$L_k^{(n)}(\beta) = P_k^{(n)}(\beta)/(n+\beta+k)^{k-1}, \tag{1.91}$$

and then inserting this in (1.90), third part, we get a 3-term recurrence formula for the $L_k^{(n)}(\beta)$

$$L_{k+1}^{(n)}(\beta) = L_k^{(n+1)} - \frac{(n+\beta)(n+\beta+k)^{k-1}}{(n+\beta+k+1)^k} L_k^{(n)}(\beta), \quad k, n \in \mathbb{N}_0. \tag{1.92}$$

If we use the starting values

$$L_0^{(n)}(\beta) = S_n/w_n, \quad n \in \mathbb{N}_0, \tag{1.93}$$

the recurrence formula (1.92) will produce the numerator of the general Levin transformation, and if we use the staring values

$$L_0^{(n)}(\beta) = 1/w_n, \quad n \in \mathbb{N}_0, \tag{1.94}$$

we obtain the denominator of the general Levin transformation.

With the help of the recurrence formula (1.92) both numerator and denominator of generalized Levin transformation can be computed if we use the starting values

$$L_0^{(n)}(\beta) = S_n/[(n+\beta)^l w_n], \quad l, n \in \mathbb{N}_0, \tag{1.95}$$

then (1.92) will produce the numerator of (1.82) and if we take the starting values

$$L_0^{(n)}(\beta) = 1/[(n+\beta)^l w_n], \quad l, n \in {}_0, \tag{1.96}$$

then (1.92) will produce the denominator of (1.82).

The 3-term recurrence formula (1.90), third part, was first published by Longman in [76], Longman's recurrence formula is based upon Sister Celine's technique [112] and not upon the properties of the difference operator, Δ, as the derivation by Fessler, Ford and Smith [55].

In order to give some derivations for Levin transformation it was necessary the knowledge of the sequence $\{w_n\}$ of remainder estimates, which it is not determined in a unique way from the formula (1.76). However, in most practical examples are no information about the analytical structure of the sequence of remainders $\{r_n\}$ and only the numerical values of a relatively small number of sequence elements $S_m, S_{m+1}, \ldots, S_{m+l}$ are known. The sequence of remainder estimates $\{w_n\}$ will be directly find from the numerical values of the elements of the sequence $\{S_n\}$. If such a sequence of remainder estimates is used in equation (1.76), the Levin transformation is a nonlinear sequence transformation because each remainder estimate w_n depends explicitly upon at least one element of the sequence $\{S_n\}$.

In [75], Levin suggested the following reminder estimate

$$w_n = (n+\beta)a_n, \quad n \in \mathbb{N}_0. \tag{1.97}$$

If, the above remainder estimate is used in (1.82) yields Levin's u transformation

$$u_k^{(n)}(\beta, S_n) = \frac{\sum_{j=0}^{k}(-1)^j \binom{k}{j} \frac{(n+\beta+j)^{k-2}}{(n+\beta+k)^{k-1}} \frac{S_{n+j}}{a_{n+j}}}{\sum_{j=0}^{k}(-1)^j \binom{k}{j} \frac{(n+\beta+j)^{k-2}}{(n+\beta+k)^{k-1}} \frac{1}{a_{n+j}}}, \quad k, n \in \mathbb{N}_0. \tag{1.98}$$

In the same book [75], Levin suggest the remainder estimate for the alternating series

$$w_n = a_n, \qquad n \in \mathbb{N}_0. \tag{1.99}$$

If the above remainder estimate will be used in (1.82) yields the Levin's t transformation

$$t_k^{(n)}(\beta, S_n) = \frac{\sum_{j=0}^{k}(-1)^j \binom{k}{j} \frac{(n+\beta+j)^{k-1}}{(n+\beta+k)^{k-1}} \frac{S_{n+j}}{a_{n+j}}}{\sum_{j=0}^{k}(-1)^j \binom{k}{j} \frac{(n+\beta+j)^{k-1}}{(n+\beta+k)^{k-1}} \frac{1}{a_{n+j}}}, \quad k, n \in \mathbb{N}_0. \tag{1.100}$$

In [113], Smith and Ford proposed the following remainder estimate for a convergent series with strictly alternating terms a_ν

$$w_n = a_{n+1}, \quad n \in \mathbb{N}_0. \tag{1.101}$$

Using the above relation in (1.82) we obtain the modification of Levin's t transformation

$$d_k^{(n)}(\beta, S_n) = \frac{\sum_{j=0}^{k}(-1)^j \binom{k}{j} \frac{(n+\beta+j)^{k-1}}{(n+\beta+k)^{k-1}} \frac{S_{n+j}}{a_{n+j+1}}}{\sum_{j=0}^{k}(-1)^j \binom{k}{j} \frac{(n+\beta+j)^{k-1}}{(n+\beta+k)^{k-1}} \frac{1}{a_{n+j+1}}}, \quad k, n \in \mathbb{N}_0. \tag{1.102}$$

Also in [75], Levin suggested the following remainder estimate

$$w_n = \frac{a_n a_{n+1}}{a_n - a_{n+1}}, \quad n \in \mathbb{N}_0. \tag{1.103}$$

If it will be used in the (1.82) yields Levin's v transformation

$$v_k^{(n)}(\beta, S_n) = \frac{\sum\limits_{j=0}^{k} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} \frac{(n+\beta+j)^{k-1}}{(n+\beta+k)^{k-1}} \frac{a_{n+j}-a_{n+j+1}}{a_{n+j}a_{n+j+1}} S_{n+j}}{\sum\limits_{j=0}^{k} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} \frac{(n+\beta+j)^{k-1}}{(n+\beta+k)^{k-1}} \frac{a_{n+j}-a_{n+j+1}}{a_{n+j}a_{n+j+1}}}, \quad k, n \in \mathbb{N}_0. \tag{1.104}$$

Levin's remainder estimates (1.97), (1.99), (1.103) as well as Smith and Ford's modification (1.101) were derived using simple heuristic arguments. One important observation is that these remainder estimates give rise to very powerful sequence transformations [55], [59], [113], [114], [117].

The sequence transformations $u_k^{(n)}(\beta, S_n)$, $t_k^{(n)}(\beta, S_n)$ require for their computation the sequence elements $S_{n-1}, S_n, S_{n+1}, \ldots, S_{n+k}$, while $d_k^{(n)}(\beta, S_n)$ requires the sequence elements $S_n, S_{n+1}, \ldots,$ S_{n+k+1}. So, they are transformations of order $l = k + 1$. The sequence transformation $v_k^{(n)}(\beta, S_n)$ requires the sequence elements $S_{n-1}, S_n, S_{n+1}, \ldots,$ S_{n+k+1} which implies that it is a transformation of order $l = k+2$. In the case when the superscript $n = 0$, $u_k^{(0)}(\beta, S_0)$ and $t_k^{(0)}(\beta, S_0)$ are transformations of order $l = k$, when $d_k^{(n)}(\beta, S_0)$ and $v_k^{(0)}(\beta, S_0)$ are transformations of order $l = k + 1$. Because, the computation of $L_k^{(n)}(\beta)$ requires the sequence elements $S_n, S_{n+1}, \ldots, S_{n+k}$ is a transformation of order $l = k$.

The elements $L_k^{(n)}(\beta)$ of the general Levin transformation, which are computed with the help of the 3-term recurrence formula (1.92) can be arranged in a rectangular scheme in such a way that the superscript n indicates the row and the subscript k indicates the column of the 2-dimensional array

$$\begin{array}{cccccc} L_0^{(0)} & L_1^{(0)} & L_2^{(0)} & \ldots & L_n^{(0)} & \ldots \\ L_0^{(1)} & L_1^{(1)} & L_2^{(1)} & \ldots & L_n^{(1)} & \ldots \\ L_0^{(2)} & L_1^{(2)} & L_2^{(2)} & \ldots & L_n^{(2)} & \ldots \\ L_0^{(3)} & L_1^{(3)} & L_2^{(3)} & \ldots & L_n^{(3)} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ L_0^{(n)} & L_1^{(n)} & L_2^{(n)} & \ldots & L_n^{(n)} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array} \tag{1.105}$$

where the entries in the first column of the array are the starting values of the recursion. If the staring values are chosen according to (1.93), the 3-term recurrence formula (1.92) will produce a table of $L_k^{(n)}(\beta)$ of numerators of the Levin transformation and if the staring values are chosen according to (1.94) a table of denominators will be computed. Every three elements, which are connected with the nonlinear recursion (1.92) form a triangle in the Levin table

$$\begin{array}{ll} L_k^{(n)}(\beta) & L_{k+1}^{(n)}(\beta) \\ L_k^{(n+1)}(\beta). & \end{array} \tag{1.106}$$

1.8 Convergence order for the iterative methods

The Definition 1.15 of the convergence order for a numeric sequence can be extended directly to a sequence from $\mathbb{R}^n$, by replacing the module with the norm. This definition has the disadvantage that it can not be applied to any sequence from $\mathbb{R}^n$ or $\mathbb{R}$, even if it is convergent. In the sequel we will introduce two new convergence orders that can be applied to any sequence from $\mathbb{R}^n$.

Definition 1.15 *[83] The sequence $\{x_k\}$ has r order of convergence if*

$$\lim_{k\to\infty} \frac{|x^* - x_{k+1}|}{|x^* - x_k|^r} = \rho$$

where $0 < \rho < \infty$ and ρ is called ***the asymptotic error of the method****.*

1.8.1 Q-order of convergence

Let $\{x^k\}$ be a sequence from $\mathbb{R}^n$ convergent to x^*. We assume that $\{x^k\}$ is generated by an iterative process, $x^{k+1} = \Phi x^k$ and that x^* is a fixed point for Φ. In the hypothesis that for some k_0 we have $x^{k_0} = x^*$, it results that for every $k > k_0$ we have $x^k = x^*$, in this case we can say that the method converges in a finite number of steps.

Definition 1.16 *[83] Let $\{x^k\}$ be a sequence from $\mathbb{R}^n$ convergent to x^* and $Q : [1,\infty) \to \mathbb{R}$ a mapping defined by*

$$Q(t) = \begin{cases} 0, & \text{if} \quad x^k = x^* \text{ for } \quad k \geq k_0, \\ \limsup\limits_{k\to\infty} \frac{\|x^{k+1}-x^*\|}{\|x^k - x^*\|^t}, & \text{if} \quad x^k \neq x^* \text{ for } \quad k \geq k_0. \end{cases}$$

We say that $Q(t)$ are the *convergence ratio factors* or *Q-factors* for $\{x^k\}$ with respect to the norm $\|\cdot\|$. A certain method can generate different sequences converging to x^* or even sequences that have different limits. It is natural to consider the convergence order of the method the lowest convergence order of the sequences generated by this method. So, the *Q-factors* should be considered for all sequences generated by the respective iterative process. To simplify the method we define both Q-factors and the convergence order for a certain sequence.

Theorem 1.16 *[83] Let $Q(t)$ be the Q-factors of a sequence $\{x^k\}$ with respect to the norm $\|\cdot\|$. Then one of the followings is true:*
(a) $Q(t) = 0$, $\forall\ t \in [1,\infty)$;
(b) $Q(t) = \infty$, $\forall\ t \in [1,\infty)$;
(c) There $\exists\ q \in [1,\infty)$ such that $Q(t) = 0, \forall t \in [1,q)$ and $Q(t) = \infty, \forall t \in (q,\infty)$.

The q-order of convergence is a number defined by the following relation

$$q = \inf\{t \in [1,\infty) | Q(t) = \infty\}. \tag{1.107}$$

Lemma 1.1 *[83] The relationships $Q(t) = 0,\ 0 < Q(t) < \infty,\ Q(t) = \infty$ are independent from the norm.*

Definition 1.17 *[83] Let $\{x^k\}$ be a convergent sequence from $\mathbb{R}^n$ and let $Q(t)$ be the Q-factors of the sequence. The number q defined by the relation (1.107) is called the* ***q-order of convergence of the sequence*** *$\{x^k\}$.*

From Lemma1.1 we have the following result.

Corollary 1.3 *[83] The q-order of convergence of a sequence does not depend upon the norm.*

If $q = 1$ holds, the convergence is called *q-linearly*; if $q > 1$ holds, the convergence is called *q-superlinearly* (usually this name is used for $1 < q < 2$), and if $q = 2$ holds, the convergence is called *q-square.* The q-order of convergence for a sequence allows the comparison of two methods in terms of the speed of convergence. Let $\{x^k\}$ and $\{x'^k\}$ be two sequences converging to x^* and let q and q', their corresponding orders. If $q < q'$ then we say that the sequence $\{x'^k\}$ is q-convergent faster than $\{x^k\}$. If $q = q'$ we compare the Q-factors; in case there exits a t for which $Q'(t) = 0 < Q(t)$ or $Q'(t) < Q(t) < \infty$ we say that the sequence $\{x'^k\}$ is q-convergent faster than $\{x^k\}$; and if $0 < Q'(t) < Q(t) < \infty$, then $\{x'^k\}$ is q-convergent faster in that norm, but there are norm in which the relation can be inverse.

1.8.2 R-order of convergence

Let $\{x^k\}$ be a sequence from $\mathbb{R}^n$ convergent to x^*.

Definition 1.18 *[83] Let $R : [1, \infty) \to \mathbb{R}$ be a mapping, defined by:*

$$R(t) = \begin{cases} \lim\limits_{k\to\infty} \sup \left\|x^k - x^*\right\|^{\frac{1}{k}}, & \text{for } t = 1, \\ \lim\limits_{k\to\infty} \sup \left\|x^k - x^*\right\|^{\frac{1}{t^k}}, & \text{for } t > 1. \end{cases}$$

We say that $R(t)$ are the *root convergence factors* or *R-factors* for $\{x^k\}$. Because, x^k converges to x^*, there exists a k_0 such that $\left\|x^k - x^*\right\| < 1, k \geq k_0$ and so $0 \leq R(t) \leq 1$, $\forall t \in [1, \infty)$. Unlike the Q-factors of a sequence, the R-factors do not depend upon norm. Indeed let $\|\cdot\|$ and $\|\cdot\|'$ be two norms from $\mathbb{R}^n$ and let α_k be a numeric sequence with the limit zero, $\lim\limits_{k\to\infty} \alpha_k = 0$. Considering

$$c\,\|x\| \leq \|x'\| \leq d\,\|x\|\,, \forall x \in \mathbb{R}^n, \exists \text{ the constants } c, d \text{ such that } 0 < c \leq d,$$

we have

$$\lim_{k\to\infty} \sup \left\|x^k - x^*\right\|^{\alpha_k} \leq \lim_{k\to\infty} \frac{1}{c^{\alpha_k}} \lim_{k\to\infty} \sup \|x_k - x^*\|'^{\alpha_k} = \lim_{k\to\infty} \sup \|x_k - x^*\|'^{\alpha_k},$$

and so $Q(t) \leq Q'(t), \forall t \in [1, \infty)$. Analogous $Q'(t) \leq Q(t), \forall t \in [1, \infty)$, from where we have $Q(t) = Q'(t), \forall t \in [1, \infty)$, so, the R-factors do not depend upon the norm.

Theorem 1.17 *[83] Let $R(t)$ be the R-factors of a sequence $\{x^k\}$. Then one of the following is true:*
(a) $R(t) = 0$, $\forall\ t \in [1, \infty)$;
(b) $R(t) = 1$, $\forall\ t \in [1, \infty)$;
(c) There $\exists\ r \in [1, \infty)$ such that $R(t) = 0, \forall t \in [1, r)$ and $R(t) = 1, \forall t \in (r, \infty)$.

The r-order of convergence is a number defined by the following relation

$$r = \inf\{t \in [1, \infty | R(t) = 1\}. \tag{1.108}$$

Definition 1.19 *[83] Let* $\{x^k\}$ *be a convergent sequence from* $\mathbb{R}^n$ *and let* $R(t)$ *be the* R*-factors of the sequence. The number* r *defined by the relation* (1.108) *is called* r***-order of convergence of the sequence*** $\{x^k\}$*.*

Like the q-order of convergence of a sequence, the r-order of convergence characterises the convergence speed of a sequence. If for two sequences $\{x^k\}$ and $\{x'^k\}$ we have $r < r'$, then we say that the sequence $\{x'^k\}$ is r-convergent faster than $\{x^k\}$. If $r = r' = t_0$ holds, we compare the R-factors; in case $R'(t_0) < R(t_0)$ we also say that the sequence $\{x'^k\}$ is r-convergent faster than $\{x^k\}$. If $r = 1$ holds, the convergence is called r*-linearly*, if $r > 1$ holds, the convergence is called r*-superlinearly* (usually this name is used for the case $1 < r < 2$), and if $r = 2$ holds, the convergence is called r*-square*.

2 A Padé-type acceleration technique

In this chapter we present a technique for accelerating Picard iteration associated to a nonlinear equation.

2.1 A Padé-type acceleration technique for the Picard iteration

Author's original contribution in this paragraph are: Theorem 2.1, Observation 2.1 and the Examples 2.1, 2.2, 2.3.

Recently, Biazar and Amirteimoori considered in [16] a Padé-type technique to accelerate Picard iteration method for solving a scalar equations of the form

$$f(x) = 0 \tag{2.1}$$

which is equivalently written as a fixed point problem

$$x = g(x), \tag{2.2}$$

where $g : [a, b] \to [a, b]$ is the iteration function.

Under appropriate assumptions on f (and therefore on g), the Picard iteration (or the sequence of successive approximations, as it is generally known), i.e.,

$$x_{n+1} = g(x_n), \quad n \geq 0, \tag{2.3}$$

converges to the (unique) fixed point of g, say α, which is the (unique) solution of (2.1) in the interval (a, b).

As we said in the first chapter, for a certain nonlinear equation (2.1), the fixed point problem (2.2) is not uniquely defined. For example, the equation $x^3 + 4x^2 - 10 = 0$ can be written under a fixed point problem as $x = \frac{1}{2}\sqrt{10 - x^3}$ or $x = \sqrt[3]{10 - 4x^2}$.

Because the convergence order for the Picard iteration (2.3) is generally linear (see Berinde [12]), it converges rather slowly to the fixed point α.

In order to improve the convergence order for (2.3), the authors in [16] considered the following equivalent fixed point problem

$$x = g_\lambda(x) \tag{2.4}$$

with g_λ of the form

$$g_\lambda(x) = \frac{g(x) + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \ldots + \lambda_k x^k}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \ldots + \lambda_k x^{k-1}}, \tag{2.5}$$

where $k \in \mathbb{N},\ k \geq 2$ and $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k \in \mathbb{R}$ are parameters that should be determined in such a way that the new iteration function g_λ will yield a faster Picard iteration.

We observe that the method of constructing (2.5) is rather similar to the way in which the Padé approximant of order (m, n), $[m/n]_f(x)$, is obtained, see for example [6]:

$$[m/n]_f(x) = \frac{p_0 + p_1 x + p_2 x^2 + \ldots + p_m x^m}{1 + q_1 x + q_2 x^2 + \ldots + q_n x^n}. \tag{2.6}$$

For this reason we called it [35] as a Padé type transform.

Based on the fact that the fixed point equation

$$x = g(x),$$

is equivalent to

$$x + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \ldots + \lambda_k x^k = g(x) + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \ldots + \lambda_k x^k,$$

and this can be written under the form if $1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \ldots + \lambda_k x^{k-1} \neq 0$

$$x = g_\lambda(x) = \frac{g(x) + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \ldots + \lambda_k x^k}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \ldots + \lambda_k x^{k-1}}, \tag{2.7}$$

we get exactly the fixed point problem (2.4).

We assumed that $g_\lambda(x)$ is well defined on the interval $[a, b]$ where the original equation is solved, that is, the equation

$$1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \ldots + \lambda_k x^{k-1} = 0,$$

has no real root on $[a, b]$.

The main idea in constructing such an acceleration method is to determine the parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$ in such a way that the new iteration function g_λ satisfies

$$g_\lambda^{(i)}(\alpha) = 0, \quad i = 1, 2, \ldots, k, \tag{2.8}$$

where α is the unique solution of (2.1) and (2.2) in the interval $[a, b]$.

Using (2.7), the equation (2.8) yields an upper diagonal linear system of equations with the unknowns $\lambda_1, \lambda_2, \ldots, \lambda_k$ which always has a unique solution as in the case of the original Padé transform.

Indeed, from (2.7) we have

$$g_\lambda(x)(1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \ldots + \lambda_k x^{k-1}) = g(x) + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \ldots + \lambda_k x^k$$

which, by differentiating with respect to x, gives

$$g'_\lambda(x)(1 + \lambda_1 + \lambda_2 x + \ldots + \lambda_k x^{k-1}) + g_\lambda(x)(\lambda_2 + 2\lambda_3 x + \ldots + (k-1)\lambda_k x^{k-2}) =$$

$$= g'(x) + \lambda_1 + 2\lambda_2 x + \ldots + k\lambda_k x^{k-1}. \tag{2.9}$$

If we take $x = \alpha$ in (2.9) and use the fact that $g_\lambda(\alpha) = \alpha$ and $g'_\lambda(\alpha)$ is required to be zero, we obtain the nonlinear equation

$$\lambda_1 + 2\lambda_2 \alpha + \ldots + k\lambda_k \alpha^{k-1} = -g'(\alpha).$$

Now we differentiate again (2.9) and then, taking $x = \alpha$, we are lead to the nonlinear equation

$$2\lambda_2 + 3\lambda_3 \alpha + \ldots + k(k-1)\lambda_k \alpha^{k-1} = -g''(\alpha),$$

and so on. The generic formula for the i^{th} derivative of g_λ is given by

$$-g^{(j)}(\alpha) = \sum_{i=j}^{k} i(i-1)(i-2)\ldots(i-j+1)\lambda_i \alpha^{i-j}, \quad j = 1, 2, \ldots, k. \tag{2.10}$$

If we rewrite the linear $k \times k$ system (2.10) in a matrix form we have

$$\begin{pmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^{k-1} \\ 0 & 1 & 2\alpha & \dots & (k-1)\alpha^{k-2} \\ 0 & 0 & 2 & \dots & (k-1)(k-2)\alpha^{k-3} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & (k-1)! \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \\ \vdots \\ -g^{(k)}(\alpha) \end{pmatrix}. \qquad (2.11)$$

By solving (2.11), we can uniquely find the values of $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k$ and hence the iteration function g_λ of the accelerated process

$$x_{n+1} = g_\lambda(x_n),\ n \geq 0.$$

In the sequel we use the concept of Q-order of convergence defined in the first chapter by the Definition 1.17. The next theorem shows how the fixed point iteration defined by the function g_λ accelerates the fixed point iteration defined by the function g.

Theorem 2.1 *(Bumbariu [35]) Let $g \in C^{k+1}[a,b]$ be such that the associated iteration function, g_λ, satisfies* (2.8)*, where α is the unique solution in the interval $[a,b]$ of* (2.2)*. Then the accelerated Picard iteration*

$$x_{n+1}^\lambda = g(x_n^\lambda), \quad n \geq 0$$

has the Q-order of convergence $k+1$.

Proof From the Taylor expansion of g_λ at x we have

$$g_\lambda(x_n) = g_\lambda(x) + \frac{g'_\lambda(x)}{1!}(x_n - x) + \ldots + \frac{g_\lambda^{(k)}(x)}{k!}(x_n - x)^k + \ldots$$

which yields, in view of $g_\lambda(\alpha) = \alpha$ and (2.8)

$$g_\lambda(x_n) - \alpha = \frac{g_\lambda^{(k+1)}(x)}{(k+1)!}(x_n - x)^{k+1} + \ldots.$$

This means that

$$\lim_{n\to\infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|^{k+1}} = \frac{|g_\lambda^{(k+1)}(\alpha)|}{(k+1)!},$$

which completes the proof.

Remark 2.1 *(Bumbariu [35]) Note that, generally, $g'(\alpha) \neq 0$, so $\{x_n\}$ has the Q-order of convergence equal to* 1*, see the following examples.*

In order to illustrate the practical implication of the Padé-type acceleration technique we will apply the algorithm to the following examples.

Example 2.1 *(Bumbariu [35]) Let be the function $f(x) = (\sin x)^2 - x^2 + 1 = 0$. $f(x)$ is continuous on $(1,2)$ and $f(1) \cdot f(2) < 0$. By Weierstrass' theorem, α, the root of $f(x)$, lies in $(1,2)$. We present the new Padé-type acceleration technique for $k = \overline{2,6}$.*

Let be $g(x) = \sqrt{1+(\sin x)^2}$ the equivalent operator of the fixed point problem $g(x) = x$.
First of all we show that g is a contraction on the interval $I = [1,2]$, where $g(I) \subset I$. In order to prove that g is a contraction we have to demonstrate the following relation:

$$|g(x) - g(y)| = |g'(t)||(x-y)|,$$

where $x, y, t \in (1,2)$. Replacing the function g in the above relation we obtain

$$1 + (\sin x)^2 - 1 - (\sin y)^2 = \frac{2\sin t\cos t}{2\sqrt{1+(\sin t)^2}}(x-y),$$

$$|(\sin x)^2 - (\sin y)^2| = \left|\frac{2\sin t\cos t}{2\sqrt{1+(\sin t)^2}}\right| |x-y|,$$

because $|g'(t)| \leq C = 0,3478 < 1$ for all $t \in (1,2)$ we obtain that g is a contraction on $[1,2]$.
In order to apply the new Padé-type acceleration technique we must known α, since $\alpha \in (1,2)$ we use an approximate value for α, $\alpha \cong 1.5$.
The values of the parameters λ_i involved in the accelerated technique are:
For $k = 2$ we have

$$g_\lambda(x) = \frac{\sqrt{1+(\sin x)^2} + \lambda_1 x + \lambda_2 x^2}{1 + \lambda_1 + \lambda_2 x},$$

where λ_1, and λ_2 are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.5 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \end{pmatrix} = \begin{pmatrix} -0.04995598832 \\ 0.7026746168 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1.103967914 \\ 0.7026746168 \end{pmatrix}.$$

So, the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\sqrt{1+(\sin x_n)^2} + \lambda_1 x_n + \lambda_2 x_n^2}{1 + \lambda_1 + \lambda_2 x_n}, \quad n \geq 0.$$

For $k = 3$ we have

$$g_\lambda(x) = \frac{\sqrt{1+(\sin x)^2} + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2},$$

where λ_1, λ_2 and λ_3 are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.5 & 1.5^2 \\ 0 & 1 & 2\cdot 1.5 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \end{pmatrix} = \begin{pmatrix} -0.04995598832 \\ 0.7026746168 \\ 0.1252663337 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -0.9630432881 \\ 0.5147751162 \\ 0.06263316685 \end{pmatrix}.$$

So, the accelerated iterative method is given in this case by:

$$g_\lambda(x_n) = \frac{\sqrt{1+(\sin x_n)^2} + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2}, \quad n \geq 0.$$

For $k = 4$ we have

$$g_\lambda(x) = \frac{\sqrt{1+(\sin x)^2} + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3},$$

where λ_1, λ_2, λ_3 and λ_4 are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.5 & 1.5^2 & 1.5^3 \\ 0 & 1 & 2 \cdot 1.5 & 3 \cdot 1.5^2 \\ 0 & 0 & 2 & 6 \cdot 1.5 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \\ -g^{(4)}(\alpha) \end{pmatrix} = \begin{pmatrix} -0.04995598832 \\ 0.7026746168 \\ 0.1252663337 \\ -1.772634866 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0.03406382402 \\ -1.479439108 \\ 1.392109316 \\ -0.2954391443 \end{pmatrix}.$$

So, the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\sqrt{1+(\sin x)^2} + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3 + \lambda_4 x_n^4}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2 + \lambda_4 x_n^3}, \quad n \geq 0.$$

For $k = 5$ we have

$$g_\lambda(x) = \frac{\sqrt{1+(\sin x)^2} + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4 + \lambda_5 x^5}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3 + \lambda_5 x^4},$$

where λ_i, $i = \overline{1,5}$ are determined from the following linear system of equations:

$$\begin{pmatrix} 1 & 1.5 & 1.5^2 & 1.5^3 & 1.5^4 \\ 0 & 1 & 2 \cdot 1.5 & 3 \cdot 1.5^2 & 4 \cdot 1.5^3 \\ 0 & 0 & 2 & 6 \cdot 1.5 & 12 \cdot 1.5^2 \\ 0 & 0 & 0 & 6 & 24 \cdot 1.5 \\ 0 & 0 & 0 & 0 & 24 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \\ -g^{(4)}(\alpha) \\ -g^{(5)}(\alpha) \end{pmatrix} =$$

$$= \begin{pmatrix} -0.04995598832 \\ 0.7026746168 \\ 0.1252663337 \\ -1.772634866 \\ 0.1373671362 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} -0.4994602957 \\ -0.4317081221 \\ 0.7193783305 \\ -0.1631142617 \\ 0.005723630675 \end{pmatrix}.$$

So, the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\sqrt{1+(\sin x_n)^2} + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3 + \lambda_4 x_n^4 + \lambda_5 x_n^5}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2 + \lambda_4 x_n^3 + \lambda_5 x_n^4}, \quad n \geq 0.$$

For $k = 6$ we have

$$g_\lambda(x) = \frac{\sqrt{1+(\sin x)^2} + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4 + \lambda_5 x^5 + \lambda_6 x^6}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3 + \lambda_5 x^4 + \lambda_6 x^5},$$

where λ_i, $i = \overline{1,6}$ are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.5 & 1.5^2 & 1.5^3 & 1.5^4 & 1.5^5 \\ 0 & 1 & 2\cdot 1.5 & 3\cdot 1.5^2 & 4\cdot 1.5^3 & 5\cdot 1.5^4 \\ 0 & 0 & 2 & 6\cdot 1.5 & 12\cdot 1.5^2 & 20\cdot 1.5^3 \\ 0 & 0 & 0 & 6 & 24\cdot 1.5 & 60\cdot 1.5^2 \\ 0 & 0 & 0 & 0 & 24 & 120\cdot 1.5 \\ 0 & 0 & 0 & 0 & 0 & 120 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \\ -g^{(4)}(\alpha) \\ -g^{(5)}(\alpha) \\ -g^{(6)}(\alpha) \end{pmatrix} =$$

$$= \begin{pmatrix} -0.04995598832 \\ 0.7026746168 \\ 0.1252663337 \\ -1.772634866 \\ 0.1373671362 \\ -1.931518358 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} = \begin{pmatrix} -0.3772313996 \\ -0.8391377758 \\ 1.262617869 \\ -0.5252739538 \\ 0.1264435280 \\ -0.01609598632 \end{pmatrix}.$$

So, the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\sqrt{1+(\sin x_n)^2} + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3 + \lambda_4 x_n^4 + \lambda_5 x_n^5 + \lambda_6 x_n^6}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2 + \lambda_4 x_n^3 + \lambda_5 x_n^4 + \lambda_6 x_n^5}, \quad n \geq 0.$$

Now we perform all computations and compare the results obtained for $k \in \{2, 3, 4, 5, 6\}$ to the initial Picard iteration (last column in Table 1).

Table 1

n	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	
	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g(x_n)$
0	1.5	1.5	1.5	1.5	1.5	1.5
1	1.407839387	1.407839386	1.407839387	1.407839385	1.407839388	1.412443361
2	1.404493085	1.404495094	1.404495963	1.404495477	1.404495477	1.405394334
3	1.404491648	1.404491648	1.404491651	1.404491648	1.404491650	1.404496296
4	1.404491648		1.404491647		1.404491648	1.404493062
5			1.404491649			1.404491813
6			1.404491646			
⋮						
9						1.404491648

The conclusion is that all g_λ-iterations accelerate the original g-iteration and that, amongst the accelerated iterations, the best empirical order of convergence is obtained in the cases $k \in \{3,5\}$.

Example 2.2 *(Bumbariu [35]) Let be the function* $f(x) = (x-1)^3 - 1$ *which has a unique root in the interval* $(1,3)$. *We observe that the function* $g(x) = \sqrt[3]{3x^2 - 3x + 2}$ *is a contraction on* $[1,3]$. *The new Padé-type acceleration technique is presented for* $k = \overline{2,6}$

First of all we show that g is a contraction on the interval $I = [1,3]$, where $g(I) \subset I$. In order to prove that g is a contraction we have to demonstrate the following relation:

$$|g(x) - g(y)| = |g'(t)||(x-y)|,$$

where $x, y, t \in (1,3)$. Replacing the function g in the above relation we obtain

$$\sqrt[3]{3x^2 - 3x + 2} - \sqrt[3]{3y^2 - 3y + 2}| = |\frac{2t-1}{\sqrt[3]{(3t^2 - 3t + 2)^2}}||(x-y)|,$$

because $|g'(t)| \leq C = 0,7646 < 1$ for all $t \in (1,3)$ we obtain that g is a contraction on $[1,3]$.
In order to apply the new Padé-type acceleration technique we must known α, since $\alpha \in (1,3)$ we use an approximate value for α, $\alpha \cong 1.7$.
The values of the parameters λ_i involved in the accelerated technique are:
For $k = 2$ we have

$$g_\lambda(x) = \frac{\sqrt[3]{3x^2 - 3x + 2} + \lambda_1 x + \lambda_2 x^2}{1 + \lambda_1 + \lambda_2 x},$$

where λ_1 and λ_2 are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.7 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \end{pmatrix} = \begin{pmatrix} -0.7637908387 \\ 0.0217115888 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -0.8007005397 \\ 0.0217115888 \end{pmatrix}.$$

So, the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\sqrt[3]{3x_n^2 - 3x_n + 2} + \lambda_1 x_n + \lambda_2 x_n^2}{1 + \lambda_1 + \lambda_2 x_n}, \quad n \geq 0.$$

For $k = 3$ we have

$$g_\lambda(x) = \frac{\sqrt[3]{3x^2 - 3x + 2} + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2},$$

where λ_1, λ_2 and λ_3 are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.7 & 1.7^2 \\ 0 & 1 & 2 \cdot 1.7 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \end{pmatrix} = \begin{pmatrix} -0.7637908387 \\ 0.0217115888 \\ 0.227476233 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -0.4719973830 \\ -0.3649980073 \\ 0.1137381165 \end{pmatrix}.$$

So, the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\sqrt[3]{3x_n^2 - 3x_n + 2} + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2}, \quad n \geq 0.$$

For $k = 4$ we have

$$g_\lambda(x) = \frac{\sqrt[3]{3x^2 - 3x + 2} + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3},$$

where λ_1, λ_2, λ_3 and λ_4 are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.7 & 1.7^2 & 1.7^3 \\ 0 & 1 & 2 \cdot 1.7 & 3 \cdot 1.7^2 \\ 0 & 0 & 2 & 6 \cdot 1.7 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \\ -g^{(4)}(\alpha) \end{pmatrix} = \begin{pmatrix} -0.7637908387 \\ 0.0217115888 \\ 0.227476233 \\ -0.830894561 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0.2083667801 \\ -1.565640648 \\ 0.8199984934 \\ -0.1384824268 \end{pmatrix}.$$

So, the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\sqrt[3]{3x_n^2 - 3x_n + 2} + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3 + \lambda_4 x_n^4}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2 + \lambda_4 x_n^3}, \quad n \geq 0.$$

For $k = 5$ we have

$$g_\lambda(x) = \frac{\sqrt[3]{3x^2 - 3x + 2} + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4 + \lambda_5 x^5}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3 + \lambda_5 x^4},$$

where λ_i, $i=\overline{1,5}$ are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.7 & 1.7^2 & 1.7^3 & 1.7^4 \\ 0 & 1 & 2\cdot 1.7 & 3\cdot 1.7^2 & 4\cdot 1.7^3 \\ 0 & 0 & 2 & 6\cdot 1.7 & 12\cdot 1.7^2 \\ 0 & 0 & 0 & 6 & 24\cdot 1.7 \\ 0 & 0 & 0 & 0 & 24 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \\ -g^{(4)}(\alpha) \\ -g^{(5)}(\alpha) \end{pmatrix} = \begin{pmatrix} -0.7637908387 \\ 0.0217115888 \\ 0.227476233 \\ -0.830894561 \\ 2.79466462 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} -576.5167080 \\ 675.7902651 \\ -197.0562299 \\ -0.9303040692 \\ 0.1164443592 \end{pmatrix}.$$

So, the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\sqrt[3]{3x_n^2 - 3x_n + 2} + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3 + \lambda_4 x_n^4 + \lambda_5 x_n^5}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2 + \lambda_4 x_n^3 + \lambda_5 x_n^4}, \quad n \geq 0.$$

For $k=6$ we have

$$g_\lambda(x) = \frac{\sqrt[3]{3x^2 - 3x + 2} + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4 + \lambda_5 x^5 + \lambda_6 x^6}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3 + \lambda_5 x^4 + \lambda_6 x^5},$$

where λ_i, $i=\overline{1,6}$ are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.7 & 1.7^2 & 1.7^3 & 1.7^4 & 1.7^5 \\ 0 & 1 & 2\cdot 1.7 & 3\cdot 1.7^2 & 4\cdot 1.7^3 & 5\cdot 1.7^4 \\ 0 & 0 & 2 & 6\cdot 1.7 & 12\cdot 1.7^2 & 20\cdot 1.7^3 \\ 0 & 0 & 0 & 6 & 24\cdot 1.7 & 60\cdot 1.7^2 \\ 0 & 0 & 0 & 0 & 24 & 120\cdot 1.7 \\ 0 & 0 & 0 & 0 & 0 & 120 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \\ -g^{(4)}(\alpha) \\ -g^{(5)}(\alpha) \\ -g^{(6)}(\alpha) \end{pmatrix} = \begin{pmatrix} -0.7637908387 \\ 0.0217115888 \\ 0.227476233 \\ -0.830894561 \\ 2.79466462 \\ -9.39964151 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} = \begin{pmatrix} -3878.575861 \\ 4558.603071 \\ -1336.173719 \\ -3.194051066 \\ 0.7822522995 \\ -0.07833034592 \end{pmatrix}.$$

So, the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\sqrt[3]{3x_n^2 - 3x_n + 2} + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3 + \lambda_4 x_n^4 + \lambda_5 x_n^5 + \lambda_6 x_n^6}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2 + \lambda_4 x_n^3 + \lambda_5 x_n^4 + \lambda_6 x_n^5}, \quad n \geq 0.$$

Now we perform all computations and compare the results obtained for $k \in \{2, 3, 4, 5, 6\}$ to the initial Picard iteration (last column in Table 2).

Table 2

n	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	
	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g(x_n)$
0	1.7	1.7	1.7	1.7	1.7	1.7
1	2.007486966	2.007486965	2.007486965	2.007483132	2.007504812	1.772631238
2	1.999773454	2.000100489	1.999981347	2.007583551	2.007519698	1.828035437
3	2.000006791	2.000001178	2.000000064	2.007685218	2007534450	1.870174554
4	1.999999796	2.000000017	1.999999996	2.007788187	2.0075549345	1.902133792
5	2.000000006	1.999999998	2.000000008	2.007892466	2.007564277	1.926313267
6	1.999999999	2.000000004	2.000000004	2.007998129	2.007579226	1.944570353
7	1.999999997	2.000000000	1.999999996	2.008105061	2.007594173	1.958333871
8	1.999999997		2.000000008	2.008213428	2.007609148	1.968697038
9			2.000000004	2.008323118		1.976492529
10			1.999999996	2008434163		1.982352284
11			2.000000008			1.986754546
⋮						
32						1.999976320

The conclusion is that all g_λ-iterations accelerate the original g-iteration and that, amongst the accelerated iterations, the best empirical order of convergence is obtained in the case $k = 3$.

Example 2.3 *(Bumbariu [35]) Let be the function* $f(x) = \cos(x) - x$ *which has a unique root in the interval* $(0, 1)$*. We observe that* $g(x) = \cos(x)$ *is a contraction on* $[0, 1]$*. We present the new Padé-type acceleration technique for* $k = \overline{2, 6}$

First of all we show that g is a contraction on the interval $I = [0, 1]$, where $g(I) \subset I$. In order to prove that g is a contraction we have to demonstrate the following relation:

$$|g(x) - g(y)| = |g'(t)||(x - y)|,$$

where $x, y, t \in (0, 1)$. Replacing the function g in the above relation we obtain

$$|\cos x - \cos y| = |-sint||(x - y)|,$$

because $|g'(t)| \leq C = 0,8414 < 1$ for all $t \in (0, 1)$ we obtain that g is a contraction on $[0, 1]$.

In order to apply the new Padé-type acceleration technique we must known α, since $\alpha \in (0,1)$ we use an approximate value for α, $\alpha \cong 0.5$.
The values of the parameters λ_i involved in the accelerated technique are:
For $k = 2$ we have

$$g_\lambda(x) = \frac{\cos x + \lambda_1 x + \lambda_2 x^2}{1 + \lambda_1 + \lambda_2 x},$$

where λ_1 and λ_2 are determined from the following system of linear equation:

$$\begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \end{pmatrix} = \begin{pmatrix} 0.4794255386 \\ 0.8775825619 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0.04063425765 \\ 0.8775825619 \end{pmatrix}.$$

So, the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\cos x_n + \lambda_1 x_n + \lambda_2 x_n^2}{1 + \lambda_1 + \lambda_2 x_n}, \quad n \geq 0.$$

For $k = 3$ we have

$$g_\lambda(x) = \frac{\cos x + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2},$$

where λ_1, λ_2 and λ_3 are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 0.5 & 0.5^2 \\ 0 & 1 & 2 \cdot 0.5 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \end{pmatrix} = \begin{pmatrix} 0.4794255386 \\ 0.8775825619 \\ -0.4794255386 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -0.01929393468 \\ 1.117295331 \\ -0.2397127693 \end{pmatrix}.$$

So, the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\cos x_n + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2}, \quad n \geq 0.$$

For $k = 4$ we have

$$g_\lambda(x) = \frac{\cos x + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3},$$

where λ_1, λ_2, λ_3 and λ_4 are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 0.5 & 0.5^2 & 0.5^3 \\ 0 & 1 & 2 \cdot 0.5 & 3 \cdot 0.5^2 \\ 0 & 0 & 2 & 6 \cdot 0.5 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \\ -g^{(4)}(\alpha) \end{pmatrix} = \begin{pmatrix} 0.4794255386 \\ 0.8775825619 \\ -0.4794255386 \\ -0.8775825619 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} -0.001010964635 \\ 1.007597511 \\ -0.02031712882 \\ -0.1462637603 \end{pmatrix}.$$

So, the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\cos x_n + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3 + \lambda_4 x_n^4}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2 + \lambda_4 x_n^3}, \quad n \geq 0.$$

For $k = 5$ we have

$$g_\lambda(x) = \frac{\cos x + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4 + \lambda_5 x^5}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3 + \lambda_5 x^4},$$

where λ_i, $i = \overline{1,5}$ are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 0.5 & 0.5^2 & 0.5^3 & 0.5^4 \\ 0 & 1 & 2 \cdot 0.5 & 3 \cdot 0.5^2 & 4 \cdot 0.5^3 \\ 0 & 0 & 2 & 6 \cdot 0.5 & 12 \cdot 0.5^2 \\ 0 & 0 & 0 & 6 & 24 \cdot 0.5 \\ 0 & 0 & 0 & 0 & 24 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \\ -g^{(4)}(\alpha) \\ -g^{(5)}(\alpha) \end{pmatrix} = \begin{pmatrix} 0.4794255386 \\ 0.8775825619 \\ -0.4794255386 \\ -0.8775825619 \\ 0.4794255386 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} 0.0002375393714 \\ 0.9976094789 \\ 0.00964697338 \\ -0.1862158885 \\ 0.01997606411 \end{pmatrix}.$$

So, the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\cos x_n + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3 + \lambda_4 x_n^4 + \lambda_5 x_n^5}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2 + \lambda_4 x_n^3 + \lambda_5 x_n^4}, \quad n \geq 0.$$

For $k = 6$ we have

$$g_\lambda(x) = \frac{\cos x + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4 + \lambda_5 x^5 + \lambda_6 x^6}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3 + \lambda_5 x^4 + \lambda_6 x^5},$$

where λ_i, $i = \overline{1,6}$ are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 0.5 & 0.5^2 & 0.5^3 & 0.5^4 & 0.5^5 \\ 0 & 1 & 2\cdot 0.5 & 3\cdot 0.5^2 & 4\cdot 0.5^3 & 5\cdot 0.5^4 \\ 0 & 0 & 2 & 6\cdot 0.5 & 12\cdot 0.5^2 & 20\cdot 0.5^3 \\ 0 & 0 & 0 & 6 & 24\cdot 0.5 & 60\cdot 0.5^2 \\ 0 & 0 & 0 & 0 & 24 & 120\cdot 0.5 \\ 0 & 0 & 0 & 0 & 0 & 120 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \\ -g^{(4)}(\alpha) \\ -g^{(5)}(\alpha) \\ -g^{(6)}(\alpha) \end{pmatrix} = \begin{pmatrix} 0.4794255386 \\ 0.8775825619 \\ -0.4794255386 \\ -0.8775825619 \\ 0.4794255386 \\ 0.8775825619 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{pmatrix} = \begin{pmatrix} 0.0000090022458559 \\ 0.9998948502 \\ 0.0005054823177 \\ -0.1679329185 \\ 0.001693094069 \\ 0.007313188016 \end{pmatrix}.$$

So, the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\cos x_n + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3 + \lambda_4 x_n^4 + \lambda_5 x_n^5 + \lambda_6 x_n^6}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2 + \lambda_4 x_n^3 + \lambda_5 x_n^4 + \lambda_6 x_n^5}, \quad n \geq 0.$$

Now we perform all computations and compare the results obtained for $k \in \{2, 3, 4, 5, 6\}$ to the initial Picard iteration (last column in Table 3).

Table 3

n	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	
	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g(x_n)$
0	0.5	0.5	0.5	0.5	0.5	0.5
1	0.7552224168	0.7552224168	0.7552224168	0.7552224173	0.7552224180	0.8775825619
2	0.7393111553	0.7391639529	0.7391407837	0.73914159935	0.7391416688	0.6390124942
3	0.7390872396	0.739085247	0.7390851314	0.7390851334	0.7390851349	0.8026851007
4	0.7390851525	0.7390851333	0.7390851332	0.7390851335	0.7390851334	0.6947780268
5	0.7390851331	0.7390851327		0.7390851341	0.7390851334	0.7681958313
6		0.7390851332		0.7390851335		0.719165449
7				0.7390851341		
⋮						
52						0.7390852281

The conclusion is that all g_λ-iterations accelerate the original g-iteration and that, amongst the accelerated iterations, the best empirical order of convergence is obtained in the case $k = 4$.

2.2 A Padé-type acceleration technique for solving initial value problems

In this section we show how to use the technique presented in the previous section to a differential equation by means of the Picard iteration.
Author's original contribution in this section are: the Padé-type acceleration techniques (2.17), (2.18) and the Example 2.4.
Consider the initial value problem for a first order ordinary differential equation

$$y' = f(x, y), \quad y(x_0) = y_0, \tag{2.12}$$

which can be written equivalently as a fixed point problem

$$y = Ty, \tag{2.13}$$

where the operator T is defined on $C[a, b]$ by

$$(Ty)(x) = y_0 + \int_{x_0}^{x} f(s, y(s))ds, \quad x \in [a, b]. \tag{2.14}$$

In the sequel we present a result, taken from [12], which ensures, under some assumptions, the existence and the uniqueness of the solution for an ordinary differential equation.

Theorem 2.2 *[12] Consider the initial value problem* (2.12)*, where* (x_0, y_0) *is an arbitrary point in the plane. Let use assume that:*
(1) f *is continuous on the domain* $D \subset \mathbb{R}^2$*, where*

$$D = \{(x, y) \in \mathbb{R}^2 |\ |x - x_0| \leq a, |y - y_0| \leq b\},$$

(2) f *satisfies the Lipschitz condition with respect to the second variable, that is, there exists* $L > 0$ *such that*

$$|f(x, y_1) - f(x, y_2)| \leq L \cdot |y_1 - y_2|, for\ each (x, y_1), (x, y_2) \in D,$$

(3) The following relation is fulfilled

$$h \cdot L < 1, where\ h = min\left(a, \frac{b}{M}\right)\ and\ M = \max_{(x,y)\in D} |f(x, y)|.$$

In these conditions there is one and only one function $\varphi(x)$*, defined and differentiable on the interval* $[x_0 - h, x_0 + h]$ *that is the solution of the equation* (2.12)*, this means*

$$\varphi'(x) = f(x, \varphi(x)),\ x \in [x_0 - h, x_0 + h],$$

and which satisfies the initial condition

$$\varphi(x_0) = y_0.$$

The integral equation (2.14) can be solved by means of the Picard iteration, generating a sequence of functions $y_i(x), i = \overline{0, n}$ of the form

$$y_{i+1}(x) = y_0 + \int_{x_0}^{x} f(s, y_i(s))ds, \tag{2.15}$$

where, under suitable assumptions, the sequence of iterates $\{y_i(x)\}$ will converge to the solution of the ordinary differential equation (2.12). Generally, the iterative scheme (2.15) has linear order of convergence. Therefore our aim in this section is to try to accelerate (2.15). To this end, we introduce two different techniques of acceleration, of the form:

$$y_{n+1} = (T_\lambda y_n)(x), \tag{2.16}$$

where $T_\lambda : C[a,b] \to C[a,b]$ is given as follows.
Case I. With T given by (2.14) the Padé-type accelerated iteration operator will be

$$T_\lambda y_n(x) = \frac{Ty_{n-1}(x) + \lambda_1 x + \lambda_2 x^2 + \ldots + \lambda_k x^k}{1 + \lambda_1 + \lambda_2 x + \ldots + \lambda_k x^{k-1}}. \tag{2.17}$$

Case II. With T given by (2.14) the Padé-type accelerated iteration operator will be

$$T_\lambda y_n(x) = \frac{Ty_{n-1}(x) + \lambda_1 y_{n-1}(x) + \lambda_2 y_{n-1}^2(x) + \ldots + \lambda_k y_{n-1}^k(x)}{1 + \lambda_1 + \lambda_2 y_{n-1}(x) + \ldots + \lambda_k y_{n-1}^{k-1}(x)}. \tag{2.18}$$

In both cases, the parameters, $\lambda_i, i = \overline{1,k}$, will be determined in a unique way, as in the previous paragraph, from the upper diagonal linear system of equations

$$\begin{pmatrix} 1 & \alpha & \alpha^2 & \ldots & \alpha^{k-1} \\ 0 & 1 & 2\alpha & \ldots & (k-1)\alpha^{k-2} \\ 0 & 0 & 2 & \ldots & (k-1)(k-2)\alpha^{k-3} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & (k-1)! \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} -Ty'(\alpha) \\ -Ty^{(2)}(\alpha) \\ -Ty^{(3)}(\alpha) \\ \vdots \\ -T^{(k)}(\alpha) \end{pmatrix}. \tag{2.19}$$

To derive the solutions of the system, we need α which is unknown. Since $\alpha \in (a,b)$, we consider the approximation $\alpha \approx \frac{a+b}{2}$. In order to see the performance of the new acceleration techniques, in the sequel we apply the two methods to a concrete initial value problem. To see the efficiency of the acceleration method we shall compare the sequence that arises when applying the Padé-type acceleration technique, in both cases, with the initial sequence of successive approximations (2.15) and with the 4th order Runge-Kutta method [67] for the general ordinary differential equation, (2.12), defined by

$$\begin{aligned} \breve{y}(x) &= y(x) + \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)h \\ K_1 &= f(x, y(x)) \\ K_2 &= f\left(x + \frac{h}{2}, y(x) + K_1\frac{h}{2}\right) \\ K_3 &= f\left(x + \frac{h}{2}, y(x) + K_2\frac{h}{2}\right) \\ K_4 &= f(x+h, y(x) + K_3 h), \end{aligned} \tag{2.20}$$

where $h = \frac{b-a}{N}$, $N \in \mathbb{N}$ is the step size. All the numerical computations listed in the tables were done with Maple 13 using 9 digit floating point arithmetic.

Example 2.4 *(Bumbariu [40]) Consider the Cauchy problem for the Riccati equation*

$$y' = ycosx, \quad y(0) = 1. \tag{2.21}$$

The sequence of successive approximations associated to problem (2.4) is

$$y_{n+1}(x) = 1 + \int_0^x y_n(s)\cos s ds, \ n \geq 0. \tag{2.22}$$

Using $\alpha \approx 0.5$, $(Ty)(x) = 1 + \int_0^x y(s)\cos s ds$, we present the techniques for $k = 2$ and for $k = 3$.
For $k = 2$, λ_1 and λ_2 are determined from the following system of linear equations

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -Ty'(\alpha) \\ -Ty"(\alpha) \end{pmatrix},$$

which has the unique solution

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0.5y'(0.5)\cos 0.5 - 0.5y(0.5)\sin 0.5 - y(0.5)\cos 0.5 \\ -y'(0.5)\cos 0.5 + y(0.5)\sin 0.5 \end{pmatrix}.$$

Method 1.
The iterative method is given by

$$y_{n+1}(x) = \frac{1 + \int_0^x y_n(s)\cos s ds + \lambda_1 x + \lambda_2 x^2}{1 + \lambda_1 + \lambda_2 x}, \quad n \geq 0.$$

Method 2.
The iterative method is given by

$$y_{n+1}(x) = \frac{1 + \int_0^x y_n(s)\cos s ds + \lambda_1 y_n(x) + \lambda_2 y_n^2(x)}{1 + \lambda_1 + \lambda_2 y_n(x)}, \quad n \geq 0.$$

Note that the exact solution for the Riccati equation (2.21) is $y(x) = e^{\sin x}$. For $k = 2$, in both cases, the acceleration technique is not effective. For $k = 3$, λ_1, λ_2 and λ_3 are determined from the following system of linear equations

$$\begin{pmatrix} 1 & \alpha & \alpha^2 \\ 0 & 1 & 2\alpha \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -Ty'(\alpha) \\ -Ty"(\alpha) \\ -Ty^{(3)}(\alpha) \end{pmatrix},$$

which has the unique solution

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}0.5^2 y"(0.5)\cos 0.5 + 0.5^2 y'(0.5)\sin 0.5 + \frac{1}{2}0.5^2 y(0.5)\cos 0.5 + \\ +0.5y'(0.5)\cos 0.5 - 0.5y(0.5)\sin 0.5 - y(0.5)\cos 0.5 \\ 0.5y"(0.5)\cos 0.5 - 2 * 0.5y'(0.5)\sin 0.5 - 0.5y(0.5)\cos 0.5 - y'(0.5)\cos 0.5 + \\ +y(0.5)\sin 0.5 \\ -\frac{1}{2}y"(0.5)\cos 0.5 + y'(0.5)\sin 0.5 + \frac{1}{2}y(0.5)\cos 0.5 \end{pmatrix}.$$

Method 1.
The iterative method is

$$y_{n+1}(x) = \frac{1 + \int_0^x y_n(s)\cos s ds + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2}, \quad n \geq 0.$$

Method 2.
The iterative method is

$$y_{n+1}(x) = \frac{1 + \int\limits_0^x y_n(s)\cos s ds + \lambda_1 y_n(x) + \lambda_2 y_n^2(x) + \lambda_3 y_n^3(x)}{1 + \lambda_1 + \lambda_2 y_n(x) + \lambda_3 y_n^2(x)}, \quad n \geq 0.$$

The same conclusion, as for the case $k = 2$, is obtained for $k = 3$: the acceleration technique is not effective.

As we said, in the previous section, the convergence order for the Picard iteration is generally linear, so the iterative method converges rather slowly to the solution of the initial value problem (2.12). Motivated by the results obtained in Paragraph 2.1, our aim in this section was to give two different techniques of acceleration, of the form (2.16), in order to improve the convergence speed of (2.12). As a global conclusion of this section we can say, for all our empirical studies, that the proposed acceleration technique, in the first case gives a divergent sequence and in the second case gives a sequence that has not a better convergence speed than the Picard iteration. So, there is no useful to apply the acceleration technique of the form (2.16) to IVP. (initial value problem)

2.3 A Padé-type acceleration technique for the Krasnoselskij iteration

In this section we present a Padé-type acceleration technique, given in two ways, for accelerating the convergence speed for the Krasnoselskij iteration associated to an operator.
Author's original contribution in this paragraph are: the Padé-type acceleration technique, given in two ways, for accelerating the Krasnoselskij iteration procedure (2.25), (2.26) and the Example 2.6.
Let K be a nonempty closed convex and bounded subset of a real uniformly convex Banach space and $T : K \to K$ be a nonexpansive mapping (i.e. $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$). Then T has a fixed point $x^* \in K$ (see [12]). Different from Banach contraction mapping principle, simple examples indicate that the sequence of successive approximations $x_{n+1} = Tx_n,\ x_0 \in K,\ n \geq 0$, for a nonexpansive map T, which has a unique fixed point may not converge to the fixed point. For example, we present the following application:

Example 2.5 *[43] Let $B := \{x \in \mathbb{R}^2 | \|x\| \leq 1\}$ and let T denote an anticlockwise rotation of $\frac{\pi}{4}$ about the origin of coordinates. Then T is nonexpansive with the origin as the only fixed point. Moreover, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n,\ x_0 \in (0,1) \in B,\ n \geq 0$, does not converge to zero.*

Krasnoselskij [73] showed that in the above example we can obtain a convergent sequence of successive approximations if instead of T one takes the averaged nonexpansive mapping $\frac{1}{2}(I + T)$, where I denotes the identity transformation of the plane, if the sequence of successive approximations is defined as follows:

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n),\ x_0 \in K,\ n \geq 0 \tag{2.23}$$

in place of the Picard iteration ($x_{n+1} = Tx_n,\ x_0 \in K,\ n \geq 0$). It is easy to check that the mapping T and $\frac{1}{2}(I + T)$ have the same set of fixed points, consequently the limit

of the sequence (2.23) must be a fixed point of T.
Let E be a normed linear space and K a convex subset of E, a generalization of (2.23) which has turned out to be successful in the approximation of fixed points of nonexpansive mappings $T : K \to K$ (when they exist), is the next relation [107]

$$x_{n+1} = (1-\mu)x_n + \mu T x_n, \;\; x_0 \in K, \;\; \mu \in (0,1), \;\; n \geq 0. \tag{2.24}$$

the above relation is called *Krasnoselskij iteration* or *Krasnoselskij iteration procedure*. Krasnoselskij iteration is mainly associated with Lipschitzian and pseudocontractive type conditions, one important observation about Krasnoselskij iteration is that we do not have theoretical results concerning its convergence order.
In order to improve the convergence speed for the Krasnoselskij iteration (2.24) we consider the following two Padé-type acceleration methods: First method

$$x_{n+1} = G_\lambda(x_n), \tag{2.25}$$

where $G_\lambda(x_n)$ is of the form

$$G_\lambda(x_n) = \frac{(1-\mu)x_n + \mu g(x_n) + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \ldots + \lambda_k x^k}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \ldots + x^{k-1}}$$

Second method

$$x_{n+1} = (1-\mu)x_n + \mu g_\lambda(x_n), \tag{2.26}$$

where g_λ is of the form

$$g_\lambda(x) = \frac{g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3 + \ldots + \overline{\lambda_k}x^k}{1 + \overline{\lambda_1} + \overline{\lambda_2}x + \overline{\lambda_3}x^2 + \ldots + \overline{\lambda_k}x^{k-1}}, \tag{2.27}$$

$k \in \mathbb{N}, \;\; k \geq 2$ and $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k; \overline{\lambda_1}, \overline{\lambda_2}, \overline{\lambda_3}, \ldots, \overline{\lambda_k} \in \mathbb{R}$ are parameters that should be determined in such a way that the new iteration functions g_λ and G_λ yield faster Krasnoselskij iterations.

To apply the technique we must know α, since α is in $\in (a,b)$, we consider $\alpha \cong \frac{a+b}{2}$. In the sequel we present the technique for both cases, for $k = \overline{2,4}$.
Method P1 (i) $k = 2$. In this case, we have

$$x + \lambda_1 x + \lambda_2 x^2 = G(x) + \lambda_1 x + \lambda_2 x^2,$$
$$g_\lambda(x) = \frac{G(x) + \lambda_1 x + \lambda_2 x^2}{1 + \lambda_1 + \lambda_2 x},$$

where $G(x) = (1-\mu)x + \mu g(x)$, which leads to the following system of linear equations:

$$\begin{cases} \lambda_1 + \alpha\lambda_2 = -G'(\alpha) \\ \lambda_2 = -G^{(2)}(\alpha) \end{cases}$$

the parameters λ_i:

$$\begin{cases} \lambda_1 = -G'(\alpha) + \alpha G^{(2)}(\alpha) \\ \lambda_2 = -G^{(2)}(\alpha). \end{cases}$$

So, the accelerated iterative method is given by:

$$x_{n+1} = \frac{G(x_n) + \lambda_1 x_n + \lambda_2 x_n^2}{1 + \lambda_1 + \lambda_2 x_n}, \quad n \geq 0$$

where

$$G(x_n) = (1-\mu)x_n + \mu g(x_n).$$

(ii) $k = 3$. In this case, we have

$$x + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 = G(x) + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3,$$
$$g_\lambda(x) = \frac{G(x) + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2},$$

where $G(x) = (1-\mu)x + \mu g(x)$, which leads to the following system of linear equations:

$$\begin{cases} \lambda_1 + \alpha\lambda_2 + \alpha^2\lambda_3 = -G'(\alpha) \\ \lambda_2 + 2\alpha\lambda_3 = -G^{(2)}(\alpha) \\ 2\lambda_3 = -G^{(3)}(\alpha) \end{cases}$$

the parameters λ_i:

$$\begin{cases} \lambda_1 = -G'(\alpha) + \alpha G^{(2)}(\alpha) - \frac{\alpha^2}{2} G^{(3)}(\alpha) \\ \lambda_2 = -G^{(2)}(\alpha) + \alpha G^{(3)}(\alpha) \\ \lambda_3 = -\frac{G^{(3)}(\alpha)}{2}. \end{cases}$$

So, the accelerated iterative method is given by:

$$x_{n+1} = \frac{G(x_n) + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2}, \quad n \geq 0$$

where $G(x_n)$ is given by (2.28). (iii) k=4. In this case, we have

$$x + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4 = G(x) + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4,$$
$$g_\lambda(x) = \frac{G(x) + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3},$$

where $G(x) = (1-\mu)x + \mu g(x)$, which leads to the following system of linear equations:

$$\begin{cases} \lambda_1 + \alpha\lambda_2 + \alpha^2\lambda_3 + \alpha^3\lambda_4 = -G'(\alpha) \\ \lambda_2 + 2\alpha\lambda_3 + 3\alpha^2\lambda_4 = -G^{(2)}(\alpha) \\ 2\lambda_3 + 6\alpha\lambda_4 = -G^{(3)}(\alpha) \\ 6\lambda_4 = -G^{(4)}(\alpha) \end{cases}$$

from where we determine the parameters λ_i in a unique way.
So, the accelerated iterative method is given by:

$$x_{n+1} = \frac{G(x_n) + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3 + \lambda_4 x_n^4}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2 + \lambda_4 x_n^3}, \quad n \geq 0$$

where $G(x_n)$ is given by (2.28).
Method P2 (i) $k = 2$. In this case, we have

$$x + \overline{\lambda_1}x + \overline{\lambda_2}x^2 = g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2,$$

$$g_\lambda(x) = \frac{g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2}{1 + \overline{\lambda_1} + \overline{\lambda_2}x},$$

which leads to the following system of linear equations:

$$\begin{cases} \overline{\lambda_1} + \alpha\overline{\lambda_2} = -g'(\alpha) \\ \overline{\lambda_2} = -g^{(2)}(\alpha), \end{cases}$$

the parameters λ_i:

$$\begin{cases} \overline{\lambda_1} = -g'(\alpha) + \alpha g^{(2)}(\alpha) \\ \overline{\lambda_2} = -g^{(2)}(\alpha). \end{cases}$$

So, the accelerated iterative method is given by:

$$x_{n+1} = (1-\mu)x_n + \mu\frac{g(x_n) + \overline{\lambda_1}x_n + \overline{\lambda_2}x_n^2}{1 + \overline{\lambda_1} + \overline{\lambda_2}x_n}, \quad n \geq 0.$$

(ii) $k = 3$. In this case, we have

$$x + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3 = g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3,$$

$$g_\lambda(x) = \frac{g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3}{1 + \overline{\lambda_1} + \overline{\lambda_2}x + \overline{\lambda_3}x^2},$$

which leads to the following system of linear equations:

$$\begin{cases} \overline{\lambda_1} + \alpha\overline{\lambda_2} + \alpha^2\overline{\lambda_3} = -g'(\alpha) \\ \overline{\lambda_2} + 2\alpha\overline{\lambda_3} = -g^{(2)}(\alpha) \\ 2\overline{\lambda_3} = -g^{(3)}(\alpha), \end{cases}$$

the parameters λ_i:

$$\begin{cases} \overline{\lambda_1} = -g'(\alpha) + \alpha g^{(2)}(\alpha) - \frac{\alpha^2}{2}g^{(3)}(\alpha) \\ \overline{\lambda_2} = -g^{(2)}(\alpha) + \alpha g^{(3)}(\alpha) \\ \overline{\lambda_3} = -\frac{g^{(3)}(\alpha)}{2}. \end{cases}$$

So, the accelerated iterative method is given by:

$$x_{n+1} = (1-\mu)x_n + \mu\frac{g(x_n) + \overline{\lambda_1}x_n + \overline{\lambda_2}x_n^2 + \overline{\lambda_3}x_n^3}{1 + \overline{\lambda_1} + \overline{\lambda_2}x_n + \overline{\lambda_3}x_n^2}, \quad n \geq 0.$$

(iii) k=4. In this case, we have

$$x + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3 + \overline{\lambda_4}x^4 = g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3 + \overline{\lambda_4}x^4,$$

$$g_\lambda(x) = \frac{g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3 + \overline{\lambda_4}x^4}{1 + \overline{\lambda_1} + \overline{\lambda_2}x + \overline{\lambda_3}x^2 + \overline{\lambda_4}x^3},$$

which leads to the following system of linear equations:

$$\begin{cases} \overline{\lambda_1} + \alpha\overline{\lambda_2} + \alpha^2\overline{\lambda_3} + \alpha^3\overline{\lambda_4} = -g'(\alpha) \\ \overline{\lambda_2} + 2\alpha\overline{\lambda_3} + 3\alpha^2\overline{\lambda_4} = -g^{(2)}(\alpha) \\ 2\overline{\lambda_3} + 6\alpha\overline{\lambda_4} = -g^{(3)}(\alpha) \\ 6\overline{\lambda_4} = -g^{(4)}(\alpha), \end{cases}$$

from where we can determine in a unique way the parameters $\overline{\lambda_i}$.
So, the accelerated iterative method is given by:

$$x_{n+1} = (1-\mu)x_n + \mu\frac{g(x_n) + \overline{\lambda_1}x_n + \overline{\lambda_2}x_n^2 + \overline{\lambda_3}x_n^3 + \overline{\lambda_4}x_n^4}{1 + \overline{\lambda_1} + \overline{\lambda_2}x_n + \overline{\lambda_3}x_n^2 + \overline{\lambda_4}x_n^3}, \quad n \geq 0.$$

In order to see the practical performance of the techniques presented above, we will apply the method, in both cases, for the Lipschitzian operator $Tx = \frac{1}{x}$, $T : \left[\frac{1}{2}, 2\right] \to \left[\frac{1}{2}, 2\right]$.

Example 2.6 *[12] Let H be the real line $\mathbb{R}$ endowed with the Euclidean inner product and norm, $K = \left[\frac{1}{2}, 2\right]$ and $T : K \to K$ be a function given by $Tx = \frac{1}{x}$ for all $x \in K$. Then T is Lipschitzian with constant $s = 4$ (so T is a generalized pseudo-contractive operator with constant $r = 4$).*
Moreover, T is a generalized pseudocontractive operator with any constant $r > 0$. It is easy to see that T has a unique fixed point, $F_T = \{1\}$, and that, for any initial choice $x_0 = a \neq 1$, The Picard iteration yields a oscillatory sequence

$$a, \frac{1}{a}, a, \frac{1}{a}, \ldots.$$

Method P1

• $\mu = \frac{1}{4}$, $1 - \mu = \frac{3}{4}$, $x_{n+1} = \frac{3}{4}x_n + \frac{1}{4x_n}$
for the initial value $x_0 = 1.25$ the best convergence rate is obtained for $k = 3$.

$$g_\lambda(x_n) = \frac{G(x_n) + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2}, \quad n \geq 0$$

λ_1 and λ_2 are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.25 & 1.25^2 \\ 0 & 1 & 2 \cdot 1.25 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \end{pmatrix} = \begin{pmatrix} -0.59 \\ -0.256 \\ 0.6144 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0.21 \\ -1.024 \\ 0.3072 \end{pmatrix}.$$

So, the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\frac{3}{4}x_n + \frac{1}{4x_n} + 0.21x_n - 1.024x_n^2 + 0.3072x_n^3}{1 + 0.21 - 1.024x_n + 0.3072x_n^2}, \quad n \geq 0.$$

For the initial value $x_0 = 1.5$ the best convergence rate is obtained for $k = 4$.

$$g_\lambda(x_n) = \frac{G(x_n) + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3 + \lambda_4 x_n^4}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2 + \lambda_4 x_n^3}, \quad n \geq 0$$

λ_1, λ_2, λ_3 and λ_4 are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.5 & 1.5^2 & 1.5^3 \\ 0 & 1 & 2\cdot 1.5 & 3\cdot 1.5^2 \\ 0 & 0 & 2 & 6\cdot 1.5 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \\ -g^{(4)}(\alpha) \end{pmatrix} = \begin{pmatrix} -0.6388888889 \\ -0.1481481482 \\ 0.2962962963 \\ -0.7901234568 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0.361111112 \\ -1.481481482 \\ 0.7407407408 \\ -0.1316872428 \end{pmatrix}.$$

So, the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\frac{3}{4}x_n + \frac{1}{4x_n} + 0.361111112x_n - 1.481481482x_n^2 + 0.7407407408x_n^3 - 0.1316872428x_n^4}{1 + 0.361111112 - 1.481481482x_n + 0.7407407408x_n^2 - 0.1316872428x_n^3},$$

$n \geq 0$. The results are listed in Table 1.

Table 1

	$x_{n+1} = G_\lambda(x_n)$	$x_{n+1} = G_\lambda(x_n)$	*Kr. it.*	*Kr. it.*
0	1.25	1.5	1.25	1.5
1	0.9756097561	0.9230769245	1.137500000	1.291666667
2	1.000139402	0.9999491490	1.072905220	1.162298387
3	0.9999980722	1.000001176	1.037691110	1.086614857
4	1.000000026	0.9999999738	1.019187810	1.045141123
5	1.000000000	1.000000000	1.009684215	1.023057989
			...	...
...				
27			1.000000001	
28				1.000000001

When comparing the results listed in Table 1, we can see that the new acceleration technique, in the first case, really accelerates the convergence speed of the Krasnoselskij iteration.

• $\mu = \frac{1}{2}$, $1-\mu = \frac{1}{2}$, $x_{n+1} = \frac{1}{2}x_n + \frac{1}{2x_n}$
for the initial value $x_0 = 1.25$ the accelerated iterative method is given by

$$g_\lambda(x_n) = \frac{\frac{1}{2}x_n + \frac{1}{2x_n} + 1.42x_n - 2.048x_n^2 + 0.6144x_n^3}{1 + 1.42 - 2.048x_n + 0.6144x_n^2}, \quad n \geq 0$$

for the initial value $x_0 = 1.5$ the accelerated iterative method is given by

$$g_\lambda(x_n) = \frac{\frac{1}{2}x_n + \frac{1}{2x_n} + 1.722222222x_n - 2.962962963x_n^2 + 1.481481482x_n^3 - 0.2633744857x_n^4}{1 + 1.722222222 - 2.962962963x_n + 1.481481482x_n^2 - 0.2633744857x_n^3},$$

$n \geq 0$
when we have the initial value 1.25 the best convergence rate is obtained for $k = 3$, and for the initial value 1.5 the best convergence rate is obtained for $k = 4$, the results are listed in Table 2.

Table 2

	$x_{n+1} = G_\lambda(x_n)$	$x_{n+1} = G_\lambda(x_n)$	$Kr.\ it.$	$Kr.\ it.$
0	1.25	1.5	1.25	1.5
1	0.9756097561	0.9230769228	1.025000000	1.083333333
2	1.000139402	0.9999491479	1.000304878	1.003205128
3	0.999998721	1.000001177	1.000000046	1.000005120
4	1.000000027	0.9999999724	1.000000000	1.000000000
5	0.9999999986	1.000000000		
6	1.000000000			

When comparing the results listed in Table 2, we can see that the new acceleration technique, in the first case, does not accelerate the convergence speed of the Krasnoselskij iteration.

• $\mu = \frac{2}{3}$, $1-\mu = \frac{1}{3}$, $x_{n+1} = \frac{1}{3}x_n + \frac{2}{3x_n}$
for the initial value $0 = 1.25$ the best convergence rate is obtained for $k = 4$

$$g_\lambda(x_n) = \frac{G(x_n) + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3 + \lambda_4 x_n^4}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2 + \lambda_4 x_n^3}, \quad n \geq 0$$

λ_1, λ_2, λ_3 and λ_4 are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.25 & 1.25^2 & 1.25^3 \\ 0 & 1 & 2\cdot 1.25 & 3\cdot 1.25^2 \\ 0 & 0 & 2 & 6\cdot 1.25 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \\ -g^{(4)}(\alpha) \end{pmatrix} = \begin{pmatrix} 0.0933333334 \\ -0.6826666666 \\ 1.6384 \\ -5.24288 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 3.933333333 \\ -6.826666667 \\ 4.096 \\ -0.8738133333 \end{pmatrix},$$

the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\frac{1}{3}x_n + \frac{2}{3x_n} + 3.933333333x_n - 6.826666667x_n^2 + 4.096x_n^3 - 0.8738133333x_n^4}{1 + 3.933333333 - 6.826666667x_n + 4.096x_n^2 - 0.8738133333x_n^3},$$

$n \geq 0$
and for the initial value $x_0 = 1.5$ the best convergence rate is obtained for $k = 4$, the iterative method is given by:

$$g_\lambda(x_n) = \frac{\frac{1}{3}x_n + \frac{2}{3x_n} + 2.629629630x_n - 3.950617284x_n^2 + 1.975308642x_n^3 - 0.3511659808x_n^4}{1 + 2.629629630 - 3.950617284x_n + 1.975308642x_n^2 - 0.3511659808x_n^3},$$

$n \geq 0$.
The results are listed in Table 3.

Table 3

	$x_{n+1} = G_\lambda(x_n)$	$x_{n+1} = G_\lambda(x_n)$	*Kr. it.*	*Kr. it.*
0	1.25	1.5	1.25	1.5
1	0.97560975539	0.9230769242	0.950000000	0.944444445
2	0.9998138976	0.9999491484	1.018421053	1.020697168
3	1.000000612	1.000001176	0.9940817805	0.9933807349
4	0.9999999977	0.9999999724	1.001996229	1.002235826
5	1.000000000	1.000000000	0.9993372417	0.9992580499
			...	...
⋮				
17			1.000000000	1.000000000

When comparing the results listed in Table 3, we can see that the new acceleration technique, in the first case, accelerates the convergence speed of the Krasnoselskij iteration.

- $\mu = \frac{2}{5}$, $1 - \mu = \frac{3}{5}$, $x_{n+1} = \frac{3}{5}x_n + \frac{2}{5x_n}$

for the initial value $x_0 = 1.25$ the best convergence is obtained for $k = 4$

$$g_\lambda(x_n) = \frac{G(x_n) + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3 + \lambda_4 x_n^4}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2 + \lambda_4 x_n^3}, \quad n \geq 0$$

λ_1, λ_2, λ_3 and λ_4 are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.25 & 1.25^2 & 1.25^3 \\ 0 & 1 & 2 \cdot 1.25 & 3 \cdot 1.25^2 \\ 0 & 0 & 2 & 6 \cdot 1.25 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \\ -g^{(4)}(\alpha) \end{pmatrix} = \begin{pmatrix} -0.344 \\ -0.4096 \\ 0.98304 \\ -3.145728 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 1.96 \\ -4.096 \\ 2.4576 \\ -0.524288 \end{pmatrix},$$

the accelerated iterative method is given by

$$g_\lambda(x_n) = \frac{\frac{1}{3}x_n + \frac{2}{3x_n} + 1.96x_n - 4.096x_n^2 + 2.4576x_n^3 - 0.524288x_n^4}{1 + 1.96 - 4.096x_n + 2.4576x_n^2 - 0.524288x_n^3}, \quad n \geq 0$$

and for the initial value $x_0 = 1.5$ the best convergence rate is obtained for $k = 4$

$$g_\lambda(x_n) = \frac{G(x_n) + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3 + \lambda_4 x_n^4}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2 + \lambda_4 x_n^3}, \quad n \geq 0$$

λ_1, λ_2, λ_3 and λ_4 are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.5 & 1.5^2 & 1.5^3 \\ 0 & 1 & 2\cdot 1.5 & 3\cdot 1.5^2 \\ 0 & 0 & 2 & 6\cdot 1.5 \\ 0 & 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \\ -g^{(4)}(\alpha) \end{pmatrix} = \begin{pmatrix} -0.422222222 \\ -0.2370370370 \\ 0.4740740741 \\ -1.264197531 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 1.177777778 \\ -2.370370371 \\ 1.185185185 \\ -0.2106995885 \end{pmatrix},$$

the accelerated iterative method is given by

$$g_\lambda(x_n) = \frac{\frac{1}{3}x_n + \frac{2}{3x_n} + 1.177777778x_n - 2.370370371x_n^2 + 1.185185185x_n^3 - 0.2106995885x_n^4}{1 + 1.177777778 - 2.370370371x_n + 1.185185185x_n^2 - 0.2106995885x_n^3},$$

$n \geq 0$.
The results are listed in Table 4.

Table 4

	$x_{n+1} = G_\lambda(x_n)$	$x_{n+1} = G_\lambda(x_n)$	*Kr. it.*	*Kr. it.*
0	1.25	1.5	1.25	1.5
1	0.9756097561	0.9230769206	1.070000000	1.666666667
2	0.9998138978	0.9999491496	1.015831776	1.042857143
3	1.000000614	1.000001177	1.003265051	1.009275930
4	0.9999999975	0.9999999719	1.000657261	1.001889287
5	0.9999999991	0.999999999	1.000131625	1.000379282
6	0.9999999981	1.000000000	...	...
7	1.000000000		...	...
...			...	...
12			1.000000000	
13				1.000000000

When comparing the results listed in Table 4, we can see that the new acceleration technique, in the first case, accelerates the convergence speed of the Krasnoselskij iteration.

- $\mu = \frac{1}{5}$, $1-\mu = \frac{4}{5}$, $x_{n+1} = \frac{4}{5}x_n + \frac{1}{5x_n}$

for the initial value $x_0 = 1.25$ the best convergence rate is obtained for $k = 3$

$$g_\lambda(x_n) = \frac{G(x_n) + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2}, \quad n \geq 0$$

λ_1, λ_2 and λ_3 are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.25 & 1.25^2 \\ 0 & 1 & 2\cdot 1.25 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \end{pmatrix} = \begin{pmatrix} -0.672 \\ -0.2048 \\ 0.49152 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -0.032 \\ -0.8192 \\ 0.24576 \end{pmatrix},$$

the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\frac{1}{3}x_n + \frac{2}{3x_n} - 0.032x_n - 0.8192x_n^2 + 0.24576x_n^3}{1 - 0.032 - 0.8192x_n + 0.24576x_n^2}, \quad n \geq 0$$

and for the initial value $x_0 = 1.5$ the best convergence is obtained for $k = 4$

$$g_\lambda(x_n) = \frac{G(x_n) + \lambda_1 x_n + \lambda_2 x_n^2 + \lambda_3 x_n^3 + \lambda_4 x_n^4}{1 + \lambda_1 + \lambda_2 x_n + \lambda_3 x_n^2 + \lambda_4 x_n^3},$$

$n \geq 0$.
λ_1, λ_2, λ_3 and λ_4 are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.5 & 1.5^2 & 1.5^3 \\ 0 & 1 & 2\cdot 1.5 & 3\cdot 1.5^2 \\ 0 & 0 & 2 & 6\cdot 1.5 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \\ -g^{(4)}(\alpha) \end{pmatrix} = \begin{pmatrix} -0.711111111 \\ -0.1185185185 \\ 0.2370370370 \\ -0.6320987654 \end{pmatrix},$$

which has the unique solutions:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0.08888888881 \\ -1.185185185 \\ 0.592595926 \\ -0.1053497942 \end{pmatrix},$$

the accelerated iterative method is given by:

$$g_\lambda(x_n) = \frac{\frac{1}{3}x_n + \frac{2}{3x_n} + 0.08888888881x_n - 1.185185185x_n^2 + 0.592595926x_n^3 - 0.1053497942x_n^4}{1 + 0.08888888881 - 1.185185185x_n + 0.592595926x_n^2 - 0.1053497942_n^3},$$

$n \geq 0$.
The results are listed in Table 5.

Table 5

	$x_{n+1} = G_\lambda(x_n)$	$x_{n+1} = G_\lambda(x_n)$	*Kr. it.*	*Kr. it.*
0	1.25	1.5	1.25	1.5
1	0.9756097561	0.9230769248	1.160000000	1.333333333
2	1.000139401	0.9999491464	1.100413793	1.216666666
3	0.9999980713	1.000001177	1.062080846	1.137716894
4	1.000000027	0.9999999703	1.037974259	1.085964172
5	0.999999999	0.9999999987	1.023062413	1.052939476
6	1.000000000	0.9999999982	...	...
7		1.000000000	...	...
			...	...
⋮				
37			1.000000001	
38				1.000000001

When comparing the results listed in Table 5, we can see that the new acceleration technique, in the first case, accelerates the convergence speed of the Krasnoselskij iteration.

If we compare the results from Tables 1-5, for this example, we observe that the new proposed Padé-type acceleration technique, in the first case, accelerates the convergence speed for the Krasnoselskij iteration, exception is for $\mu = \frac{1}{2}$ (μ is the middle of the interval where the operator is defined).

Method P2

• $\mu = \frac{1}{4}$, $1-\mu = \frac{3}{4}$, $x_{n+1} = \frac{3}{4}x_n + \frac{1}{4x_n}$
for the initial value $x_0 = 1.25$ the best convergence rate is obtained for $k = 2$

$$g_\lambda(x_n) = \frac{g(x_n) + \overline{\lambda_1}x_n + \overline{\lambda_2}x_n^2}{1 + \overline{\lambda_1} + \overline{\lambda_2}x_n}, \quad n \geq 0$$

$\overline{\lambda_1}$ and $\overline{\lambda_2}$ are determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.25 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{\lambda_1} \\ \overline{\lambda_2} \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \end{pmatrix} = \begin{pmatrix} 0.64 \\ -1.024 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \overline{\lambda_1} \\ \overline{\lambda_2} \end{pmatrix} = \begin{pmatrix} 1.92 \\ -1.024 \end{pmatrix},$$

the accelerated iterative method is given by:

$$x_{n+1} = \frac{3}{4}x_n + \frac{1}{4}g_\lambda(x_n),$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.92x_n - 1.024x_n^2}{1 + 1.92 - 1.024x_n},$$

and for the initial value $x_0 = 1.5$ the best convergence rate is obtained for $k = 2$

$$g_\lambda(x_n) = \frac{g(x_n) + \overline{\lambda_1}x_n + \overline{\lambda_2}x_n^2}{1 + \overline{\lambda_1} + \overline{\lambda_2}x_n},$$

$\overline{\lambda_1}$ and $\overline{\lambda_2}$ can be determined from the following system of linear equations:

$$\begin{pmatrix} 1 & 1.5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{\lambda_1} \\ \overline{\lambda_2} \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \end{pmatrix} = \begin{pmatrix} 0.444444444 \\ -0.5925925926 \end{pmatrix},$$

which has the unique solution:

$$\begin{pmatrix} \overline{\lambda_1} \\ \overline{\lambda_2} \end{pmatrix} = \begin{pmatrix} 1.333333333 \\ -0.5925925926 \end{pmatrix},$$

the accelerated iterative method is given by:

$$x_{n+1} = \frac{3}{4}x_n + \frac{1}{4}g_\lambda(x_n),$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.333333333x_n - 0.5925925926x_n^2}{1 + 1.333333333 - 0.5925925926x_n},$$

in case of the initial value 1.25 the best convergence rate is obtained for $k = 2$, and for the initial value 1.5 the best convergence rate is obtained, also, for $k = 2$, the results are listed in Table 6.

Table 6

	$x_{n+1} = \frac{3}{4}x_n + \frac{1}{4}g_\lambda(x_n)$	$x_{n+1} = \frac{3}{4}x_n + \frac{1}{4}g_\lambda(x_n)$	*Kr. it.*	*Kr. it.*
0	1.25	1.5	1.25	1.5
1	1.181402439	1.355769231	1.137500000	1.291666667
2	1.132440004	1.254753689	1.072905220	1.162298387
3	1.097022820	1.182764830	1.037691110	1.086614857
4	1.071215777	1.131110693	1.019187810	1.045141123
5	1.052333252	1.093976423	1.009684215	1.023057989
	...	...	...	...
⋮				
27			1.000000001	
28				1.000000001
⋮				
56	...	1.000000002		
	...			
⋮				
60	1.000000002			

When comparing the results listed in Table 6, we can see that the new acceleration technique, in the second case, does not accelerate the convergence speed of the Krasnoselskij iteration.

• $\mu = \frac{1}{2}$, $1 - \mu = \frac{1}{2}$, $x_{n+1} = \frac{1}{2}x_n + \frac{1}{2x_n}$
(the parameters $\overline{\lambda_1}$ and $\overline{\lambda_2}$ have the same values as above, in both cases)
for the initial value $x_0 = 1.25$ the iterative method is given by:

$$x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.92x_n - 1.024x_n^2}{1 + 1.92 - 1.024x_n}, \quad n \geq 0$$

and for the initial value $x_0 = 1.5$ the iterative method is given by:

$$x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.333333333x_n - 0.5925925926x_n^2}{1 + 1.333333333 - 0.5925925926x_n}, \quad n \geq 0$$

in the case of the initial value 1.25 the best convergence rate is obtained for $k = 2$, and for the initial value 1.5 the best convergence rate is obtained, also, for $k = 2$, the results are listed in Table 7.

Table 7

	$x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}g_\lambda(x_n)$	$x_{n+1} = \frac{1}{2}x + \frac{1}{2}g_\lambda(x_n)$	*Kr. it.*	*Kr. it.*
0	1.25	1.5	1.25	1.5
1	1.112804878	1.211538462	1.025000000	1.083333333
2	1.052659917	1.092018431	1.000304878	1.003205128
3	1.024787702	1.039746520	1.000000046	1.000005120
4	1.011696881	1.017042634	1.000000000	1.000000000
5	1.005524309	1.007277560		
	...	...	...	...
...				
24		1.000000000		
25	...			
26	1.000000000			

When comparing the results listed in Table 7, we can see that the new acceleration technique, in the second case, does not accelerate the convergence speed of the Krasnoselskij iteration.

• $\mu = \frac{2}{3}$, $1 - \mu = \frac{1}{3}$, $x_{n+1} = \frac{1}{3}x_n + \frac{2}{3x_n}$
(the parameters $\overline{\lambda_1}$ and $\overline{\lambda_2}$ as above, in both cases)
for the initial value $x_0 = 1.25$ the accelerated iterative method is given by:

$$x_{n+1} = \frac{1}{3}x_n + \frac{2}{3}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.92x_n - 1.024x_n^2}{1 + 6.4 - 10.24x_n}, \quad n \geq 0$$

and for the initial value $x_0 = 1.5$ the accelerated iterative method is given by:

$$x_{n+1} = \frac{1}{3}x_n + \frac{2}{3}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.333333333x_n - 0.5925925926x_n^2}{1 + 1.333333333 - 0.5925925926x_n}, \quad n \geq 0$$

in case of the initial value 1.25 the best convergence rate is obtained for $k = 2$, and for the initial value 1.5 the best convergence rate is obtained, also, for $k = 2$, the results are listed in Table 8.

Table 8

	$x_{n+1} = \frac{1}{3}x_n + \frac{2}{3}g_\lambda(x_n)$	$x_{n+1} = \frac{1}{3}x_n + \frac{2}{3}g_\lambda(x_n)$	*Kr. it.*	*Kr. it.*
0	1.25	1.5	1.25	1.5
1	1.067073171	1.115384616	0.950000000	0.944444445
2	1.019670227	1.028149698	1.018421053	1.020697168
3	1.005823761	1.006677628	0.9940817805	0.9933807349
4	1.001727261	1.001568198	1.001996229	1.002235826
5	1.000512505	1.000367324	0.9993372417	0.9992580499
	...	...	...	...
...				
14		1.000000000		
15				
16	1.000000000		...	...
17			1.000000000	1.000000000

When comparing the results listed in Table 8, we can see that the new acceleration technique, in the second case, accelerates the convergence speed of the Krasnoselskij iteration.

- $\mu = \frac{2}{5}$, $1 - \mu = \frac{3}{5}$, $x_{n+1} = \frac{3}{5}x_n + \frac{2}{5x_n}$

(the parameters $\overline{\lambda_1}$ and $\overline{\lambda_2}$ have the same values as above, in both cases)
for the initial value $x_0 = 1.25$ the iterative method is given by:

$$x_{n+1} = \frac{3}{5}x_n + \frac{2}{5}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.92x_n - 1.024x_n^2}{1 + 1.92 - 1.024x_n}, \quad n \geq 0$$

and for the initial value $x_0 = 1.5$ the iterative method is given by:

$$x_{n+1} = \frac{3}{5}x_n + \frac{2}{5}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.333333333x_n - 0.5925925926x_n^2}{1 + 1.333333333 - 0.5925925926x_n}, \quad n \geq 0$$

in case of the initial value 1.25 the best convergence rate is obtained for $k = 2$, and for the initial value 1.5 the best convergence rate is obtained, also, for $k = 2$, the results are listed in Table 9.

Table 9

	$x_{n+1} = \frac{3}{5}x_n + \frac{2}{5}g_\lambda(x_n)$	$x_{n+1} = \frac{3}{5}x_n + \frac{2}{5}g_\lambda(x_n)$	*Kr. it.*	*Kr. it.*
0	1.25	1.5	1.25	1.5
1	1.140243902	1.269230770	1.070000000	1.666666667
2	1.080157159	1.147461728	1.015831776	1.042857143
3	1.046116842	1.080694990	1.003265051	1.009275930
4	1.026601085	1.043985781	1.000657261	1.001889287
5	1.015360931	1.023896067	1.000131625	1.000379282
...	...	...	...	...
12			1.000000000	
13				1.000000000
...				
33		1.000000000		
34	1.000000001			

When comparing the results listed in Table 9, we can see that the new acceleration technique, in the second case, does not accelerate the convergence speed of the Krasnoselskij iteration.

- $\mu = \frac{1}{5}$, $1 - \mu = \frac{4}{5}$, $x_{n+1} = \frac{4}{5}x_n + \frac{1}{5x_n}$

(the parameters $\overline{\lambda_1}$ and $\overline{\lambda_2}$ have the same values as above, in both cases)
for the initial value $x_0 = 1.25$ the accelerated iterative method is given by:

$$x_{n+1} = \frac{4}{5}x_n + \frac{1}{5}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.92x_n - 1.024x_n^2}{1 + 1.92 - 1.024x_n}, \quad n \geq 0$$

and for the initial value $x_0 = 1.5$ the accelerated iterative method is given by:

$$x_{n+1} = \frac{4}{5}x_n + \frac{1}{5}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.333333333x_n - 0.5925925926x_n^2}{1 + 1.333333333 - 0.5925925926x_n}, \quad n \geq 0$$

in case of the initial value 1.25 the best convergence rate is obtained for $k = 2$, and for the initial value 1.5 the best convergence rate is obtained, also, for $k = 2$, the results are listed in Table 10.

Table 10

	$x_{n+1} = \frac{4}{5}x_n + \frac{1}{5}g_\lambda(x_n)$	$x_{n+1} = \frac{4}{5}x_n + \frac{1}{5}g_\lambda(x_n)$	*it. Kr.*	*it. Kr.*
0	1.25	1.5	1.25	1.5
1	1.195121951	1.384615385	1.160000000	1.333333333
2	1.152864164	1.297044763	1.100413793	1.216666666
3	1.120042700	1.229804196	1.062080846	1.137716894
4	1.094412714	1.177868899	1.037974259	1.085964172
5	1.074329515	1.137647466	1.023062413	1.052939476
...	...	...	...	...
37			1.000000001	
38				1.000000001
...				
73		1.000000002		
...				
77	1.000000002			

When comparing the results listed in Table 10, we can see that the new acceleration technique, in the second case, does not accelerate the convergence speed of the Krasnoselskij iteration.
If we compare the results listed in the above tables we observe that the new Padé-type acceleration technique, in the second case, accelerates the convergence speed for the Krasnoselskij iteration in four of five cases, but the convergence speed is small, two or three iterations.
We have done test on ten operators, starting from different initial values and for different values for the constant μ that appears in the definition of Krasnoselskij iteration and the results were the same as those presented in the Tables 1-10.

2.4 A Padé-type acceleration technique for the Mann iteration

Chronologically Mann iteration was introduced two years earlier than the Krasnoselskij iteration, it is a generalization of the Krasnoselskij iteration and its normal form is obtained by replacing the parameter μ in the Krasnoselskij iteration formula by a sequence of real numbers $\{\alpha_n\} \subset [0, 1]$.
The reason way Mann iteration was introduced, is that, in the literature are some theorems of the following type: T is a selfmap of a complete metric space E, satisfying a contractive condition that may or may not be strong enough to guarantee the convergence to a fixed point of the Picard iteration associated to T.
In this conditions it is assumed that the Mann iteration associated to T converges, for some sequence $\{\alpha_n\}$, and it is shown that in this case, it converges to a fixed point of T.
Let E be a normed linear space and $T : E \to E$ a given operator. The sequence

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n g(x_n),\ x_0 \in E,\ \alpha_n \in [0, 1],\ n \geq 0 \tag{2.28}$$

is called the **Mann iteration** or **Mann iterative procedure**, see [12].
There exists a large literature on the convergence of the Mann iteration for different

classes of operators considered on different spaces, for an extensive matter about the Mann iteration see the papers due Babu [3], [4], [5], Berinde [12], [14], Chidume [42], Rhoades [99], Xue [132].
Two important observations concerning Mann iteration are: firstly, the theoretical convergence order it is not known, it was established only empirically in [12], [99] and secondly, this iterative process is very slow, see [12], [99], consequently an acceleration process has to be used.
Author's original contribution in this paragraph are: the Padé-type acceleration technique for accelerating the Mann iteration procedure (2.29) and Example 2.7.
In order to improve the convergence speed for the Mann iteration (2.28) we consider the following acceleration method:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n g_\lambda(x_n), \tag{2.29}$$

where g_λ is of the form:

$$g_\lambda(x) = \frac{g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3 + \ldots + \overline{\lambda_k}x^k}{1 + \overline{\lambda_1} + \overline{\lambda_2}x + \overline{\lambda_3}x^2 + \ldots + \overline{\lambda_k}x^{k-1}}, \tag{2.30}$$

$k \in \mathbb{N},\ k \geq 2$ and $\overline{\lambda_1}, \overline{\lambda_2}, \overline{\lambda_3}, \ldots, \overline{\lambda_k} \in \mathbb{R}$ are parameters that should be determined in such a way that the new iteration function, g_λ, will yield a faster Mann iteration.
To apply the technique we must know α, since α is in (a,b), we consider $\alpha \cong \frac{a+b}{2}$. In the sequel we show the technique for $k = \overline{2,4}$.
(i) $k = 2$. In this case, we have

$$x + \overline{\lambda_1}x + \overline{\lambda_2}x^2 = g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2,$$
$$g_\lambda(x) = \frac{g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2}{1 + \overline{\lambda_1} + \overline{\lambda_2}x},$$

with the following system of linear equations:

$$\begin{cases} \overline{\lambda_1} + \alpha\overline{\lambda_2} = -g'(\alpha) \\ \overline{\lambda_2} = -g^{(2)}(\alpha), \end{cases}$$

from where we can determine the parameters $\overline{\lambda_i}$:

$$\begin{cases} \overline{\lambda_1} = -g'(\alpha) + \alpha g^{(2)}(\alpha) \\ \overline{\lambda_2} = -g^{(2)}(\alpha). \end{cases}$$

So, the accelerated iterative method is given by:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \frac{g(x_n) + \overline{\lambda_1}x_n + \overline{\lambda_2}x_n^2}{1 + \overline{\lambda_1} + \overline{\lambda_2}x_n}, \quad n \geq 0.$$

(ii) $k = 3$. In this case, we have

$$x + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3 = g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3,$$
$$g_\lambda(x) = \frac{g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3}{1 + \overline{\lambda_1} + \overline{\lambda_2}x + \overline{\lambda_3}x^2},$$

with the following system of linear equations:

$$\begin{cases} \overline{\lambda_1} + \alpha\overline{\lambda_2} + \alpha^2\overline{\lambda_3} = -g'(\alpha) \\ \overline{\lambda_2} + 2\alpha\overline{\lambda_3} = -g^{(2)}(\alpha) \\ 2\overline{\lambda_3} = -g^{(3)}(\alpha), \end{cases}$$

from where we determine the parameters $\overline{\lambda_i}$:

$$\begin{cases} \overline{\lambda_1} = -g'(\alpha) + \alpha g^{(2)}(\alpha) - \dfrac{\alpha^2}{2} g^{(3)}(\alpha) \\ \overline{\lambda_2} = -g^{(2)}(\alpha) + \alpha g^{(3)}(\alpha) \\ \overline{\lambda_3} = -\dfrac{g^{(3)}(\alpha)}{2}. \end{cases}$$

So, the accelerated iterative method is given by:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \frac{g(x_n) + \overline{\lambda_1}x_n + \overline{\lambda_2}x_n^2 + \overline{\lambda_3}x_n^3}{1 + \overline{\lambda_1} + \overline{\lambda_2}x_n + \overline{\lambda_3}x_n^2}, \quad n \geq 0.$$

(iii) $k = 4$. In this case, we have

$$x + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3 + \overline{\lambda_4}x^4 = g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3 + \overline{\lambda_4}x^4,$$

$$g_\lambda(x) = \frac{g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3 + \overline{\lambda_4}x^4}{1 + \overline{\lambda_1} + \overline{\lambda_2}x + \overline{\lambda_3}x^2 + \overline{\lambda_4}x^3},$$

with the following system of linear equations:

$$\begin{cases} \overline{\lambda_1} + \alpha\overline{\lambda_2} + \alpha^2\overline{\lambda_3} + \alpha^3\overline{\lambda_4} = -g'(\alpha) \\ \overline{\lambda_2} + 2\alpha\overline{\lambda_3} + 3\alpha^2\overline{\lambda_4} = -g^{(2)}(\alpha) \\ 2\overline{\lambda_3} + 6\alpha\overline{\lambda_4} = -g^{(3)}(\alpha) \\ 6\overline{\lambda_4} = -g^{(4)}(\alpha), \end{cases} \tag{2.31}$$

which has a unique solution.
The accelerated iterative method is given by:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \frac{g(x_n) + \overline{\lambda_1}x_n + \overline{\lambda_2}x_n^2 + \overline{\lambda_3}x_n^3 + \overline{\lambda_4}x_n^4}{1 + \overline{\lambda_1} + \overline{\lambda_2}x_n + \overline{\lambda_3}x_n^2 + \overline{\lambda_4}x_n^3}, \quad n \geq 0.$$

with $\overline{\lambda_i}$ the solution of the system (2.31).

Example 2.7 *[12] Let $g : [\frac{1}{2}, 2] \to [\frac{1}{2}, 2]$, $g(x) = \frac{1}{x}$, g has a unique fixed point $F_g = \{1\}$.*

Because g is a strongly pseudo-contraction operator, with any constant $k \in (0,1)$ a certain Mann iteration associated to the operator g will converge to the unique fixed point of g, $x^* = 1$, for any initial approximation $x_0 \in \left[\frac{1}{2}, 2\right]$ and $\alpha_n \in (0,1)$, consequently we can apply the Padé-type acceleration technique in order to try to accelerate its convergence speed.
(the parameters $\overline{\lambda_i}$ have been calculated above)

• $\alpha_n = \frac{n}{4n+1}$, $1-\alpha_n = \frac{3n+1}{4n+1}$, $x_{n+1} = \frac{3n+1}{4n+1}x_n + \frac{n}{(4n+1)x_n}$, $\quad n \geq 0$
for the initial value $x_0 = 1.25$ the accelerated iterative method is given by:

$$x_{n+1} = \frac{3n+1}{4n+1}x_n + \frac{n}{4n+1}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.92x_n - 1.024x_n^2}{1 + 1.92 - 1.024x_n}, \quad n \geq 0$$

and for the initial value $x_0 = 1.5$ the accelerated iterative method is given by

$$x_{n+1} = \frac{3n+1}{4n+1}x_n + \frac{n}{4n+1}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.333333333x_n - 0.5925925926x_n^2}{1 + 1.333333333 - 0.5925925926x_n}, \quad n \geq 0$$

when the initial value is 1.25 the best convergence rate is obtained for $k = 2$, and for the initial value 1.5 the best convergence rate is obtained, also, for $k = 2$, the results are listed in Table 11.

Table 11

	$x_{n+1} = \frac{3n+1}{4n+1}x_n + \frac{n}{4n+1}g_\lambda(x_n)$	$x_{n+1} = \frac{3n+1}{4n+1}x_n + \frac{n}{4n+1}g_\lambda(x_n)$	*M. it.*	*M. it.*
0	1.25	1.5	1.25	1.5
1	1.195121951	1.384615385	1.160000000	1.333333333
2	1.148168855	1.287314694	1.093793103	1.203703704
3	1.111492810	1.212300323	1.052360007	1.117641902
4	1.083523981	1.155853300	1.028332984	1.065194624
5	1.062402217	1.113865317	1.015026954	1.035099612
...	...	...	...	...
31			1.000000001	
32				1.000000001
...				
60		1.000000002		
...				
62	1.000000002			

When comparing the results listed in Table 11, we can see that the new acceleration technique does not accelerate the convergences speed of the Mann iteration.

• $\alpha_n = \frac{n}{2n+1}$, $1-\alpha_n = \frac{n+1}{2n+1}$, $x_{n+1} = \frac{n+1}{2n+1}x_n + \frac{n}{(2n+1)x_n}$, $\quad n \geq 0$
for the initial value $x_0 = 1.25$ the accelerated iterative method is given by:

$$x_{n+1} = \frac{n+1}{2n+1}x_n + \frac{n}{2n+1}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.92x_n - 1.024x_n^2}{1 + 1.92 - 1.024x_n}, \quad n \geq 0$$

and for the initial value $x_0 = 1.5$ the accelerated iterative is given by:

$$x_{n+1} = \frac{n+1}{2n+1}x_n + \frac{n}{2n+1}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.333333333x_n - 0.5925925926x_n^2}{1 + 1.333333333 - 0.5925925926x_n}, \quad n \geq 0$$

when the initial value is 1.25 the best convergence is obtained for $k = 2$, and for the initial value 1.5 the best convergence rate is obtained, also, for $k = 2$, the results are listed in Table 12.

Table 12

	$x_{n+1} = \frac{n+1}{2n+1}x_n + \frac{n}{2n+1}g_\lambda(x_n)$	$x_{n+1} = \frac{n+1}{2n+1}x_n + \frac{n}{2n+1}g_\lambda(x_n)$	*it. M.*	*it. M.*
0	1.25	1.5	1.25	1.5
1	1.1158536585	1.307692308	1.100000000	1.222222222
2	1.090385058	1.168322483	1.023636364	1.060606060
3	1.049207253	1.086735764	1.003610528	1.010142239
4	1.026063656	1.042918862	1.000406943	1.001172174
5	1.013549259	1.020640453	1.000037070	1.000107185
...	...	...	...	...
10			1.000000000	1.000000000
...				
28		1.000000000		
...				
30	1.000000000			

When comparing the results listed in Table 12, we can see that the new acceleration technique does not accelerate the convergence speed of the Mann iteration.

- $\alpha_n = \frac{2n}{3n+1}$, $1 - \alpha_n = \frac{n+1}{3n+1}$, $x_{n+1} = \frac{n+1}{3n+1}x_n + \frac{2n}{(3n+1)x_n}$, $n \geq 0$

for the initial value $x_0 = 1.25$ the accelerated iterative method is given by:

$$x_{n+1} = \frac{n+1}{3n+1}x_n + \frac{2n}{3n+1}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.92x_n - 1.024x_n^2}{1 + 1.92 - 1.024x_n}, \quad n \geq 0$$

and for the initial value $x_0 = 1.5$ the accelerated iterative is given by:

$$x_{n+1} = \frac{n+1}{3n+1}x_n + \frac{2n}{3n+1}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.333333333x_n - 0.5925925926x_n^2}{1 + 1.333333333 - 0.5925925926x_n}, \quad n \geq 0$$

when the initial value is 1.25 the best rate of convergence is obtained for $k = 2$, and for the initial value 1.5 the best rate of convergence is obtained, also, for $k = 2$, the results are listed in Table 13.

Table 13

	$x_{n+1} = \frac{n+1}{3n+1}x_n + \frac{2n}{3n+1}g_\lambda(x_n)$	$x_{n+1} = \frac{n+1}{3n+1}x_n + \frac{2n}{3n+1}g_\lambda(x_n)$	*M. it.*	*M. it.*
0	1.25	1.5	1.25	1.5
1	1.112804878	1.211538462	1.025000000	1.083333333
2	1.044067779	1.074944141	0.9967770035	0.9917582418
3	1.016099740	1.023776044	1.000650852	1.001689446
4	1.005640608	1.007025108	0.9998500639	0.9996118813
5	1.001920950	1.001986028	1.000037498	1.000097124
...	...	...	...	...
15				1.000000000
16			1.000000000	
...				
18		1.000000000		
19	1.000000000			

When comparing the results listed in Table 13, we can see that the new acceleration technique does not accelerate the convergence speed of the Mann iteration.

- $\alpha_n = \frac{2n}{5n+1}$, $1-\alpha_n = \frac{3n+1}{5n+1}$, $x_{n+1} = \frac{3n+1}{5n+1}x_n + \frac{2n}{(5n+1)x_n}$, $n \geq 0$

for the initial value $x_0 = 1.25$ the accelerated iterative method is given by:

$$x_{n+1} = \frac{3n+1}{5n+1}x_n + \frac{2n}{5n+1}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.92x_n - 1.024x_n^2}{1 + 1.92 - 1.024x_n}, \quad n \geq 0$$

and for the initial value $x_0 = 1.5$ the accelerated iterative method is given by:

$$x_{n+1} = \frac{3n+1}{5n+1}x_n + \frac{2n}{5n+1}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.333333333x_n - 0.5925925926x_n^2}{1 + 1.333333333 - 0.5925925926x_n}, \quad n \geq 0$$

when the initial value is 1.25 the best rate of convergence is obtained for $k = 2$, and for the initial value 1.5 the best rate of convergence is, also, obtained for $k = 2$, the results are listed in Table 14.

Table 14

	$x_{n+1} = \frac{3n+1}{5n+1}x_n + \frac{2n}{5n+1}g_\lambda(x_n)$	$x_{n+1} = \frac{3n+1}{5n+1}x_n + \frac{2n}{5n+1}g_\lambda(x_n)$	*it. M.*	*it. M.*
0	1.25	1.5	1.25	1.5
1	1.158536585	1.307692308	1.100000000	1.222222222
2	1.096580652	1.180992467	1.030578512	1.075298439
3	1.058048807	1.104259250	1.007984866	1.020801917
4	1.034627598	1.059180300	1.001925255	1.005114324
5	1.020550583	1.033236532	1.000445712	1.001190238
	...	...	...	...
14			1.000000000	
15				1.000000000
...				
33		1.000000000		
...				
38	1.000000001			

When comparing the results listed in Table 14, we can see that the new acceleration technique does not accelerate the convergence speed of the Mann iteration.

- $\alpha_n = \frac{n}{5n+1}$, $1-\alpha_n = \frac{4n+1}{5n+1}$, $x_{n+1} = \frac{4n+1}{5n+1}x_n + \frac{n}{(5n+1)x_n}$, $n \geq 0$

for the initial value $x_0 = 1.25$ the accelerated iterative method is given by:

$$x_{n+1} = \frac{4n+1}{5n+1}x_n + \frac{n}{5n+1}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.92x_n - 1.024x_n^2}{1 + 1.92 - 1.024x_n}, \quad n \geq 0$$

and for the initial value $x_0 = 1.5$ the accelerated iterative method is given by:

$$x_{n+1} = \frac{4n+1}{5n+1}x_n + \frac{n}{5n+1}g_\lambda(x_n), \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.333333333x_n - 0.5925925926x_n^2}{1 + 1.333333333 - 0.5925925926x_n}, \quad n \geq 0$$

when the initial value is 1.25 the best convergence rate is obtained for $k = 2$, and for the initial value 1.5 the best convergence rate is, also, obtained for $k = 2$, the results are listed in Table 15.

Table 15

	$x_{n+1} = \frac{4n+1}{5n+1}x_n + \frac{n}{5n+1}g_\lambda(x_n)$	$x_{n+1} = \frac{4n+1}{5n+1}x_n + \frac{n}{5n+1}g_\lambda(x_n)$	*M. it.*	*M. it.*
0	1.25	1.5	1.25	1.5
1	1.204268293	1.403846154	1.175000000	1.361111111
2	1.163967987	1.320105391	1.116102514	1.247217069
3	1.130892818	1.252096147	1.074828614	1.163698566
4	1.104230420	1.197822335	1.047314761	1.105723423
5	1.082882807	1.120918505	1.029527843	1.067004552
	...	...	...	...
...				
41			1.000000001	
42				1.000000002
...				
78		1.000000002		
...				
80	1.000000002			

When comparing the results listed in Table 15, we can see that the new acceleration technique does not accelerate the convergence speed of the Mann iteration.
If we compare the results listed in the above tables we observe that the new Padé-type acceleration technique does not accelerates the convergence speed of the Mann iteration. We have done test on ten operators, starting from different initial values and for different values for the real sequence $\{\alpha_n\}$ that appears in the definition of Mann iteration and the results were the same as those presented in the Tables 11-15.

2.5 A Padé-type acceleration technique for the Ishikawa iteration

It is well known that if T is a continuous map and if the Mann iterative process converges, then it converges to a fixed point of T. Contrary, if T it is continuous, than it is not warranty that, even if Mann iteration converges, it will converge to the fixed point of T, as it is shown in the following example:

Example 2.8 *[12] Let $T : [0,1] \to [0,1]$ be given by $T0 = T1 = 0$ and $Tx = 1,\ 0 < x < 1$. Then $F_T = \{0\}$ and the Mann iteration $M(x_1, \alpha_n, T)$ (where x_1 is the initial guess, $\{\alpha_n\}$ is a sequence in $(0,1)$ and T is an operator to whom the process is associated) with $0 < x_1 < 1$ and $\alpha_n = \frac{1}{n},\ n \geq 1$, converges to 1, which is not a fixed point of T.*

Consequently another iterative process was needed for approximating the fixed point of some classes of contractive mappings T for which Mann iteration it is not able to converge to a fixed point of T, this process is called **Ishikawa iteration** or **Ishikawa iteration procedure** and its given by the following recursive formula:

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n g(x_n),\ n \geq 0 \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n g(y_n),\ n \geq 0 \end{cases} \tag{2.32}$$

where $g : [a, b] \to [a, b]$ is an operator, $x_0 \in [a, b]$, $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are two sequences of real numbers from $[0, 1]$, see [12].
The above relation is in some sense a two-step Mann iteration and it was first used to establish the strong convergence to a fixed point for a Lipschitzian and pseudo-contractive selfmap of a convex compact subset of a Hilbert space. An extended matter about Ishikawa iteration can be found in papers due Berinde [12], Ishikawa [69], Rhoades [101], Xue [132].
Author's original contribution in this paragraph are: the Padé-type acceleration technique for accelerating the Ishikawa iteration procedure (2.33) and Example 2.9.
In order to improve the convergence speed for the Ishikawa iteration (2.32) we consider the following Padé-type acceleration technique:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n g((1 - \beta_n)x_n + \beta_n g_\lambda(x_n)), \quad n \geq 0 \tag{2.33}$$

where g_λ is of the form

$$g_\lambda(x) = \frac{g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3 + \ldots + \overline{\lambda_k}x^k}{1 + \overline{\lambda_1} + \overline{\lambda_2}x + \overline{\lambda_3}x^2 + \ldots + \overline{\lambda_k}x^{k-1}}, \tag{2.34}$$

$k \in \mathbb{N},\ k \geq 2$ and $\overline{\lambda_1}, \overline{\lambda_2}, \overline{\lambda_3}, \ldots, \overline{\lambda_k} \in \mathbb{R}$ are parameters that should be determined in such a way that the new iteration function, g_λ, yields a faster Ishikawa iteration.
To apply the technique we must know α, since α is in (a, b), we consider $\alpha \cong \frac{a+b}{2}$. In the sequel we will show the technique for $k = \overline{2, 4}$.
(i) $k = 2$. In this case, we have:

$$x + \overline{\lambda_1}x + \overline{\lambda_2}x^2 = g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2,$$
$$g_\lambda(x) = \frac{g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2}{1 + \overline{\lambda_1} + \overline{\lambda_2}x},$$

with the following system of linear equations:

$$\begin{cases} \overline{\lambda_1} + \alpha\overline{\lambda_2} = -g'(\alpha) \\ \overline{\lambda_2} = -g^{(2)}(\alpha), \end{cases}$$

from where we determine the parameters $\overline{\lambda_i}$ in a unique way:

$$\begin{cases} \overline{\lambda_1} = -g'(\alpha) + \alpha g^{(2)}(\alpha) \\ \overline{\lambda_2} = -g^{(2)}(\alpha). \end{cases}$$

The accelerated iterative method is given by:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n g((1 - \beta_n)x_n + \beta_n \frac{g(x_n) + \overline{\lambda_1}x_n + \overline{\lambda_2}x_n^2}{1 + \overline{\lambda_1} + \overline{\lambda_2}x_n}), \quad n \geq 0.$$

(ii) $k = 3$. In this case, we have:

$$x + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3 = g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3,$$
$$g_\lambda(x) = \frac{g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3}{1 + \overline{\lambda_1} + \overline{\lambda_2}x + \overline{\lambda_3}x^2},$$

with the following system of linear equations:

$$\begin{cases} \overline{\lambda_1} + \alpha\overline{\lambda_2} + \alpha^2\overline{\lambda_3} = -g'(\alpha) \\ \overline{\lambda_2} + 2\alpha\overline{\lambda_3} = -g^{(2)}(\alpha) \\ 2\overline{\lambda_3} = -g^{(3)}(\alpha), \end{cases}$$

from where we determine the parameters $\overline{\lambda_i}$ in a unique way:

$$\begin{cases} \overline{\lambda_1} = -g'(\alpha) + \alpha g^{(2)}(\alpha) - \dfrac{\alpha^2}{2} g^{(3)}(\alpha) \\ \overline{\lambda_2} = -g^{(2)}(\alpha) + \alpha g^{(3)}(\alpha) \\ \overline{\lambda_3} = -\dfrac{g^{(3)}(\alpha)}{2}. \end{cases}$$

The accelerated iterative method is given by:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n g((1-\beta_n)x_n + \beta_n \frac{g(x_n) + \overline{\lambda_1}x_n + \overline{\lambda_2}x_n^2 + \overline{\lambda_3}x_n^3}{1 + \overline{\lambda_1} + \overline{\lambda_2}x_n + \overline{\lambda_3}x_n^2}), \quad n \geq 0.$$

(iii) $k = 4$. In this case, we have:

$$x + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3 + \overline{\lambda_4}x^4 = g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3 + \overline{\lambda_4}x^4,$$

$$g_\lambda(x) = \frac{g(x) + \overline{\lambda_1}x + \overline{\lambda_2}x^2 + \overline{\lambda_3}x^3 + \overline{\lambda_4}x^4}{1 + \overline{\lambda_1} + \overline{\lambda_2}x + \overline{\lambda_3}x^2 + \overline{\lambda_4}x^3},$$

with the following system of linear equations:

$$\begin{cases} \overline{\lambda_1} + \alpha\overline{\lambda_2} + \alpha^2\overline{\lambda_3} + \alpha^3\overline{\lambda_4} = -g'(\alpha) \\ \overline{\lambda_2} + 2\alpha\overline{\lambda_3} + 3\alpha^2\overline{\lambda_4} = -g^{(2)}(\alpha) \\ 2\overline{\lambda_3} + 6\alpha\overline{\lambda_4} = -g^{(3)}(\alpha) \\ 6\overline{\lambda_4} = -g^{(4)}(\alpha), \end{cases}$$

from where we obtain parameters $\overline{\lambda_i}$ in a unique way.
The accelerated iterative method is given by:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n g((1-\beta_n)x_n + \beta_n \frac{g(x_n) + \overline{\lambda_1}x_n + \overline{\lambda_2}x_n^2 + \overline{\lambda_3}x_n^3 + \overline{\lambda_4}x_n^4}{1 + \overline{\lambda_1} + \overline{\lambda_2}x_n + \overline{\lambda_3}x_n^2 + \overline{\lambda_4}x_n^3}), \quad n \geq 0.$$

Example 2.9 *(Bumbariu [37]) Let* $g : [\frac{1}{2}, 2] \to [\frac{1}{2}, 2]$, $g(x) = \frac{1}{x}$, *g has a unique fixed point* $F_g = \{1\}$.

Because g is a Lipschitzian pseudo-contraction operator, a certain Ishikawa iteration associated to the operator g will converge to the unique fixed point of g, $x^* = 1$, for any initial approximation $x_0 \in \left[\frac{1}{2}, 2\right]$ and $\{\alpha_n\}, \{\beta_n\} \in (0,1)$, consequently we can apply the

Padé-type acceleration technique in order to accelerate its convergence speed. (the parameters $\overline{\lambda_i}$ have the same values as above)

• $\alpha_n = \frac{n}{4n+1}$, $\beta_n = \frac{3n}{4n+1}$, $x_{n+1} = \frac{3n+1}{4n+1}x_n + \frac{n}{4n+1}\frac{1}{\frac{n+1}{4n+1}x_n + \frac{3n}{4n+1}\frac{1}{x_n}}$, $n \geq 0$

for the initial value $x_0 = 1.25$ the accelerated iterative method is:

$$x_{n+1} = \frac{3n+1}{4n+1}x_n + \frac{n}{4n+1}\frac{1}{\frac{n+1}{4n+1}x_n + \frac{3n}{4n+1}g_\lambda(x_n)}, \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 6.4x_n - 10.24x_n^2 + 6.144x_n^3 - 1.31072x_n^4, \quad n \geq 0}{1 + 6.4 - 10.24x_n + 6.144x_n^2 - 1.31072x_n^3}, \quad n \geq 0$$

and for the initial value $x_0 = 1.5$ the accelerated iterative method is:

$$x_{n+1} = \frac{3n+1}{4n+1}x_n + \frac{n}{4n+1}\frac{1}{\frac{n+1}{4n+1}x_n + \frac{3n}{4n+1}g_\lambda(x_n)}, \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 4.444444444x_n - 5.925925925x_n^2 + 2.962962963x_n^3 - 0.5267489712x_n^4}{1 + 4.444444444 - 5.925925925x_n + 2.962962963x_n^2 - 0.5267489712x_n^3}, \quad n \geq 0$$

when the initial value is 1.25 the best rate of convergence is obtained for $k = 4$, and for the initial value 1.5 the best rate of convergence is obtained, also, for $k = 4$, the results are listed in Table 16.

Table 16

	Is. it. with Padé	*Is. it. with* Padé	*Is. it.*	*Is. it.*
0	1.25	1.5	1.25	1.5
1	1.184269663	1.373333333	1.204081633	1.400000000
2	1.132336688	1.271606328	1.169193357	1.324579125
3	1.093909474	1.195073917	1.141815734	1.266985905
4	1.066129794	1.138853936	1.119782779	1.221936400
5	1.046316019	1.098175694	1.101749032	1.186007313
...	...	...	...	...
51	1.000000001			
...				
53		1.000000002		
...				
128			1.000000004	
...				
133				1.000000004

When comparing the results listed in Table 16, we can see that the new acceleration technique does accelerate the convergence speed of the Ishikawa iteration.

• $\alpha_n = \frac{n}{2n+1}$, $\beta_n = \frac{n}{2n+1}$, $x_{n+1} = \frac{n+1}{2n+1}x_n + \frac{n}{2n+1}\frac{1}{\frac{n+1}{2n+1}x_n + \frac{n}{2n+1}\frac{1}{x_n}}$, $n \geq 0$.

For the initial value $x_0 = 1.25$ the accelerated iterative method is:

$$x_{n+1} = \frac{n+1}{2n+1}x_n + \frac{n}{2n+1}\frac{1}{\frac{n+1}{2n+1}x_n + \frac{n}{2n+1}g_\lambda(x_n)}, \quad n \geq 0$$

where $g_\lambda(x_n) = \frac{\frac{1}{x_n} + 6.4x - 10.24x_n^2 + 6.144x_n^3 - 1.31072x_n^4}{1 + 6.4 - 10.24x_n + 6.144x_n^2 - 1.31072x_n^3}$, $n \geq 0$.

For the initial value $x_0 = 1.5$ the accelerated iterative method is:

$$x_{n+1} = \frac{n+1}{2n+1}x_n + \frac{n}{2n+1}\frac{1}{\frac{n+1}{2n+1}x_n + \frac{n}{2n+1}g_\lambda(x_n)}, \quad n \geq 0$$

where $g_\lambda(x_n) = \frac{\frac{1}{x_n}+4.444444444x_n-5.925925925x_n^2+2.962962963x_n^3-0.5267489712x_n^4}{1+4.444444444-5.925925925x_n+2.962962963x_n^2-0.5267489712x_n^3}, \quad n \geq 0$
when the initial value is 1.25 the best rate of convergence is obtained for $k = 4$, and for the initial value 1.5 the best rate of convergence is obtained, also, for $k = 4$, the results are listed in Table 17.

Table 17

	Is. it. with Padé	*Is. it. with* Padé	*Is. it.*	*Is. it.*
0	1.25	1.5	1.25	1.5
1	1.121052632	1.254901961	1.164772727	1.329545455
2	1.046455485	1.102985601	1.106105284	1.209750852
3	1.015655063	1.035971168	1.068529805	1.133755808
4	1.004896496	1.011512290	1.044527668	1.086013469
5	1.001465632	1.003505907	1.029085651	1.055759865
...	...	...	...	...
17	1.000000000	1.000000000		
...				
46			1.000000001	
...				
48				1.000000001

When comparing the results listed in Table 17, we can see that the new acceleration technique does accelerate the convergence speed of the Ishikawa iteration.

- $\alpha_n = \frac{2n}{3n+1}$, $\beta_n = \frac{4n}{5n+1}$, $x_{n+1} = \frac{n+1}{3n+1}x_n + \frac{2n}{3n+1}\frac{1}{\frac{n+1}{5n+1}x_n+\frac{4n}{5n+1}\frac{1}{x_n}}, \quad n \geq 0$

for the initial value $x_0 = 1.25$ the accelerated iterative method is:

$$x_{n+1} = \frac{n+1}{3n+1}x_n + \frac{2n}{3n+1}\frac{1}{\frac{n+1}{5n+1}x_n + \frac{4n}{5n+1}g_\lambda(x_n)}, \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 6.4x - 10.24x_n^2 + 6.144x_n^3 - 1.31072x_n^4}{1 + 6.4 - 10.24x_n + 6.144x_n^2 - 1.31072x_n^3}, \quad n \geq 0$$

for the initial value $x_0 = 1.5$ the accelerated iterative method is:

$$x_{n+1} = \frac{n+1}{3n+1}x_n + \frac{2n}{3n+1}\frac{1}{\frac{n+1}{5n+1}x_n + \frac{4n}{5n+1}g_\lambda(x_n)}, \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 4.444444444x_n - 5.925925925x_n^2 + 2.962962963x_n^3 - 0.5267489712x_n^4}{1 + 4.444444444 - 5.925925925x_n + 2.962962963x_n^2 - 0.5267489712x_n^3}, \quad n \geq 0$$

when the initial value is 1.25 the best convergence rate is obtained for $k = 4$, and for the initial value 1.5 the best rate of convergence is, also, obtained for $k = 4$, the results are listed in Table 18.

Table 18

	Is. it. with Padé	*Is. it. with* Padé	*Is. it*	*Is. it.*
0	1.25	1.5	1.25	1.5
1	1.093571428	1.198275862	1.151315790	1.279411764
2	1.027481919	1.061997174	1.097670313	1.171215619
3	1.007087695	1.016801104	1.065580339	1.111459895
4	1.001710916	1.004232637	1.045091694	1.075167579
5	1.000398321	1.0011027109	1.031481118	1.051819052
	...	...	...	...
...				
14	1.000000000			
...				
16		1.000000000		
...				
58			1.000000001	
...				
60				1.000000001

When comparing the results listed in Table 18, we can see that the new acceleration technique does accelerate the convergence speed of the Ishikawa iteration.

- $\alpha_n = \frac{2n}{5n+1}$, $\beta_n = \frac{2n}{3n+1}$, $x_{n+1} = \frac{3n+1}{5n+1}x_n + \frac{2n}{5n+1}\frac{1}{\frac{n+1}{3n+1}x_n+\frac{2n}{3n+1}\frac{1}{x_n}}$, $n \geq 0$

for the initial value $x_0 = 1.25$ the accelerated iterative method is:

$$x_{n+1} = \frac{3n+1}{5n+1}x_n + \frac{2n}{5n+1}\frac{1}{\frac{n+1}{3n+1}x_n + \frac{2n}{3n+1}g_\lambda(x_n)}, \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 1.92x_n - 1.024x_n^2}{1 + 1.92 - 1.024x_n}, \quad n \geq 0$$

and for the initial value $x_0 = 1.5$ the accelerated iterative method is:

$$x_{n+1} = \frac{3n+1}{5n+1}x_n + \frac{2n}{5n+1}\frac{1}{\frac{n+1}{3n+1}x_n + \frac{2n}{3n+1}g_\lambda(x_n)}, \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 4.444444444x_n - 5.925925925x_n^2 + 2.962962963x_n^3 - 0.5267489712x_n^4}{1 + 4.444444444 - 5.925925925x_n + 2.962962963x_n^2 - 0.5267489712x_n^3}, \quad n \geq 0$$

when the initial value is 1.25 the best convergence rate is obtained for $k = 2$, and for the initial value 1.5 the best convergence rate is obtained for $k = 4$, the results are listed in Table 19.

Table 19

	Is. it. with Padé	*Is. it. with* Padé	*Is. it.*	*Is. it.*
0	1.25	1.5	1.25	1.5
1	1.132876712	1.275132275	1.158536585	1.307692308
2	1.066698441	1.141637804	1.104653250	1.196746870
3	1.032798841	1.070357113	1.071111919	1.130597247
4	1.015984808	1.034229000	1.049236898	1.088959668
5	1.007756521	1.016462540	1.034524439	1.061673910
...	...	...	...	...
26	1.000000001			
27		1.000000001		
...				
57			1.000000002	
...				
59				1.000000002

When comparing the results listed in Table 19, we can see that the new acceleration technique does accelerate the convergence speed of the Ishikawa iteration.

- $\alpha_n = \frac{2n}{3n+1}$, $\beta_n = \frac{3n}{4n+1}$, $x_{n+1} = \frac{n+1}{3n+1}x_n + \frac{2n}{n+1}\frac{1}{\frac{n+1}{4n+1}x_n + \frac{3n}{4n+1}\frac{1}{x_n}}$, $n \geq 0$

for the initial value $x_0 = 1.25$ the accelerated iterative method is:

$$x_{n+1} = \frac{n+1}{3n+1}x_n + \frac{2n}{3n+1}\frac{1}{\frac{n+1}{4n+1}x_n + \frac{3n}{4n+1}g_\lambda(x_n)}, \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 6.4x_n - 10.24x_n^2 + 6.144x_n^3 - 1.31072x_n^4}{1 + 6.4 - 10.24x_n + 6.144x_n^2 - 1.31072x_n^3}, \quad n \geq 0$$

and for the initial value $x_0 = 1.5$ the accelerated iterative method is:

$$x_{n+1} = \frac{n+1}{3n+1}x_n + \frac{2n}{3n+1}\frac{1}{\frac{n+1}{4n+1}x_n + \frac{3n}{4n+1}g_\lambda(x_n)}, \quad n \geq 0$$

where

$$g_\lambda(x_n) = \frac{\frac{1}{x_n} + 4.444444444x_n - 5.925925925x_n^2 + 2.962962963x_n^3 - 0.5267489712x_n^4}{1 + 4.444444444 - 5.925925925x_n + 2.962962963x_n^2 - 0.5267489712x_n^3}, \quad n \geq 0$$

when the initial value is 1.25 the best convergence is $k = 4$, and for the initial value 1.5 the best convergence rate is, also, obtained for $k = 4$, the results are listed in Table 20.

Table 20

	Is. it. with Padé	*Is. it. with* Padé	*Is. it.*	*Is. it.*
0	1.25	1.5	1.25	1.5
1	1.085674157	1.183333333	1.135204082	1.250000000
2	1.022087283	1.050676310	1.078260816	1.137218045
3	1.004903754	1.011859767	1.047426934	1.080743670
4	1.001011585	1.002560027	1.029529310	1.049427350
5	1.000200482	1.000530548	1.018696882	1.030978334
...	...	...	...	...
13	1.000000000			
14		1.000000000		
...				
45			1.000000000	
46				1.000000000

When comparing the results listed in Table 20, we can see that the new acceleration technique does accelerate the convergence speed of the Ishikawa iteration.
If we compare the results listed in the Tables 16-20 we observe that the new Padé-type acceleration technique does accelerates the convergence speed of the Ishikawa iteration. We have done test on ten operators, starting from different initial values and for different values for the real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that appear in the definition of Ishikawa iteration and the results were the same as those presented in the Tables 16-20.
We recall the fact that the operator $T = \frac{1}{x}, x \in [\frac{1}{2}, 2]$, $T : \left[\frac{1}{2}, 2\right] \to \left[\frac{1}{2}, 2\right]$, presented in the second part of this chapter has the following properties:
(1) T is Lipschitzian with constant $L = 4$;
(2) T is strictly pseudocontractive;
(3) $Fix(T) = \{1\}$, where $Fix(T) = \{x \in K | Tx = x\}$;
(4) The Picard iteration associated to T does not converge to the fixed point of T, for any $x_0 \in K\ \{1\}$;
(5) The Krasnoselskij iteration associated to T converges to the fixed point $x^* = 1$, for any $x_0 \in K$ and $\mu \in \left(o, \frac{1}{16}\right)$;

(6) The Mann iteration associated to T with $\alpha_n = \frac{n}{2n+1}$, $n \geq 0$ and $x_0 = 2$ converges to 1, the unique fixed point of T.
In the first chapter, Paragraph 1.2.6, Theorem 1.14 and Theorem 1.15, we have presented a result for the Krasnoselskij, Mann and Ishikawa iteration in class of Lipschitz pseudocontractive mappings taken from [3], we remained that these results shown that the Krasnoselskij iteration converges faster than Mann iteration to the fixed point of a Lipschitz pseudocontractive mapping and that Mann iteration converges faster than Ishikawa iteration to the fixed point of a Lipschitz pseudocontractive mapping. Consequently, for our presented operator T, that is a Lipschitz pseudocontractive mapping we expected that the accelerated Padé-type acceleration technique applied to the Krasnoselskij iteration has a better convergence speed than the accelerated Padé-type acceleration technique applied to the Mann iteration and that the accelerated Padé-type acceleration technique applied to the Mann iteration has a better convergence speed than the accelerated Padé-type acceleration technique applied to the Ishikawa iteration. From the results listed in Tables 1-15 we can say that the first affirmation is true and in the second case the accelerated Padé-type acceleration technique applied to Ishikawa iteration has a better convergence than the accelerated Padé-type acceleration technique applied to the Mann iteration.

3 New acceleration methods for sequences that converge linearly

All chapter is Author's original result.

In the last decade an important domain of numerical analysis has became the development and the improvement of several extrapolation algorithms, an extrapolation method is in fact a sequence transformation constructed by extrapolation. Among such sequence transformations, the most well known are Richardson's extrapolation [97], Aitken's Δ^2 process [1], others sequence transformations are: Romberg's process [102], Shanks transformation [108], Wynn's ϵ and ρ-algorithms [126], Brezinski θ-algorithm [21] and Levin's transformation [75]. Motivated by this fact, in this chapter we present some new acceleration methods inspired from Aitken's Δ^2 process, defined by the relation (1.27) and by its iterated form, defined by the relation (1.30). As we said in the second chapter by an extrapolation process a scalar sequence $\{S_n\}$, convergent to S, is transformed into a new sequence $\{T_n\}$, through a sequence transformation $T : \{S_n\} \to \{T_n\}$, and it is said that the transformation T accelerates the convergence speed of the sequence $\{S_n\}$ if it is satisfying the relation (1.20).

Similarly to Chapter 1, we consider a nonlinear equation

$$f(x) = 0, \tag{3.1}$$

which is equivalently written as a fixed point problem

$$g(x) = x, \tag{3.2}$$

where $g : [a, b] \to [a, b]$ is the iteration function.

We assume that α is the unique solution of (3.1) in the interval (a, b) and let $x_0 \in [a, b]$ be an initial approximation sufficiently close to α. Under appropriate assumptions on f (and therefore on g), the Picard iteration

$$x_{n+1} = g(x_n), \quad n = 0, 1, \ldots \tag{3.3}$$

converges to the (unique) fixed point, x^*, of g, which is the (unique) solution of (3.1) in the interval (a, b), as it said in the first chapter, Theorem 1.1 (The contraction mapping principle), we remind here the properties of the operator g:
(i) g has a unique fixed point, that is, $F_g = \{x^*\}$;
(ii) The Picard iteration associated to g, i.e., the sequence $\{x_n\}_{n\geq 0}$ defined by

$$x_n = g(x_{n-1}) = g^n(x_0), \quad n = 1, 2, \ldots \tag{3.4}$$

converges to x^*, for any initial guess $x_0 \in [a, b]$;
(iii) The following a priori and a posteriori error estimates hold:

$$d(x_n, x^*) \leq \frac{a^n}{1-a} d(x_0, x_1), n = 0, 1, 2, \ldots \tag{3.5}$$

$$d(x_n, x^*) \leq \frac{a}{1-a} d(x_{n-1}, x_n), n = 0, 1, 2, \ldots \tag{3.6}$$

(iv) The rate of convergence is given by

$$d(x_n, x^*) \leq a d(x_{n-1}, x^*) \leq a^n d(x_0, x_1), n = 1, 2, \ldots \tag{3.7}$$

It is well known that the convergence order of the Picard iteration is generally linear, see Berinde [12], so the method converges slowly to the fixed point x^*. In order to improve the convergence speed of (3.3) we must use a convergence acceleration process. In the sequel we present a new acceleration method for the Picard iteration, that we called the B-algorithm, then we give some representations for the new proposed method, an iterated form for the B-algorithm and for its representations, we give some extensions for the iterated form of the B-algorithm and we end this chapter with some examples in which we apply the B-algorithm, its iterated form, the representations for the B-algorithm and their iterated forms and the extensions for the new proposed technique to the Krasnoselskij, Mann and Ishikawa iterations.

3.1 The B-algorithm for accelerating Picard iteration

Author's original contribution in this paragraph are: the new acceleration method, called the B-algorithm (3.8), Theorems 3.1, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, Observation 3.3 and the Examples 3.2-3.12.
In this paragraph we propose a new acceleration technique obtained by a suitable manipulation of the expression (1.27), the method is called the B-algorithm. The new proposed acceleration technique is given by following nonlinear recursive scheme

$$B_1^{(n)} = S_{n+3} - \frac{[\overline{\Delta} S_{n+1}][\Delta S_{n+2}]}{\overline{\Delta} S_{n+1} - \overline{\Delta} S_n}, \quad n = 0, 1, \dots \tag{3.8}$$

where $\{S_n\}$ is the sequence to be accelerated, Δ denotes the forward difference operator, defined by $\Delta S_n = S_{n+1} - S_n$ and we denote $\overline{\Delta}$, the forward difference operator with two steps, defined by $\overline{\Delta} S_n = S_{n+2} - S_n$.

The difference between B-algorithm and Aitken's Δ^2 process, defined by

$$A_1^{(n)} = S_n - \frac{[\Delta S_n]^2}{\Delta S_{n+1} - \Delta S_n}, \quad n = 0, 1, \dots \tag{3.9}$$

where $\{S_n\}$ is the sequence to be accelerated and Δ was defined above is that in the proposed acceleration method appears the forward difference operator with two steps. Consequently, the new introduced algorithm needs four therms for computing the recursion (3.8), in spite of Aitken's Δ^2 process that needs three therms for computing the recursion (3.9), so our algorithm is more complicated but more effective as will see in the last part of this chapter.
In what follows we present some theoretical results for the new proposed acceleration technique.

Theorem 3.1 *(Bumbariu, [32]) Let $\{S_n\}_{n\in\mathbb{N}}$ be a sequence converging to S such that*

$$\lim_{n\to\infty} \frac{S_{n+1} - S}{S_n - S} = \lambda, \quad |\lambda| < 1.$$

If the sequence $\{\alpha_n\}$, $\alpha_n = S_n - S$ satisfies

$$\lim_{n\to\infty} \frac{\alpha_n \alpha_{n+2}}{\alpha_{n+3} - \alpha_{n+2} - \alpha_{n+1} + \alpha_n} = 0, \tag{3.10}$$

then the sequence $\{B_n\}_{n\in\mathbb{N}}$ defined by the relation (3.8) *is an accelerating sequence of $\{S_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ and $\{A_n\}_{n\in\mathbb{N}}$, defined by* (3.9), *have the same rate of convergence.*

Proof In order to compare the rate of convergence of two scalar sequences we will use the Definition 1.13, given in Chapter 2, Paragraph 1.4.
Let be

$$\frac{S_{n+1}-S}{S_n-S}=\lambda+e_n, \tag{3.11}$$

and

$$\frac{S_{n+3}-S}{S_{n+2}-S}=\lambda+e_{n+2}, \tag{3.12}$$

where $e_n,\ n\in\mathbb{N}_0$ is the remainder sequence, that is, $\epsilon_n\to 0$ as $n\to\infty$.
From (3.11) and (3.12) we can write

$$\frac{S_{n+1}-S}{S_n-S}-e_n=\frac{S_{n+3}-S}{S_{n+2}-S}-e_{n+2}.$$

Doing some elementary computations we get

$$S_nS_{n+3}-S_{n+1}S_{n+2}=S[(S_{n+3}-S_{n+1})-(S_{n+2}-S_n)]+U_n, \tag{3.13}$$

where

$$\begin{aligned}U_n&=S^2-e_nS_nS_{n+2}+e_nS_nS+e_nSS_{n+2}-e_nS^2+e_{n+2}S_nS_{n+2}-e_{n+2}SS_n-S^2-e_{n+2}SS_{n+2}+\\&+e_{n+2}S^2\\&=(e_{n+2}-e_n)(S-S_n)(S-S_{n+2}).\end{aligned} \tag{3.14}$$

The relation (3.13) can be written in the following form

$$\frac{S_nS_{n+3}-S_{n+1}S_{n+2}}{S_{n+3}-S_{n+2}-S_{n+1}+S_n}=S+\frac{U_n}{S_{n+3}-S_{n+2}-S_{n+1}+S_n}.$$

We know that $\lim_{n\to\infty}S_n=S$, where $S_n=S+\alpha_n,\ \alpha_n\longrightarrow 0$. Consequently we have

$$S_{n+3}-S_{n+2}=\alpha_{n+3}-\alpha_{n+2},$$

and

$$S_{n+1}-S_n=\alpha_{n+1}-\alpha_n.$$

Further we have to prove the following limit

$$\lim_{n\to\infty}\frac{(e_{n+2}-e_n)\alpha_n\alpha_{n+2}}{\alpha_{n+3}-\alpha_{n+2}-\alpha_{n+1}+\alpha_n}=0. \tag{3.15}$$

From the theorem's hypothesis we have that $\lim_{n\to\infty}\frac{\alpha_n\alpha_{n+2}}{\alpha_{n+3}-\alpha_{n+2}-\alpha_{n+1}+\alpha_n}=0$ therefore we have to prove that the sequence $\{e_{n+2}-e_n\}$ is bounded, which follows from

$$|e_{n+2}-e_n|\le|\epsilon_{n+2}|+|\epsilon_n|\le M,\quad \forall M\in\mathbb{R}.$$

So, the limit (3.15) is proved, consequently B-algorithm converges to ,S, the limit of the sequence $\{S_n\}$, when $\alpha_n=S_n-S$ satisfies the relation (3.10).

Further we prove that $\{B_n\}_{n\in\mathbb{N}}$ converges faster than $\{S_n\}_{n\in\mathbb{N}}$ to S.

$$\frac{B_n-S}{S_n-S}=\frac{S_{n+3}-S+\frac{(S_{n+3}-S_{n+1})(S_{n+3}-S_{n+2})}{(S_{n+2}-S_n)-(S_{n+3}-S_{n+1})}}{S_n-S}.$$

Adding $S-S$ in parenthesis and dividing numerator and denominator by S_{n+2} we obtain

$$\frac{B_n - S}{S_n - S} = \frac{S_{n+3}-S}{S_n - S} + \frac{\left(\frac{S_{n+3}-S}{S_{n+2}-S} - \frac{S_{n+1}-S}{S_{n+2}-S}\right)\left(\frac{S_{n+3}-S}{S_{n+2}-S} - 1\right)}{\left(1 - \frac{S_n - S}{S_{n+2}-S}\right) - \left(\frac{S_{n+3}-S}{S_{n+2}-S} - \frac{S_{n+1}-S}{S_{n+2}-S}\right)} \frac{S_{n+2}-S}{S_n - S}.$$

Taking the limit when $n \to \infty$ we obtain

$$\begin{aligned}\lim_{n\to\infty} \frac{B_n - S}{S_n - S} &= \lambda^3 + \frac{\left(\lambda - \frac{1}{\lambda}\right)(\lambda - 1)}{\left(1 - \frac{1}{\lambda^2}\right) - \left(\lambda - \frac{1}{\lambda}\right)}\lambda^2 \\ &= \lambda^3 - \lambda^3 \\ &= 0.\end{aligned}$$

Therefore $\{B_n\}_{n\in\mathbb{N}}$ converges faster than $\{S_n\}_{n\in\mathbb{N}}$.
In the sequel we prove that $\{B_n\}_{n\in\mathbb{N}}$ and $\{A_n\}_{n\in\mathbb{N}}$ have the same convergent speed. B-algorithm is defined as

$$B_n = S_{n+3} - \frac{(S_{n+3} - S_{n+1})(S_{n+3} - S_{n+2})}{S_{n+3} - S_{n+2} - S_{n+1} + S_n}.$$

Adding $S-S$ in parenthesis and dividing numerator and denominator by S_{n+2} we obtain

$$B_n = S_{n+3} + \frac{\left(\frac{S_{n+3}-S}{S_{n+2}-S} + \frac{S_{n+1}-S}{S_{n+2}-S}\right)(S_{n+3} - S_{n+2})}{\left(\frac{S_{n+2}-S}{S_{n+2}-S} - \frac{S_n - S}{S_{n+2}-S}\right) - \left(\frac{S_{n+3}-S}{S_{n+2}-S} - \frac{S_{n+1}-S}{S_{n+2}-S}\right)}.$$

Taking the limit when $n \to \infty$ we get

$$\lim_{n\to\infty} B_n = S + \frac{1 + \frac{1}{\lambda}}{1 - \frac{1}{\lambda^2} - \lambda + \frac{1}{\lambda}} \lim_{n\to\infty}(S_{n+3} - S_{n+2}),$$

and we can prove that

$$\lim_{n\to\infty} A_n = S + \frac{1 - \frac{1}{\lambda}}{2 - \frac{1}{\lambda} - \lambda} \lim_{n\to\infty}(S_{n+1} - S_n).$$

Taking the limit when $n \to \infty$ and dividing $B_n - S$ by $A_n - S$ we obtain

$$\lim_{n\to\infty} \frac{B_n - S}{A_n - S} = \frac{\lim_{n\to\infty}(B_n - S)}{\lim_{n\to\infty}(A_n - S)} = \frac{\frac{1+\frac{1}{\lambda}}{1-\frac{1}{\lambda^2}-\lambda+\frac{1}{\lambda}} \lim_{n\to\infty}(S_{n+3} - S_{n+2})}{\frac{1-\frac{1}{\lambda}}{2-\frac{1}{\lambda}-\lambda} \lim_{n\to\infty}(S_{n+1} - S_n)}.$$

Making some elementary calculus we obtain

$$\lim_{n\to\infty} \frac{B_n - S}{A_n - S} = \frac{(\lambda+1)\lambda}{\lambda^2 + 1} \frac{\lim_{n\to\infty}(S_{n+3} - S_{n+2})}{\lim_{n\to\infty}(S_{n+1} - S_n)}.$$

Adding $S-S$ in parenthesis and dividing numerator and denominator by S_{n+2} we obtain

$$\lim_{n\to\infty} \frac{B_n - S}{A_n - S} = \frac{(\lambda+1)\lambda}{\lambda^2 + 1} \frac{\lim_{n\to\infty}\left[\frac{S_{n+3}-S}{S_{n+2}-S} - 1\right]}{\lim_{n\to\infty}\left[\frac{S_{n+1}-S}{S_{n+2}-S} - \frac{S_n - S}{S_{n+2}-S}\right]}.$$

Taking the limit when $n \to \infty$ we get

$$\lim_{n\to\infty} \frac{B_n - S}{A_n - S} = \frac{(\lambda+1)\lambda}{\lambda^2+1} \frac{\lambda - 1}{\frac{1}{\lambda} - \frac{1}{\lambda^2}} = \frac{(\lambda+1)\lambda^3}{\lambda^2+1} \neq 0.$$

Consequently, $\{B_n\}_{n\in\mathbb{N}}$ and $\{A_n\}_{n\in\mathbb{N}}$ have the same convergent speed.

Remark 3.1 *(i) One important observation here, is that, Theorem 2.8 [81] is not valid in general.*
(ii) B-algorithm generally does not accelerate the convergence speed of the sequence $\{S_n\}$ if $\{\alpha_n\}$ does not verify the condition (3.10) *as shown by the next example.*

Example 3.1 *Let $\alpha_n = \frac{1}{n+3}$, $\alpha_{n+1} = \frac{2}{n+3}$, $\alpha_{n+2} = -\frac{2}{n}$, $\alpha_{n+3} = -\frac{1}{n}$, $n \geq 1$ be four sequences of real numbers, where $\lim_{n\to\infty} \alpha_n = 0$. Prove that* (3.10) *is not fulfilled.*

Introducing the sequences in (3.10) we obtain

$$\lim_{n\to\infty} \frac{\alpha_n \alpha_{n+2}}{\alpha_{n+3} - \alpha_{n+2} - \alpha_{n+1} + \alpha_n} = \lim_{n\to\infty} \frac{\frac{1}{n+3}\left(-\frac{2}{n}\right)}{\frac{2}{n(n+3)}} = -1 \neq 0.$$

Consequently, the relation (3.10) is not fulfilled even if $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \alpha_{n+1} = 0$, $\lim_{n\to\infty} \alpha_{n+2} = 0$ and $\lim_{n\to\infty} \alpha_{n+3} = 0$.

The B-algorithm, consists in constructing the Picard iteration, (3.3), and the B sequence transformation as follows:
Step 0: $y_0 := x_0$
Step $n \geq 1$:
(1) set $S_0 = y_0$, $S_1 = f(S_0)$, $S_2 = f(S_1)$, $S_3 = f(S_2)$,
(2) compute B_n by the relation (3.21) from $S_0,\ S_1,\ S_2,\ S_3$,
(3) set $y_n = B_n$.

In what follows we use the concept of convergence order that was defined in the Chapter 2, Paragraph 1.8.1, Definition 1.17.

It is well known that the iteration function of the Steffensen's method [68] is

$$\overline{g}(x) = \frac{xg(g(x)) - (g(x))^2}{g(g(x)) - 2g(x) + x}, \tag{3.16}$$

and let $G(x)$ be the iteration function of the B-algorithm defined by

$$G(x) = \frac{xg(g(g(x))) - g(x)g(g(x))}{g(g(g(x))) - g(g(x)) - g(x) + x}. \tag{3.17}$$

In the sequel we present a convergence result for the acceleration method (3.16).

Theorem 3.2 *[68]*
(1) *If the functional iteration applied to $x = g(x)$ is of order $p \geq 2$ for some root α of* (3.12) *then Steffensen's method has order $2p - 1$.*
(2) *If the functional iteration applied to $x = g(x)$ is of first order (but not necessarily convergent) for a simple root α of* (3.12) *then Steffensen's method is of second order.*
(3) *If as in* (2)*, the root α of* (3.12) *has multiplicity $m \geq 2$, then Steffensen's method is first order with the asymptotic convergence factor $C = 1 - \frac{1}{m}$.*

The proof can be found in [68], page 107.

A similar result, but for the B-algorithm, is given by the following theorem.

Theorem 3.3 *(Bumbariu, [33])*
(i) If the Picard iteration applied to $x = g(x)$ *is of order* $p \geq 2$ *for some root* α *of* (3.12) *then the* B*-algorithm has order* $p^2 + p - 1$.
(ii) If the Picard iteration applied to $x = g(x)$ *is of first order (but not necessarily convergent) for a simple root* α *of* (3.12) *then the* B*-algorithm is of second order.*
(iii) If as in (2)*, the root* α *of* (3.12) *has multiplicity* $m \geq 3$*, then* B*-algorithm is of first order and the asymptotic convergence factor is* $C < 1 - \frac{1}{m-1}$.

Proof We recall the fact that the iteration function for the B-algorithm is given by (3.8) and assume that $x = \alpha$ is a root of (3.12) and that:

$$\begin{aligned} &g'(\alpha) = g''(\alpha) = \ldots = g^{(p-1)}(\alpha) = 0; \\ &g^{(p)}(\alpha) = p!A \neq 0; \\ &g^{(p+1)} \ \exists \ in \ |x - \alpha| \leq \rho. \end{aligned} \tag{3.18}$$

These conditions imply that $g(x)$ determines a method of p-order. From Taylor's theorem [84] and (3.18) for every ϵ such that $|\epsilon| \leq \rho$, we have

$$\begin{aligned} g(\alpha + \epsilon) &= \alpha + A\epsilon^p + \frac{g^{(p+1)}(\alpha + \theta\epsilon)}{(p+1)!}\epsilon^{p+1}, 0 < \theta < 1 \\ &= \alpha + A\epsilon^p + B\epsilon^{p+1} \\ &= \alpha + \delta, \end{aligned} \tag{3.19}$$

where $B = \frac{g^{(p+1)}(\alpha+\theta\epsilon)}{(p+1)!}$ and $\delta = (A + B\epsilon)\epsilon^p$.
Hence,

$$\begin{aligned} g(\alpha + \delta) &= \alpha + A\delta^p + B'\delta^{p+1} \\ &= \alpha + \gamma, \end{aligned} \tag{3.20}$$

where $B' = \frac{g^{(p+1)}(\alpha+\Theta\delta)}{(p+1)!}, 0 < \Theta < 1$ and $\gamma = (A + B'\delta)\delta^p$. Therefore

$$g(\alpha + \gamma) = \alpha + A\gamma^p + B''\gamma^{p+1} \tag{3.21}$$

where $B'' = \frac{g^{(p+1)}(\alpha+\Gamma\gamma)}{(p+1)!}, 0 < \Gamma < 1$.

From (3.19)-(3.21) in (3.17) with $x = \alpha + \epsilon$ and $\epsilon \neq 0$, it results

$$\begin{aligned} G(\alpha + \epsilon) &= \frac{(\alpha + \epsilon)g(g(g(\alpha + \epsilon))) - g(\alpha + \epsilon)g(g(\alpha + \epsilon))}{g(g(g(g(\alpha + \epsilon)))) - g(g(g(\alpha + \epsilon)g)) - g(g(\alpha + \epsilon)) + (\alpha + \epsilon)} \\ &= \alpha - \frac{\delta\gamma - A\epsilon\gamma^p - B''\epsilon\gamma^{p+1}}{\epsilon - \delta - \gamma + A\gamma^p + B''\gamma^{p+1}}. \end{aligned} \tag{3.22}$$

There are two cases to be considered: a) $p \geq 2$ and b) $p = 1$.

a) First, for $p \geq 2$ the expression (3.22) can be written

$$G(\alpha+\epsilon) = \alpha - [A + B'(A+B\epsilon)\epsilon^p](A+B\epsilon)^{p+1}\epsilon^{p^2+p-1}\cdot$$
$$\cdot\left(\frac{1 - A\epsilon[A+B'(A+B\epsilon)\epsilon^p]^{p-1}(A+B\epsilon)^{p^2-p-1}\epsilon^{p^3-p^2-p} - A_1B''\epsilon^{p^3-p+1}}{1-[A+B'(A+B\epsilon)\epsilon^p](A+B\epsilon)^p\epsilon^{p^2-1} - (A+B\epsilon)\epsilon^{p-1} + A_2\epsilon^{p^3-1} + A_3\epsilon^{p^3+p^2-1}}\right), \tag{3.23}$$

where
$A_1 = [A+B'(A+B\epsilon)\epsilon^p]^p(A+B\epsilon)^{p^2-1}$,
$A_2 = A[A+B'(A+B\epsilon)\epsilon^p]^p(A+B\epsilon)^{p^2}$,
$A_3 = B''[A+B'(A+B\epsilon)\epsilon^p]^{p+1}(A+B\epsilon)^{p^2+p}$.

When ϵ approaches 0 the expression in the parenthesis approaches 1 and so the above relation, (3.23), can be written

$$G(\alpha+\epsilon) = \alpha - A^{p+2}\epsilon^{p^2+p-1} + O(\epsilon^{p^2+p}), \quad p \geq 2. \tag{3.24}$$

b) For $p = 1$ the expression (3.22) can be written

$$G(\alpha+\epsilon) = \alpha - [A+B'(A+B\epsilon)\epsilon](A+B\epsilon)\epsilon^2\cdot$$
$$\left(\frac{(B - A^2B'') - (ABB'' + A^2B'B'')\epsilon - 2ABB'B''\epsilon^2 - B^2B'B''\epsilon^3}{B_1 + B_2\epsilon + B_3\epsilon^2 + B_4\epsilon^3 + B_5\epsilon^4 + B_6\epsilon^5 + 4AB^3B'^2B''\epsilon^6 + B^4B'^2B''\epsilon^7}\right), \tag{3.25}$$

where
$B_1 = (1-A^2)(1-A)$,
$B_2 = A^4B'' + A^3B' + A^2B - B - AB - A^2B'$,
$B_3 = 2A^3BB'' + 2A^4B'B'' + 2A^2BB' - 2ABB'$,
$B_4 = 6A^3BB'B'' + A^4B'^2B'' + A^2B^2B'' + AB^2B' - B^2B'$,
$B_5 = 6A^2B^2B'B'' + 4A^3BB'^2B''$,
$B_6 = 5A^2B^2B'^2B'' + 2AB^3B'B'' + AB^2B'^2B''$.

In general, if $A \neq 1$ the expression in the parenthesis approaches $\frac{B^*}{1-A}$ when $\epsilon \to 0$ because B'' and B approach $B^* = \frac{g''(\alpha)}{2}$ and so (3.25) can be written as

$$G(\alpha+\epsilon) = \alpha - \frac{A^2B^*}{1-A}\epsilon^2 + O(\epsilon^3), \ p = 1, g'(\alpha) = A \neq 1. \tag{3.26}$$

If $g'(\alpha) = A = 1$ and α has multiplicity m then

$$G(\alpha+\epsilon) = \alpha + [1+B'(1+B\epsilon)\epsilon](1+B\epsilon)\epsilon$$
$$\left(\frac{B'' - B + (BB'' + B'B'')\epsilon + 2BB'B''\epsilon^2 + B^2B'B''\epsilon^3}{C_1 + C_2\epsilon + C_3\epsilon^2 + C_4\epsilon^3 + C_5\epsilon^4 + 4B^3B'^2B''\epsilon^5 + B^4B'^2B''\epsilon^6}\right), \tag{3.27}$$

where
$C_1 = B'' - B$,
$C_2 = 2(BB'' + B'B'')$,
$C_3 = B^2B'' + B'^2B'' + 6BB'B''$,
$C_4 = 6B^2B'B'' + 4BB'^2B''$,
$C_5 = 6B^2B'^2B'' + 2B^3B'B''$.

Let $g(\alpha) = \alpha, g'(\alpha) = 1, g''(\alpha) = \ldots = g^{(m-1)}(\alpha) = 0$ and $g^{(m)} \neq 0$. In the hypothesis that g has derivatives of order $3m$, $g(\alpha + \epsilon) = \alpha + \epsilon + B\epsilon^3$, where $m \geq 3$ and

$$B(\epsilon) = \frac{g^{(m)}(\alpha)}{m!}\epsilon^{m-3} + \frac{g^{(m+1)}(\alpha)}{(m+1)!}\epsilon^{m-2} + \ldots \tag{3.28}$$

and similarly, with $\delta = \epsilon + B\epsilon^3$ we have $g(\alpha + \delta) = \alpha + \delta + B'\delta^3$ where

$$B'(\delta) = \frac{g^{(m)}(\alpha)}{m!}\delta^{m-3} + \frac{g^{(m+1)}(\alpha)}{(m+1)!}\delta^{m-2} + \ldots \tag{3.29}$$

be observe that $\delta = \epsilon + B\epsilon^3 = \epsilon + \frac{g^{(m)}(\alpha)}{m!}\epsilon^m + \ldots$ and

$$B'(\delta) = \frac{g^{(m)}(\alpha)}{m!}\epsilon^{m-3} + (m-3)\left[\frac{g^{(m)}(\alpha)}{m!}\right]^2 \epsilon^{2m-4} + \ldots \tag{3.30}$$

$$B''(\gamma) = \frac{g^{(m)}(\alpha)}{m!}\gamma^{m-3} + \frac{g^{(m+1)}(\alpha)}{(m+1)!}\gamma^{m-2} + \ldots \tag{3.31}$$

where $\gamma = \delta + B'\delta^3 = \epsilon + 2\frac{g^{(m)}(\alpha)}{m!}\epsilon^m + m\left[\frac{g^{(m)}(\alpha)}{m!}\right]^2 \epsilon^{2m-1} + \ldots$ then

$$B''(\gamma) = \frac{g^{(m)}(\alpha)}{m!}\epsilon^{m-3} + 2(m-3)\left[\frac{g^{(m)}(\alpha)}{m)!}\right]^2 \epsilon^{2m-4} + (m-3)(2m-4)\left[\frac{g^{(m)}(\alpha)}{m)!}\right]^3 \epsilon^{3m-5} \tag{3.32}$$

and therefore

$$B'' - B = 2(m-3)\left[\frac{g^{(m)}(\alpha)}{m!}\right]^2 \epsilon^{2m-4} + \ldots \tag{3.33}$$

$$BB'' + B'B'' = 2\left[\frac{g^{(m)}(\alpha)}{m)!}\right]^2 \epsilon^{2m-6} + \ldots. \tag{3.34}$$

Introducing (3.33) and (3.34) in (3.27) be obtain

$$G(\alpha + \epsilon) = \alpha + \frac{m - 3 + \epsilon + \ldots}{m - 3 + 2\epsilon + \ldots} < \alpha + 1 - \frac{1}{m-1}. \tag{3.35}$$

Which completes the proof.

Remark 3.2 *If we compare the results from Theorem 3.2 and Theorem 3.3 we can see that B-algorithm has a better order of convergence than Steffensen's method for all* $p \geq 2$.

In the sequel let $\{S_n\}$ be a model sequence, defined by

$$S_n = S + c\lambda^n, \quad c \neq 0, \ |\lambda| < 1, \ n \in N_0, \tag{3.36}$$

where each element of the sequence $\{S_n\}$ is obtained by three unknowns c, λ and the limit S, that for a sequence transformation will need at least three elements of the model sequence (3.36) in order to obtain the limit S. Further we define the first difference of S_n and the first difference operator with two steps of S_n as follows

$$\Delta S_n = c\lambda^n(\lambda - 1),$$

$$\overline{\Delta} S_n = c\lambda^n(\lambda^2 - 1).$$

Before giving some other convergent theorems for the B-algorithm we recall the definition of an exact sequence transformation.

Definition 3.1 *[25] Let T_n be an acceleration method for the sequence $\{S_n\}$, for which $\lim_{n\to\infty} S_n = S$. We say that the transformation T_n is exact for the sequence $\{S_n\}$ if $T_n = S$ for all n.*

We have the following convergent theorems for the B-algorithm.

Theorem 3.4 *(Bumbariu, [39]) The B-algorithm is exact for the model sequence* (3.36).

Proof

$$\begin{aligned} B_1^{(n)} &= S + c\lambda^{n+3} - \frac{(S + c\lambda^{n+3} - S - c\lambda^{n+2})(S + c\lambda^{n+3} - S - c\lambda^{n+1})}{S + c\lambda^{n+3} - S - c\lambda^{n+2} - S - c\lambda^{n+1} + S + c\lambda^n} \\ &= S + c\lambda^{n+3} - \frac{c^2\lambda^{2n+3}(\lambda - 1)(\lambda^2 - 1}{c\lambda^n(\lambda^3 - \lambda^2 - \lambda + 1} \\ &= S + c\lambda^{n+3} - c\lambda^{n+3}\frac{(\lambda - 1)(\lambda^2 - 1)}{(\lambda - 1)(\lambda^2 - 1)} \\ &= S. \end{aligned}$$

As a consequence of Theorem 3.4 we can give the following theorem.

Theorem 3.5 *(Bumbariu, [39]) The B-algorithm is exact for the model sequence*

$$S_n = S + ax^n + by^n, \quad 0 < |y| < |x| < 1, \; a, b \neq 0, \tag{3.37}$$

which is a generalization of the model sequence (3.36).

Proof Doing some computations, we are lead to the fact that the B-algorithm eliminates the dominator therm ax^n from the model sequence (3.37)

$$B_1^{(n)} = S + \frac{by^n \frac{(y-x)^2(y+x)}{(x-1)^2(x+1)}}{1 + \frac{b}{a}\left(\frac{y}{x}\right)^n \frac{(y^2-1)(y+1)}{(x^2-1)(x+1)}}. \tag{3.38}$$

Because we have the assumption $0 < |y| < |x| < 1$, the transformed sequence (3.38) converges faster than the original sequence (3.37). Taking the limit when $n \to \infty$ in (3.38), $\left(\frac{y}{x}\right)^n$ vanishes and then $B_1^{(n)}$ has a better convergence speed than the sequence defined by the relation (3.37).

Theorem 3.6 *(Bumbariu, [39]) $B_1^{(n)} = S$, for $\forall n > N$ if the sequence $\{S_n\}$ is satisfying the following relation*

$$a_0(S_n - S) + a_1(S_{n+1} - S) = 0, \quad \forall n \tag{3.39}$$

where a_0 and a_1 are parameters such that $a_0 + a_1 \neq 0$ and $a_0 \cdot a_1 \neq 0$.

Proof From the relation (3.39) we obtain

$$a_0(S_{n+1} - S) + a_1(S_{n+2}) = 0,$$

and

$$a_0(S_{n+2} - S) + a_1(S_{n+3}) = 0.$$

Doing some easy computations, we can write every $S_n - S, S_{n+1} - S$ and $S_{n+2} - S$ depending on $S_{n+3} - S$ as follows

$$S_n - S = -\left(\frac{a_1}{a_0}\right)^3 (S_{n+3} - S), \tag{3.40}$$

$$S_{n+1} - S = \left(\frac{a_1}{a_0}\right)^2 (S_{n+3} - S), \tag{3.41}$$

$$S_{n+2} - S = -\frac{a_1}{a_0}(S_{n+3} - S). \tag{3.42}$$

$B_1^{(n)}$ is defined by the following relation

$$B_1^{(n)} = S_{n+3} - \frac{(S_{n+3} - S_{n+2})(S_{n+3} - S_{n+1})}{S_{n+3} - S_{n+2} - S_{n+1} + S_n}, \quad n = 0, 1, \ldots,$$

adding $S - S$ in each parenthesis of the above relation we get

$$B_1^{(n)} = S_{n+3} - \frac{[(S_{n+3} - S) - (S_{n+2} - S)][(S_{n+3} - S) - (S_{n+1} - S)]}{(S_{n+3} - S) - (S_{n+2} - S) - (S_{n+1} - S) + (S_n - S)}, \quad n = 0, 1, \ldots,$$

then replacing $S_n - S, S_{n+1} - S$ and $S_{n+2} - S$ according to the relations (3.40)-(3.42) we have

$$\begin{aligned}
B_1^{(n)} &= S_{n+3} - \frac{\left[(S_{n+3} - S) + \frac{a_1}{a_0}(S_{n+3} - S)\right]\left[(S_{n+3} - S) - \left(\frac{a_1}{a_0}\right)^2 (S_{n+3} - S)\right]}{(S_{n+3} - S) - \left(\frac{a_1}{a_0}\right)^2 (S_{n+3} - S) + \frac{a_1}{a_0}(S_{n+3} - S) - \left(\frac{a_1}{a_0}\right)^3 (S_{n+3} - S)} \\
&= S_{n+3} - \frac{\left(1 + \frac{a_1}{a_0}\right)\left[1 - \left(\frac{a_1}{a_0}\right)^2\right](S_{n+3} - S)}{\left(1 + \frac{a_1}{a_0}\right) - \left(\frac{a_1}{a_0}\right)^2 \left(1 + \frac{a_1}{a_0}\right)} \\
&= S_{n+3} - \frac{\left(1 + \frac{a_1}{a_0}\right)\left[1 - \left(\frac{a_1}{a_0}\right)^2\right](S_{n+3} - S)}{\left(1 + \frac{a_1}{a_0}\right)\left[1 - \left(\frac{a_1}{a_0}\right)^2\right]} \\
&= S_{n+3} - (S_{n+3} - S) \\
&= S.
\end{aligned}$$

Remark 3.3 *(Bumbariu, [39])*
(1) The assumption $\lambda \neq 1$ from Theorem 3.4 is equivalent with the assumption $a_0 + a_1 \neq 0$ from Theorem 3.6.
(2) B-algorithm can be interpreted as an exponential extrapolation for four consecutive terms of a sequence, S_n, S_{n+1}, S_{n+2} and S_{n+3}. The technique consists in determining the unknowns S, c and λ such that

$$S_n = S + c\lambda^n,$$

$$S_{n+1} = S + c\lambda^{n+1},$$
$$S_{n+2} = S + c\lambda^{n+2},$$
$$S_{n+3} = S + c\lambda^{n+3},$$

for that $B_1^{(n)} = S$.

The B-algorithm is a nonlinear transformation. But, we have linearity for multiplication by a constant and for adding a constant to all the terms of the sequence. This property leads to the following theorem.

Theorem 3.7 *(Bumbariu, [39]) Let* $a \neq 0$ *and* b *be two arbitrary constants, such that,* $V_n = aS_n + b$. *Then*
(i)B-algorithm is quasi-linear, that is

$$B(V_n) = aB(S_n) + b,$$

(ii)

$$B(\Delta V_n) = aB(\Delta S_n).$$

Proof (i) We prove the first part of the theorem.

$$\begin{aligned} B(V_n) &= B(aS_n + b) \\ &= aS_{n+3} + b - \frac{(aS_{n+3} + b - aS_{n+2} - b)(aS_{n+3} + b - aS_{n+1} - b)}{aS_{n+3} + b - aS_{n+2} - b - aS_{n+1} - b + aS_n + b} \\ &= a\left[S_{n+3} - \frac{(S_{n+3} - S_{n+2})(S_{n+3} - S_{n+1})}{S_{n+3} - S_{n+2} - S_{n+1} + S_n}\right] + b \\ &= aB(S_n) + b. \end{aligned} \tag{3.43}$$

(ii) Because $\Delta V_n = a(S_{n+1} - S_n)$ we have

$$\begin{aligned} B(\Delta V_n) &= \Delta V_{n+3} - \frac{[\Delta V_{n+3} - \Delta V_{n+2}][\Delta V_{n+3} - \Delta V_{n+1}]}{\Delta V_{n+3} - \Delta V_{n+2} - \Delta V_{n+1} + \Delta V_n} \\ &= \frac{a[\Delta S_{n+3}\Delta S_n - \Delta S_{n+2}\Delta S_{n+1}]}{\Delta S_{n+3} - \Delta S_{n+2} - \Delta S_{n+1} + \Delta S_n}, \end{aligned} \tag{3.44}$$

and

$$\begin{aligned} aB(\Delta S_n) &= a\left[\Delta S_{n+3} - \frac{[\Delta S_{n+3} - \Delta S_{n+2}][\Delta S_{n+3} - \Delta S_{n+1}]}{\Delta S_{n+3} - \Delta S_{n+2} - \Delta S_{n+1} + \Delta S_n}\right] \\ &= \frac{a[\Delta S_{n+3}\Delta S_n - \Delta S_{n+2}\Delta S_{n+1}]}{\Delta S_{n+3} - \Delta S_{n+2} - \Delta S_{n+1} + \Delta S_n}, \end{aligned} \tag{3.45}$$

therefore relations (3.44) and (3.45) are equivalent and the proof is completed.

Theorem 3.8 *(Bumbariu, [39]) If we apply B-algorithm to a sequence* $\{S_n\}$, *which is convergent to* S *and is satisfying the following relation*

$$\lim_{n\to\infty} \frac{S_{n+1} - S}{S_n - S} = \lim_{n\to\infty} \frac{\Delta S_{n+1}}{\Delta S_n} = \lambda \neq 1,$$

then the sequence $\{B_1^{(n)}\}$ *converges to* S *more rapidly than the sequence* $\{S_{n+1}\}$ *to* S, *that is*

$$\lim_{n\to\infty} \frac{B_1^{(n)} - S}{S_{n+1} - S} = 0.$$

Proof

$$\frac{B_1^{(n)} - S}{S_{n+1} - S} = \frac{S_{n+3} - S + \frac{(S_{n+3}-S_{n+1})(S_{n+3}-S_{n+2})}{(S_{n+2}-S_n)-(S_{n+3}-S_{n+1})}}{S_{n+1} - S}.$$

Adding $S - S$ in parenthesis and dividing numerator and denominator by S_{n+2} we obtain

$$\frac{B_1^{(n)} - S}{S_{n+1} - S} = \frac{S_{n+3} - S}{S_{n+1} - S} + \frac{\left(\frac{S_{n+3}-S}{S_{n+2}-S} - \frac{S_{n+1}-S}{S_{n+2}-S}\right)\left(\frac{S_{n+3}-S}{S_{n+2}-S} - 1\right)}{\left(1 - \frac{S_n-S}{S_{n+2}-S}\right) - \left(\frac{S_{n+3}-S}{S_{n+2}-S} - \frac{S_{n+1}-S}{S_{n+2}-S}\right)} \frac{S_{n+2} - S}{S_{n+1} - S}.$$

Taking the limit when $n \to \infty$ we have

$$\begin{aligned}\lim_{n\to\infty} \frac{B_1^{(n)} - S}{S_{n+1} - S} &= \lambda^2 + \frac{\left(\lambda - \frac{1}{\lambda}\right)(\lambda - 1)}{\left(1 - \frac{1}{\lambda^2}\right) - \left(\lambda - \frac{1}{\lambda}\right)}\lambda \\ &= \lambda^2 - \lambda^2 \\ &= 0.\end{aligned}$$

Therefore $\{B_n\}_{n\in\mathbb{N}}$ converges faster than $\{S_{n+1}\}_{n\in\mathbb{N}}$.

In order to investigate the acceleration of convergence of the method, B-algorithm has been applied to some linear and logarithmically sequences taken from [24].

Example 3.2 *(Bumbariu, [39]) Let us consider the sequence defined by*

$$S_n = n\lambda^n,$$

where $\lambda \in (0, 1)$. *Then* $\lim_{n\to\infty} \frac{B_1^{(n)}}{S_n} = 0$.

When applying B-algorithm to the sequence $\{S_n\}$ we obtain

$$\begin{aligned}B_1^{(n)} &= (n+3)\lambda^{n+3} - \frac{[(n+3)\lambda^{n+3} - (n+2)\lambda^{n+2}][(n+3)\lambda^{n+3} - (n+1)\lambda^{n+1}]}{(n+3)\lambda^{n+3} - (n+2)\lambda^{n+2} - (n+1)\lambda^{n+1} + n\lambda^n} \\ &= \frac{2\lambda^{n+3}}{(1-\lambda)[(n+2)\lambda^2 - n + \lambda(\lambda+1)]}.\end{aligned}$$

Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty} \frac{B_1^{(n)}}{S_n} = \lim_{n\to\infty} \frac{2\lambda^{n+3}}{(1-\lambda)[(n+2)\lambda^2 - n + \lambda(\lambda+1)]} \cdot \frac{1}{n\lambda^n} = 0,$$

which shows that the sequence $\{B_1^{(n)}\}$ converges faster than the sequence $\{S_n\}$.

Example 3.3 *(Bumbariu, [39]) Let us consider the sequence defined by*

$$S_n = \frac{n}{\lambda^n},$$

where $\lambda \in (1, \infty)$. *Then* $\lim_{n\to\infty} \frac{B_1^{(n)}}{S_n} = 0$.

When applying B-algorithm to the sequence $\{S_n\}$ we obtain

$$B_1^{(n)} = \frac{n+3}{\lambda^{n+3}} - \frac{\left(\frac{n+3}{\lambda^{n+3}} - \frac{n+2}{\lambda^{n+2}}\right)\left(\frac{n+3}{\lambda^{n+3}} - \frac{n+1}{\lambda^{n+1}}\right)}{\frac{n+3}{\lambda^{n+3}} - \frac{n+2}{\lambda^{n+2}} - \frac{n+1}{\lambda^{n+1}} + \frac{n}{\lambda^n}}$$
$$= \frac{2}{\lambda^n(\lambda-1)[(n+2) - n\lambda^2 + \lambda + 1]}.$$

Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty} \frac{B_1^{(n)}}{S_n} = \lim_{n\to\infty} \frac{2}{\lambda^n(\lambda-1)[(n+2) - n\lambda^2 + \lambda + 1]} \cdot \frac{\lambda^n}{n} = 0,$$

which shows that the sequence $\{B_1^{(n)}\}$ converges faster than the sequence $\{S_n\}$.

Example 3.4 *(Bumbariu, [39]) Let us consider the sequence defined by*

$$S_n = \frac{\lambda^n}{n},$$

where $\lambda \in (-1,1), \lambda \neq 0$. Then $\lim_{n\to\infty} \frac{B_1^{(n)}}{S_n} = 0$.

When applying B-algorithm to the sequence $\{S_n\}$ we obtain

$$B_1^{(n)} = \frac{\lambda^{n+3}}{n+3} - \frac{\left(\frac{\lambda^{n+3}}{n+3} - \frac{\lambda^{n+2}}{n+2}\right)\left(\frac{\lambda^{n+3}}{n+3} - \frac{\lambda^{n+1}}{n+1}\right)}{\frac{\lambda^{n+3}}{n+3} - \frac{\lambda^{n+2}}{n+2} - \frac{\lambda^{n+1}}{n+1} + \frac{\lambda^n}{n}}$$
$$= \frac{2\lambda^{n+3}}{\lambda(\lambda^2-1)(n^3+3n^2+2n) - (\lambda^2-1)(n^3+4n^2+3n) - (\lambda-1)(2n^2+4n) + 4n + 6}.$$

Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty} \frac{B_1^{(n)}}{S_n} =$$
$$\lim_{n\to\infty} \frac{2\lambda^{n+3}}{\lambda(\lambda^2-1)(n^3+3n^2+2n) - (\lambda^2-1)(n^3+4n^2+3n) - (\lambda-1)(2n^2+4n) + 4n + 6} \cdot \frac{n}{\lambda^n},$$
$$= 0,$$

which shows that the sequence $\{B_1^{(n)}\}$ converges faster than the sequence $\{S_n\}$.

Example 3.5 *(Bumbariu, [39]) Let us consider the sequence defined by*

$$S_n = 1 + \frac{1}{n},$$

where $n = 0, 1, 2, \ldots$. Then $B_1^{(n)} = \frac{1}{2n+3}$ for every $n = 0, 1, 2, \ldots$.

When we apply B-algorithm to the sequence $\{S_n\}$ we obtain

$$\begin{aligned} B_1^{(n)} &= S_{n+3} - \frac{(S_{n+3}-S_{n+2})(S_{n+3}-S_{n+1})}{S_{n+3}-S_{n+2}-S_{n+1}-S_n} \\ &= \frac{1}{n+3} - \frac{\left(\frac{1}{n+3}-\frac{1}{n+2}\right)\left(\frac{1}{n+3}-\frac{1}{n+1}\right)}{\frac{1}{n+3}-\frac{1}{n+2}-\frac{1}{n+1}+\frac{1}{n}} \\ &= \frac{1}{2n+3}. \end{aligned}$$

Example 3.6 *(Bumbariu, [39]) Let us consider the sequence defined by*

$$S_n = \frac{1}{n+1},$$

where $n = 0, 1, 2, \ldots$. Then $B_1^{(n)} = \frac{1}{2n+5}$ for every $n = 0, 1, 2, \ldots$.

When we apply B-algorithm to the sequence S_n we obtain

$$\begin{aligned} B_1^{(n)} &= S_{n+3} - \frac{(S_{n+3}-S_{n+2})(S_{n+3}-S_{n+1})}{S_{n+3}-S_{n+2}-S_{n+1}-S_n} \\ &= \frac{1}{n+4} - \frac{\left(\frac{1}{n+4}-\frac{1}{n+3}\right)\left(\frac{1}{n+4}-\frac{1}{n+2}\right)}{\frac{1}{n+4}-\frac{1}{n+3}-\frac{1}{n+2}+\frac{1}{n+1}} \\ &= \frac{1}{2n+5}. \end{aligned}$$

Example 3.7 *(Bumbariu, [39]) Let us consider the sequence defined by*

$$S_n = \frac{1}{n+\alpha},$$

where $n = 0, 1, 2, \ldots$ and α is an arbitrary constant. Then $B_1^{(n)} = \frac{1}{2(n+\alpha)+3}$ for every $n = 0, 1, 2, \ldots$.

When we apply B-algorithm to the sequence $\{S_n\}$ we obtain

$$\begin{aligned} B_1^{(n)} &= S_{n+3} - \frac{(S_{n+3}-S_{n+2})(S_{n+3}-S_{n+1})}{S_{n+3}-S_{n+2}-S_{n+1}-S_n} \\ &= \frac{1}{n+3+\alpha} - \frac{\left(\frac{1}{n+3+\alpha}-\frac{1}{n+2+\alpha}\right)\left(\frac{1}{n+3+\alpha}-\frac{1}{n+1+\alpha}\right)}{\frac{1}{n+3+\alpha}-\frac{1}{n+2+\alpha}-\frac{1}{n+1+\alpha}+\frac{1}{n+\alpha}} \\ &= \frac{1}{2(n+\alpha)+3}. \end{aligned}$$

Example 3.8 *(Bumbariu, [39]) Let us consider the sequence defined by*

$$S_n = \frac{\beta}{n+\alpha},$$

where $n = 0, 1, 2, \ldots$ and α, β are two arbitrary constants. Then $B_1^{(n)} = \frac{\beta}{2(n+\alpha)+3}$ for every $n = 0, 1, 2, \ldots$.

When we apply B-algorithm to the sequence $\{S_n\}$ we obtain

$$\begin{aligned} B_1^{(n)} &= S_{n+3} - \frac{(S_{n+3}-S_{n+2})(S_{n+3}-S_{n+1})}{S_{n+3}-S_{n+2}-S_{n+1}-S_n} \\ &= \frac{\beta}{n+3+\alpha} - \frac{\left(\frac{\beta}{n+3+\alpha}-\frac{\beta}{n+2+\alpha}\right)\left(\frac{\beta}{n+3+\alpha}-\frac{\beta}{n+1+\alpha}\right)}{\frac{\beta}{n+3+\alpha}-\frac{\beta}{n+2+\alpha}-\frac{\beta}{n+1+\alpha}+\frac{1}{n+\alpha}} \\ &= \frac{\beta}{2(n+\alpha)+3}. \end{aligned}$$

Example 3.9 *(Bumbariu, [39]) Let us consider the sequence defined by*

$$S_n = 1 + \frac{1}{n+1},$$

where $n = 0, 1, 2, \ldots$. Then $B_1^{(n)} = \frac{2n+6}{2n+5}$ for every $n = 0, 1, 2, \ldots$.

When we apply B-algorithm to the sequence $\{S_n\}$ we obtain

$$\begin{aligned} B_1^{(n)} &= S_{n+3} - \frac{(S_{n+3}-S_{n+2})(S_{n+3}-S_{n+1})}{S_{n+3}-S_{n+2}-S_{n+1}-S_n} \\ &= 1 + \frac{1}{n+4} - \frac{\left(1+\frac{1}{n+4}-1-\frac{1}{n+3}\right)\left(1+\frac{1}{n+4}-1-\frac{1}{n+2}\right)}{1+\frac{1}{n+4}-1-\frac{1}{n+3}-1-\frac{1}{n+2}+1+\frac{1}{n+1}} \\ &= \frac{2n+6}{2n+5}. \end{aligned}$$

Example 3.10 *(Bumbariu) Let us consider the sequence defined by*

$$S_n = \frac{n}{n+1},$$

where $n = 0, 1, 2, \ldots$. Then $B_1^{(n)} = \frac{2n+4}{2n+5}$ for every $n = 0, 1, 2, \ldots$.

When we apply B-algorithm to the sequence $\{S_n\}$ we obtain

$$\begin{aligned} B_1^{(n)} &= S_{n+3} - \frac{(S_{n+3}-S_{n+2})(S_{n+3}-S_{n+1})}{S_{n+3}-S_{n+2}-S_{n+1}-S_n} \\ &= \frac{n+3}{n+4} - \frac{\left(\frac{n+3}{n+4}-\frac{n+2}{n+3}\right)\left(\frac{n+3}{n+4}-\frac{n+1}{n+2}\right)}{\frac{n+3}{n+4}-\frac{n+2}{n+3}-\frac{n+1}{n+2}+\frac{n}{n+1}} \\ &= \frac{2n+4}{2n+5}. \end{aligned}$$

In [24], Brezinski considered the sequence $\{S_n\}$ given as in Example 3.9, consequently it is natural to consider the sequence S_n given in the following example.

Example 3.11 *(Bumbariu, [39]) Let us consider the sequence defined by*

$$S_n = \frac{n+1}{n+2},$$

where $n = 0, 1, 2, \ldots$. *Then* $B_1^{(n)} = \frac{2n+6}{2n+7}$ *for every* $n = 0, 1, 2, \ldots$.

When we apply B*-algorithm to the sequence* $\{S_n\}$ *we obtain*

$$\begin{aligned} B_1^{(n)} &= S_{n+3} - \frac{(S_{n+3} - S_{n+2})(S_{n+3} - S_{n+1})}{S_{n+3} - S_{n+2} - S_{n+1} - S_n} \\ &= \frac{n+4}{n+5} - \frac{\left(\frac{n+4}{n+5} - \frac{n+3}{n+4}\right)\left(\frac{n+4}{n+5} - \frac{n+2}{n+3}\right)}{\frac{n+4}{n+5} - \frac{n+3}{n+4} - \frac{n+2}{n+3} + \frac{n+1}{n+2}} \\ &= \frac{2n+6}{2n+7}. \end{aligned}$$

Examples (3.5)-(3.11) show that the sequences $\{B_1^n\}$ and $\{S_n\}$ have the same rate of convergence.

In what follows we present one numerical example to illustrate the efficiency and the performance of the new proposed algorithm. The new acceleration method is applied to some sequences arising in solving nonlinear equations that can be found in [16], [35].

Example 3.12 *(Bumbariu, [35]) Test function:* $f(x) = x^3 + 4x^2 - 10$, *which has a unique root* $x^* = 1.365230013414096845760806828981666078\ldots$. *This equation can be rewritten into a fixed point problem by* $g(x) = \frac{1}{2}\sqrt{10 - x^3}$. *To apply the new acceleration process we shall take the initial guess* $x_0 \in \{1.2, 1.3, 1.4, 1.5\}$. *The results for Example 3.12 with the initial value* $x_0 = 1.5$ *are listed in Table 1. When taking the other initial values* $x_0 \in \{1.2, 1.3, 1.4\}$ *we obtained the same results as for* $x_0 = 1.5$.

Table 1

	$B_1^{(n)},\ n = \overline{0,20}$	$x_{n+1} = g(x_n)$
0	1.5	1.5
1	$B_1^{(1)}$=1.2869537676233750394746711830246937473	1.2869537676233750394746711830246937473
2	$B_1^{(2)}$=1.4025408035395784066277799785265369811	1.4025408035395784066277799785265369811
3	$B_1^{(3)}$=1.3454583740232940979445917702990603921	1.3454583740232940979445917702990603921
4	$B_1^{(4)}$=1.3668709390541160278600394928280017504	1.3751702528160382933670186183367518412
5	$B_1^{(5)}$=1.3643886536653163386109115992881051451	1.3600941927617329278073270932797485347
⋮	⋮	⋮
20	$B_1^{(20)}$=1.3652300134140968457608068289816660783	1.3652302361581811714841436571672175091

We have done studies on twenty test functions, we here presented only one of them. For all test functions, that we have studied, the new acceleration method improves significantly the convergence speed of the sequence of successive approximations. The computations listed in the table are done with Maple 13, using 39 digits arithmetic.

3.1.1 Other representations for the B-algorithm for accelerating the Picard iteration

In this paragraph we introduce some new representations for the B-algorithm, we give a convergence result for these methods and one numerical example to see numerical implication of the these new representations.

Author's original contribution in this paragraph are: some other representations for the B-algorithm (3.46)-(3.51), Theorem 3.9 and Example 3.13. By suitable manipulations of the equation (3.8) we can derive many other representations for the B-algorithm. Examples are:

$$B_1^{(n)} = \frac{S_n S_{n+3} - S_{n+1} S_{n+2}}{\overline{\Delta} S_{n+1} - \overline{\Delta} S_n}, \quad n = 0, 1, \ldots \tag{3.46}$$

$$B_1^{(n)} = \frac{S_n[\overline{\Delta} S_{n+1}] - S_{n+1}[\overline{\Delta} S_n]}{\overline{\Delta} S_{n+1} - \overline{\Delta} S_n}, \quad n = 0, 1, \ldots \tag{3.47}$$

$$B_1^{(n)} = \frac{S_{n+2}[\overline{\Delta} S_{n+1}] - S_{n+3}[\overline{\Delta} S_n]}{\overline{\Delta} S_{n+1} - \overline{\Delta} S_n}, \quad n = 0, 1, \ldots \tag{3.48}$$

$$B_1^{(n)} = S_n - \frac{[\overline{\Delta} S_n][\Delta S_n]}{\overline{\Delta} S_{n+1} - \overline{\Delta} S_n}, \quad n = 0, 1, \ldots \tag{3.49}$$

$$B_1^{(n)} = S_{n+1} - \frac{[\overline{\Delta} S_{n+1}][\Delta S_n]}{\overline{\Delta} S_{n+1} - \overline{\Delta} S_n}, \quad n = 0, 1, \ldots \tag{3.50}$$

$$B_1^{(n)} = S_{n+2} - \frac{[\overline{\Delta} S_n][\Delta S_{n+2}]}{\overline{\Delta} S_{n+1} - \overline{\Delta} S_n}, \quad n = 0, 1, \ldots \tag{3.51}$$

where $\{S_n\}$ is the sequence to be accelerated, Δ and $\overline{\Delta}$ were defined before.

In the previous paragraph we gave a convergent result for the B-algorithm, an analogous result, but for the techniques defined by the relations (3.47)-(3.51), is given by the following theorem:

Theorem 3.9 *(Bumbariu, [41]) Let $\{S_n\}_{n\in\mathbb{N}}$ be a converging sequence to S, verifying $\lim_{n\to\infty} \frac{S_{n+1}-S}{S_n-S} = \lambda$, with $|\lambda| < 1$. The following affirmations are true:*
(i) The sequences $\{B_n\}_{n\in\mathbb{N}}$ defined by the relations (3.47)-(3.51) *are accelerating sequence $\{S_n\}_{n\in\mathbb{N}}$.*
(ii) The sequences $\{B_n\}_{n\in\mathbb{N}}$ defined by the relations (3.47)-(3.51) *and the sequence $\{B_n\}_{n\in\mathbb{N}}$ defined by the relation* (3.8) *have the same rate of convergence.*

The proof is similar with that of the Theorem 3.1 in previous paragraph.

Proof
(i) We prove that $\{B_n\}_{n\in\mathbb{N}}$, defined by de relation (3.47), converges faster to S than $\{S_n\}_{n\in\mathbb{N}}$ to S.

$$\begin{aligned}\frac{B_n - S}{S_n - S} &= \frac{\frac{S_n(S_{n+3}-S_{n+1})-S_{n+1}(S_{n+2}-S_n)}{(S_{n+3}-S_{n+1})-(S_{n+2}-S_n)} - S}{S_n - S} \\ &= \frac{(S_n - S)(S_{n+3} - S_{n+1}) - (S_{n+1} - S)(S_{n+2} - S_n)}{[(S_{n+3} - S_{n+1}) - (S_{n+2} - S_n)](S_n - S)}.\end{aligned}$$

Adding $S - S$ in parenthesis and dividing numerator and denominator by S_{n+2} we obtain

$$\frac{B_n - S}{S_n - S} = \frac{\frac{S_n-S}{S_{n+2}-S}\left(\frac{S_{n+3}-S}{S_{n+2}-S} - \frac{S_{n+1}-S}{S_{n+2}-S}\right) - \frac{S_{n+1}-S}{S_{n+2}-S}\left(1 - \frac{S_n-S}{S_{n+2}-S}\right)}{\left(\frac{S_{n+3}-S}{S_{n+2}-S} - \frac{S_{n+1}-S}{S_{n+2}-S} - 1 + \frac{S_n-S}{S_{n+2}-S}\right)\frac{S_n-S}{S_{n+2}-S}}.$$

Taking the limit when $n \to \infty$ we obtain

$$\lim_{n\to\infty}\frac{B_n - S}{S_n - S} = \frac{\frac{1}{\lambda^2}\left(\lambda - \frac{1}{\lambda}\right) - \frac{1}{\lambda}\left(1 - \frac{1}{\lambda^2}\right)}{\left(\lambda - \frac{1}{\lambda} - 1 + \frac{1}{\lambda^2}\right)\frac{1}{\lambda^2}}$$
$$= \frac{0}{\frac{(\lambda^2-1)(\lambda-1)}{\lambda^4}}$$
$$= 0.$$

Therefore $\{B_n\}_{n\in\mathbb{N}}$ defined by the relation (3.47) converges faster than $\{S_n\}_{n\in\mathbb{N}}$ to S.

We prove that $\{B_n\}_{n\in\mathbb{N}}$, defined by de relation (3.48), converges faster to S than $\{S_n\}_{n\in\mathbb{N}}$ to S.

$$\frac{B_n - S}{S_n - S} = \frac{\frac{S_{n+2}(S_{n+3}-S_{n+1}) - S_{n+3}(S_{n+2}-S_n)}{(S_{n+3}-S_{n+1})-(S_{n+2}-S_n)} - S}{S_n - S}$$
$$= \frac{(S_{n+2} - S)(S_{n+3} - S_{n+1}) - (S_{n+3} - S)(S_{n+2} - S_n)}{[(S_{n+3} - S_{n+1}) - (S_{n+2} - S_n)](S_n - S)}.$$

Adding $S - S$ in parenthesis and dividing numerator and denominator by S_{n+2} we obtain

$$\frac{B_n - S}{S_n - S} = \frac{\left(\frac{S_{n+3}-S}{S_{n+2}-S} - \frac{S_{n+1}-S}{S_{n+2}-S}\right) - \frac{S_{n+3}-S}{S_{n+2}-S}\left(1 - \frac{S_n-S}{S_{n+2}-S}\right)}{\left(\frac{S_{n+3}-S}{S_{n+2}-S} - \frac{S_{n+1}-S}{S_{n+2}-S} - 1 + \frac{S_n-S}{S_{n+2}-S}\right)\frac{S_n-S}{S_{n+2}-S}}.$$

Taking the limit when $n \to \infty$ we obtain

$$\lim_{n\to\infty}\frac{B_n - S}{S_n - S} = \frac{\lambda - \frac{1}{\lambda} - \lambda\left(1 - \frac{1}{\lambda^2}\right)}{\left(\lambda - \frac{1}{\lambda} - 1 + \frac{1}{\lambda^2}\right)\frac{1}{\lambda^2}}$$
$$= \frac{0}{\frac{(\lambda^2-1)(\lambda-1)}{\lambda^4}}$$
$$= 0.$$

Therefore $\{B_n\}_{n\in\mathbb{N}}$ defined by the relation (3.48) converges faster than $\{S_n\}_{n\in\mathbb{N}}$ to S.

We prove that $\{B_n\}_{n\in\mathbb{N}}$, defined by de relation (3.49), converges faster to S than $\{S_n\}_{n\in\mathbb{N}}$ to S.

$$\frac{B_n - S}{S_n - S} = \frac{S_n - S - \frac{(S_{n+2}-S_n)(S_{n+1}-S_n)}{(S_{n+3}-S_{n+1})-(S_{n+2}-S_n)}}{S_n - S}$$
$$= 1 - \frac{(S_{n+2} - S_n)(S_{n+1} - S_n)}{[(S_{n+3} - S_{n+1}) - (S_{n+2} - S_n)](S_n - S)}.$$

Adding $S - S$ in parenthesis and dividing numerator and denominator by S_{n+2} we obtain

$$\frac{B_n - S}{S_n - S} = 1 - \frac{\left(1 - \frac{S_n - S}{S_{n+2} - S}\right)\left(\frac{S_{n+1} - S}{S_{n+2} - S} - \frac{S_n - S}{S_{n+2} - S}\right)}{\left(\frac{S_{n+3} - S}{S_{n+2} - S} - \frac{S_{n+1} - S}{S_{n+2} - S} - 1 + \frac{S_n - S}{S_{n+2} - S}\right)\frac{S_n - S}{S_{n+2} - S}}.$$

Taking the limit when $n \to \infty$ we obtain

$$\begin{aligned}\lim_{n\to\infty} \frac{B_n - S}{S_n - S} &= 1 - \frac{\left(1 - \frac{1}{\lambda^2}\right)\left(\frac{1}{\lambda} - \frac{1}{\lambda^2}\right)}{\left(\lambda - \frac{1}{\lambda} - 1 + \frac{1}{\lambda^2}\right)\frac{1}{\lambda^2}} \\ &= 1 - 1 \\ &= 0.\end{aligned}$$

Therefore $\{B_n\}_{n\in\mathbb{N}}$ defined by the relation (3.49) converges faster than $\{S_n\}_{n\in\mathbb{N}}$ to S.

We prove that $\{B_n\}_{n\in\mathbb{N}}$, defined by de relation (3.50), converges faster to S than $\{S_n\}_{n\in\mathbb{N}}$ to S.

$$\begin{aligned}\frac{B_n - S}{S_n - S} &= \frac{S_{n+1} - S - \frac{(S_{n+3} - S_{n+1})(S_{n+1} - S_n)}{(S_{n+3} - S_{n+1}) - (S_{n+2} - S_n)}}{S_n - S} \\ &= \frac{S_{n+1} - S}{S_n - S} - \frac{(S_{n+3} - S_{n+1})(S_{n+1} - S_n)}{[(S_{n+3} - S_{n+1}) - (S_{n+2} - S_n)](S_n - S)}.\end{aligned}$$

Adding $S - S$ in parenthesis and dividing numerator and denominator by S_{n+2} we obtain

$$\frac{B_n - S}{S_n - S} = \frac{S_{n+1} - S}{S_n - S} - \frac{\left(\frac{S_{n+3} - S}{S_{n+2} - S} - \frac{S_{n+1} - S}{S_{n+2} - S}\right)\left(\frac{S_{n+1} - S}{S_{n+2} - S} - \frac{S_n - S}{S_{n+2} - S}\right)}{\left(\frac{S_{n+3} - S}{S_{n+2} - S} - \frac{S_{n+1} - S}{S_{n+2} - S} - 1 + \frac{S_n - S}{S_{n+2} - S}\right)\frac{S_n - S}{S_{n+2} - S}}.$$

Taking the limit when $n \to \infty$ we obtain

$$\begin{aligned}\lim_{n\to\infty} \frac{B_n - S}{S_n - S} &= \lambda - \frac{\left(\lambda - \frac{1}{\lambda}\right)\left(\frac{1}{\lambda} - \frac{1}{\lambda^2}\right)}{\left(\lambda - \frac{1}{\lambda} - 1 + \frac{1}{\lambda^2}\right)\frac{1}{\lambda^2}} \\ &= \lambda - \lambda \\ &= 0.\end{aligned}$$

Therefore $\{B_n\}_{n\in\mathbb{N}}$ defined by the relation (3.50) converges faster than $\{S_n\}_{n\in\mathbb{N}}$ to S.

We prove that $\{B_n\}_{n\in\mathbb{N}}$, defined by de relation (3.51), converges faster to S than $\{S_n\}_{n\in\mathbb{N}}$ to S.

$$\begin{aligned}\frac{B_n - S}{S_n - S} &= \frac{S_{n+2} - S - \frac{(S_{n+3} - S_{n+2})(S_{n+2} - S_n)}{(S_{n+3} - S_{n+1}) - (S_{n+2} - S_n)}}{S_n - S} \\ &= \frac{S_{n+2}}{S_n} - \frac{(S_{n+3} - S_{n+2})(S_{n+2} - S_n)}{[(S_{n+3} - S_{n+1}) - (S_{n+2} - S_n)](S_n - S)}.\end{aligned}$$

Adding $S - S$ in parenthesis and dividing numerator and denominator by S_{n+2} we obtain

$$\frac{B_n - S}{S_n - S} = \frac{S_{n+2} - S}{S_n - S} - \frac{\left(\frac{S_{n+3}-S}{S_{n+2}-S} - 1\right)\left(1 - \frac{S_n - S}{S_{n+2}-S}\right)}{\left(\frac{S_{n+3}-S}{S_{n+2}-S} - \frac{S_{n+1}-S}{S_{n+2}-S} - 1 + \frac{S_n-S}{S_{n+2}-S}\right)\frac{S_n-S}{S_{n+2}-S}}.$$

Taking the limit when $n \to \infty$ we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{B_n - S}{S_n - S} &= \lambda^2 - \frac{(\lambda - 1)\frac{\lambda^2-1}{\lambda^2}}{\left(\lambda - \frac{1}{\lambda} - 1 + \frac{1}{\lambda^2}\right)\frac{1}{\lambda^2}}\\ &= \lambda^2 - \lambda^2\\ &= 0.\end{aligned}$$

Therefore $\{B_n\}_{n\in\mathbb{N}}$ defined by the relation (3.51) converges faster than $\{S_n\}_{n\in\mathbb{N}}$ to S.

(ii) We prove that $\{\overline{B}_n\}_{n\in\mathbb{N}}$, defined by de relation (3.47), and $\{B_n\}_{n\in\mathbb{N}}$, defined by de relation (3.8) have the same rate of convergence.

$$\frac{\overline{B}_n - S}{B_n - S} = \frac{\frac{S_n(S_{n+3}-S_{n+1}) - S_{n+1}(S_{n+2}-S_n)}{(S_{n+3}-S_{n+1})-(S_{n+2}-S_n)} - S}{S_{n+3} - S - \frac{(S_{n+3}-S_{n+1})(S_{n+3}-S_{n+2})}{(S_{n+3}-S_{n+1})-(S_{n+2}-S_n)}}.$$

Making some elementary calculus we obtain

$$\frac{\overline{B}_n - S}{B_n - S} = \frac{S_nS_{n+3} - S_{n+1}S_{n+2} - SS_{n+3} + SS_{n+2} + SS_{n+1} - SS_n}{S_nS_{n+3} - S_{n+1}S_{n+2} - SS_{n+3} + SS_{n+2} + SS_{n+1} - SS_n} = 1. \tag{3.52}$$

Taking the limit when $n \to \infty$ we obtain

$$\lim_{n\to\infty}\frac{\overline{B}_n - S}{B_n - S} = \lim_{n\to\infty} 1 = 1.$$

We prove that $\{\overline{B}_n\}_{n\in\mathbb{N}}$, defined by de relation (3.48), and $\{B_n\}_{n\in\mathbb{N}}$, defined by de relation (3.8) have the same rate of convergence.

$$\frac{\overline{B}_n - S}{B_n - S} = \frac{\frac{S_{n+2}(S_{n+3}-S_{n+1}) - S_{n+3}(S_{n+2}-S_n)}{(S_{n+3}-S_{n+1})-(S_{n+2}-S_n)} - S}{S_{n+3} - S - \frac{(S_{n+3}-S_{n+1})(S_{n+3}-S_{n+2})}{(S_{n+3}-S_{n+1})-(S_{n+2}-S_n)}}.$$

Making some elementary calculus we obtain

$$\frac{\overline{B}_n - S}{B_n - S} = \frac{S_nS_{n+3} - S_{n+1}S_{n+2} - SS_{n+3} + SS_{n+2} + SS_{n+1} - SS_n}{S_nS_{n+3} - S_{n+1}S_{n+2} - SS_{n+3} + SS_{n+2} + SS_{n+1} - SS_n} = 1. \tag{3.53}$$

Taking the limit when $n \to \infty$ we obtain

$$\lim_{n\to\infty}\frac{\overline{B}_n - S}{B_n - S} = \lim_{n\to\infty} 1 = 1.$$

We prove that $\{\overline{B}_n\}_{n\in\mathbb{N}}$, defined by de relation (3.49), and $\{B_n\}_{n\in\mathbb{N}}$, defined by de relation (3.8) have the same rate of convergence.

$$\frac{\overline{B}_n - S}{B_n - S} = \frac{S_n - S - \frac{(S_{n+2}-S_n)(S_{n+1}-S_n)}{(S_{n+3}-S_{n+1})-(S_{n+2}-S_n)}}{S_{n+3} - S - \frac{(S_{n+3}-S_{n+1})(S_{n+3}-S_{n+2})}{(S_{n+3}-S_{n+1})-(S_{n+2}-S_n)}}.$$

Making some elementary calculus we obtain

$$\frac{\overline{B}_n - S}{B_n - S} = \frac{S_nS_{n+3} - S_{n+1}S_{n+2} - SS_{n+3} + SS_{n+2} + SS_{n+1} - SS_n}{S_nS_{n+3} - S_{n+1}S_{n+2} - SS_{n+3} + SS_{n+2} + SS_{n+1} - SS_n} = 1. \tag{3.54}$$

Taking the limit when $n \to \infty$ we obtain

$$\lim_{n\to\infty}\frac{\overline{B}_n - S}{B_n - S} = \lim_{n\to\infty} 1 = 1.$$

We prove that $\{\overline{B}_n\}_{n\in\mathbb{N}}$, defined by de relation (3.50), and $\{B_n\}_{n\in\mathbb{N}}$, defined by de relation (3.8) have the same rate of convergence.

$$\frac{\overline{B}_n - S}{B_n - S} = \frac{S_{n+1} - S - \frac{(S_{n+3}-S_{n+1})(S_{n+1}-S_n)}{(S_{n+3}-S_{n+1})-(S_{n+2}-S_n)}}{S_{n+3} - S - \frac{(S_{n+3}-S_{n+1})(S_{n+3}-S_{n+2})}{(S_{n+3}-S_{n+1})-(S_{n+2}-S_n)}}.$$

Making some elementary calculus we obtain

$$\frac{\overline{B}_n - S}{B_n - S} = \frac{S_nS_{n+3} - S_{n+1}S_{n+2} - SS_{n+3} + SS_{n+2} + SS_{n+1} - SS_n}{S_nS_{n+3} - S_{n+1}S_{n+2} - SS_{n+3} + SS_{n+2} + SS_{n+1} - SS_n} = 1. \tag{3.55}$$

Taking the limit when $n \to \infty$ we obtain

$$\lim_{n\to\infty}\frac{\overline{B}_n - S}{B_n - S} = \lim_{n\to\infty} 1 = 1.$$

We prove that $\{\overline{B}_n\}_{n\in\mathbb{N}}$, defined by de relation (3.51), and $\{B_n\}_{n\in\mathbb{N}}$, defined by de relation (3.8) have the same rate of convergence.

$$\frac{\overline{B}_n - S}{B_n - S} = \frac{S_{n+2} - S - \frac{(S_{n+3}-S_{n+2})(S_{n+2}-S_n)}{(S_{n+3}-S_{n+1})-(S_{n+2}-S_n)}}{S_{n+3} - S - \frac{(S_{n+3}-S_{n+1})(S_{n+3}-S_{n+2})}{(S_{n+3}-S_{n+1})-(S_{n+2}-S_n)}}.$$

Making some elementary calculus we obtain

$$\frac{\overline{B}_n - S}{B_n - S} = \frac{S_nS_{n+3} - S_{n+1}S_{n+2} - SS_{n+3} + SS_{n+2} + SS_{n+1} - SS_n}{S_nS_{n+3} - S_{n+1}S_{n+2} - SS_{n+3} + SS_{n+2} + SS_{n+1} - SS_n} = 1. \tag{3.56}$$

Taking the limit when $n \to \infty$ we obtain

$$\lim_{n\to\infty} \frac{\overline{B}_n - S}{B_n - S} = \lim_{n\to\infty} 1 = 1.$$

The representation (3.46) is numerically unstable as will see in the sequel, consequently we can not say that it has the same convergence speed as the other representations.

In what follows we will give a example to see the numerical implication of the other representation for the B-algorithm.

Example 3.13 *(Bumbariu, [41])[35] Test function:* $f(x) = \cos x - x$, *which has a unique root* $x^* = 0.7390851332151606416553120876738734040134\ldots$. *This equation can be rewritten into a fixed point problem by* $g(x) = \cos x$. *To apply the new acceleration technique we shall take the initial guess* $x_0 \in \{0.5, 0.7, 0.8, 0.9\}$. *The results for Example 3.13 with the initial value* 0.7 *are listed in Table 2, for the other initial values the results are the same .*

Table 2

n	$B_1^{(n)}$(3.46)	$B_1^{(n)}$(3.47)	$B_1^{(n)}$(3.48)	$B_1^{(n)}$(3.49)	$B_1^{(n)}$(3.50)	$B_1^{(n)}$(3.51)	$B_1^{(n)}$(3.8)
0	0.7	0.7	0.7	0.7	0.7	0.7	0.7
1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1
4	3	3	3	3	3	3	3
5	2	2	2	2	2	2	2
6	3	3	3	3	3	3	3
7	4	4	4	4	4	4	4
8	8	8	8	8	8	8	8
9	8	8	8	8	8	8	8
10	8	8	8	8	8	8	8
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
20	16	38	38	39	39	39	39
21	17	38	38				
22	17	39	38				
23	17		39				
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
50	22						
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
100	20						

From the results presented in Table 2 we can say that the acceleration technique (3.46) is numerically unstable, the acceleration methods (3.47) and (3.48), in majority of cases, give the same number of digits as in case of B-algorithm and when it does not, the number of iterations that differ is very little, less than five iterations. And for the representations (3.49)-(3.51) the number of iterations are the same as for B-algorithm. We have done studies for twenty test functions and the results were the same as for the example presented above. In the table are listed the exact number of decimals at the first one hundred steps, all the numerical computations are done with Maple 13, using 39 digits arithmetic.

3.1.2 Iterated form for the B-algorithm for accelerating Picard iteration

Author's original contribution in this paragraph are: the iterated form for the B-algorithm (3.57), Theorems 3.10, 3.11, 3.12, 3.13, 3.14, Examples 3.14-3.19 and Example 3.21.
In the first part of this paragraph we present the iterated form of the B-algorithm and some convergence theorems for it. And in the second part we apply the iterated B-algorithm to some sequences taken from [24] and we end this paragraph with one example in which we performed a comparison between the sequence that arises when applying the new acceleration technique and the sequence that arises when applying Aitken's iterated Δ^2 process, (1.30), in order to see which one has the best convergence speed. The iterated form for the B-algorithm is given by the following expression

$$B_{k+1}^{(n)} = B_k^{(n+3)} - \frac{[\overline{\Delta} B_k^{(n+1)}][\Delta B_k^{(n+2)}]}{\overline{\Delta} B_k^{(n+1)} - \overline{\Delta} B_k^{(n)}}, \quad n, k \in \mathbb{N} \tag{3.57}$$

where $B_0^{(n)} = S_n$, $\{S_n\}$ is the sequence to be accelerated, Δ, the forward difference operator, has been defined before and we denote $\overline{\Delta}$, the forward difference operator with two steps, by $\overline{\Delta} B_k^{(n)} = B_k^{(n+2)} - B_k^{(n)}$. Like in the case of the Aitken's iterated Δ^2 process the difference operators, Δ and $\overline{\Delta}$, act only upon superscript n and not upon the subscript k.

In the sequel we will give conditions on the $B_i^{(n)}$'s which insure that $\lim_{n\to\infty} B_k^{(n)} = S$.

Theorem 3.10 *(Bumbariu, [36]) If* $\lim_{n\to\infty} S_n = S$, *and if* $\forall i, \exists b_i \neq 1$ *such that* $\lim_{n\to\infty} \frac{B_i^{(n+1)}}{B_i^{(n)}} = b_i$, *for* $\forall i \neq j, \quad b_i \neq b_j$. *Then* $\forall k$ *we have* $\lim_{n\to\infty} B_k^{(n)} = S$.

Proof
For $k = 0$ we have

$$B_1^{(n)} = B_0^{(n+3)} - \frac{(B_0^{(n+3)} - B_0^{(n+1)})(B_0^{(n+3)} - B_0^{(n+2)})}{(B_0^{(n+3)} - B_0^{(n+1)}) - (B_0^{(n+2)} - B_0^{(n)})},$$

where $B_0^{(n)} = S_n$.

Multiplying with $\frac{B_0^{(n+2)}}{B_0^{(n+2)}}$ we get

$$B_1^{(n)} = B_0^{(n+3)} - \frac{\frac{B_0^{(n+3)}}{B_0^{(n+2)}} - \frac{B_0^{(n+1)}}{B_0^{(n+2)}}}{\frac{B_0^{(n+3)}}{B_0^{(n+2)}} - \frac{B_0^{(n+1)}}{B_0^{(n+2)}} - 1 + \frac{B_0^{(n)}}{B_0^{(n+2)}}}(B_0^{(n+3)} - B_0^{(n+2)}). \tag{3.58}$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n+3)} - \frac{b_0 - \frac{1}{b_0}}{b_0 - \frac{1}{b_0} - 1 + \frac{1}{b_0^2}} \lim_{n\to\infty} (B_0^{(n+3)} - B_0^{(n+2)}) \tag{3.59}$$

from where it results

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n+3)} - \frac{b_0}{b_0 - 1} \lim_{n\to\infty} (B_0^{(n+3)} - B_0^{(n+2)}), \tag{3.60}$$

where $b_0 \neq 1$ and $B_0^{(n)} = S_n$. Hence

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} S_{n+3} - \frac{b_0}{b_0 - 1} \lim_{n\to\infty} (S_{n+3} - S_{n+2}), \tag{3.61}$$

because $\lim_{n\to\infty} S_n = S$ we obtain $\lim_{n\to\infty} B_1^{(n)} = S$.

Suppose that $\lim_{n\to\infty} B_{k-1}^{(n)} = S$ we prove that $\lim_{n\to\infty} B_k^{(n)} = S$.

$$B_k^{(n)} = B_{k-1}^{(n+3)} - \frac{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})(B_{k-1}^{(n+3)} - B_{k-1}^{(n+2)})}{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})}$$

Multiplying with $\frac{B_{k-1}^{(n+2)}}{B_{k-1}^{(n+2)}}$ we obtain

$$B_k^{(n)} = B_{k-1}^{(n+3)} - \frac{\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+2)}} - \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}}{\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+2)}} - \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}} - 1 + \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+2)}}} (B_{k-1}^{(n+3)} - B_{k-1}^{(n+2)}). \tag{3.62}$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+3)} - \frac{b_{k-1} - \frac{1}{b_{k-1}}}{b_{k-1} - \frac{1}{b_{k-1}} - 1 + \frac{1}{b_{k-1}^2}} \lim_{n\to\infty} (B_{k-1}^{(n+3)} - B_{k-1}^{(n+2)}), \tag{3.63}$$

hence there

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+3)} - \frac{b_{k-1}}{b_{k-1} - 1} \lim_{n\to\infty} (B_{k-1}^{(n+3)} - B_{k-1}^{(n+2)}) \tag{3.64}$$

where $b_{k-1} \neq 1$ and $B_{k-1}^{(n)} = S_n$ we obtain $\lim_{n\to\infty} B_k^{(n)} = S$.

Theorem 3.11 *(Bumbariu, [36]) If the conditions of Theorem 3.10 are satisfied and if* $\lim_{n\to\infty} \frac{B_{k-1}^{(n+1)} - S}{B_{k-1}^{(n)} - S} = b_k$*. Then* $\{B_k^{(n)}\}$ *converges to S faster than* $\{B_{k-1}^{(n)}\}$*, when n tends to infinity, that is*

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = 0. \tag{3.65}$$

Moreover if $b_k \neq 0$, $\{B_k^{(n)}\}$ *converges to S faster than* $\{B_{k-1}^{(n+1)}\}$.

Proof We prove the first part of theorem.

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = \frac{B_{k-1}^{(n+3)} - S - \frac{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})(B_{k-1}^{(n+3)} - B_{k-1}^{(n+2)})}{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})}}{B_{k-1}^{(n)} - S}, \tag{3.66}$$

adding $S - S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n)}-S}-$$
$$-\frac{[(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+1)}-S)][(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+2)}-S)]}{\{[(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+1)}-S)]-[(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n)}-S)]\}(B_{k-1}^{(n)}-S)}.$$

Multiplying with $\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+2)}-S}$ it follows that

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}\cdot\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n)}-S}-\frac{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}\right]\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-1\right]}{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}-1+\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right]\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}}. \tag{3.67}$$

Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=b_{k-1}^3-\frac{\left(b_{k-1}-\frac{1}{b_{k-1}}\right)(b_{k-1}-1)}{\left(b_{k-1}-\frac{1}{b_{k-1}}-1+\frac{1}{b_{k-1}^2}\right)\frac{1}{b_{k-1}^2}}, \tag{3.68}$$

from where it follows

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=b_{k-1}^3-b_{k-1}^3=0. \tag{3.69}$$

The proof of the second part of the theorem

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=\frac{B_{k-1}^{(n+3)}-S-\frac{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})(B_{k-1}^{(n+3)}-B_{k-1}^{(n+2)})}{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})-(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})}}{B_{k-1}^{(n+1)}-S}, \tag{3.70}$$

adding $S-S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+1)}-S}-$$
$$-\frac{[(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+1)}-S)][(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+2)}-S)]}{\{[(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+1)}-S)]-[(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n+1)}-S)]\}(B_{k-1}^{(n+1)}-S)}.$$

Multiplying with $\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+2)}-S}$

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}\cdot\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+1)}-S}-$$
$$-\frac{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}\right]\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-1\right]}{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}-1+\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right]\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}}.$$

Taking the limit when n tends to infinity

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=b_{k-1}^2-\frac{\left(b_{k-1}-\frac{1}{b_{k-1}}\right)(b_{k-1}-1)}{\left(b_{k-1}-\frac{1}{b_{k-1}}-1+\frac{1}{b_{k-1}^2}\right)\frac{1}{b_{k-1}}} \tag{3.71}$$

from where it results

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=b_{k-1}^2-b_{k-1}^2=0. \tag{3.72}$$

Theorem 3.12 *(Bumbariu) If the conditions of Theorem 3.11 are satisfied. Then we have the following results for the iterated B-algorithm:*
(i) $\{B_{k+1}^{(n)}\}$ *converges to S faster than* $\{B_k^{(n+1)}\}$*, when n tends to infinity.*
(ii) $\{B_{k+2}^{(n)}\}$ *converges to S faster than* $\{B_k^{(n)}\}$*, when n tends to infinity.*

Proof (i)

$$\frac{B_{k+1}^{(n)}-S}{B_k^{(n+1)}-S}=\frac{B_k^{(n+3)}-S-\frac{(B_k^{(n+3)}-B_k^{(n+2)})(B_k^{(n+3)}-B_k^{(n+1)})}{B_k^{(n+3)}-B_k^{(n+2)}-B_k^{(n+1)}+B_k^{(n)}}}{B_k^{(n+1)}-S}, \tag{3.73}$$

adding $S-S$ in each parenthesis we obtain

$$\frac{B_{k+1}^{(n)}-S}{B_k^{(n+1)}-S}=\frac{B_k^{(n+3)}-S-\frac{[(B_k^{(n+3)}-S)(B_k^{(n+2)}-S)][(B_k^{(n+3)}-S)(B_k^{(n+1)}-S)]}{(B_k^{(n+3)}-S)-(B_k^{(n+2)}-S)-(B_k^{(n+1)}-S)+(B_k^{(n)}-S)}}{B_k^{(n+1)}-S}. \tag{3.74}$$

Multiplying with $\frac{B_k^{(n+1)}-S}{B_k^{(n+1)}-S}$ we have

$$\frac{B_{k+1}^{(n)}-S}{B_k^{(n+1)}-S}=\frac{B_k^{(n+3)}-S}{B_k^{(n+1)}-S}-\frac{\left(\frac{B_k^{(n+3)}-S}{B_k^{(n+1)}-S}-\frac{B_k^{(n+2)}-S}{B_k^{(n+1)}-S}\right)\left(\frac{B_k^{(n+3)}-S}{B_k^{(n+1)}-S}-1\right)}{\left(\frac{B_k^{(n+3)}-S}{B_k^{(n+1)}-S}-\frac{B_k^{(n+2)}-S}{B_k^{(n+1)}-S}-1+\frac{B_k^{(n)}-S}{B_k^{(n+1)}-S}\right)\cdot 1} \tag{3.75}$$

Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty}\frac{B_{k+1}^{(n)}-S}{B_k^{(n+1)}-S}=b_k^2-\frac{(b_k^2-b_k)(b_k^2-1)}{b_k^2-b_k-1+\frac{1}{b_k}} \tag{3.76}$$

from where it results

$$\lim_{n\to\infty}\frac{B_{k+1}^{(n)}-S}{B_k^{(n+1)}-S}=b_k^2-b_k^2=0. \tag{3.77}$$

(ii)

$$\frac{B_{k+2}^{(n)}-S}{B_k^{(n)}-S}=\frac{B_{k+1}^{(n+3)}-S-\frac{(B_{k+1}^{(n+3)}-B_{k+1}^{(n+2)})(B_{k+1}^{(n+3)}-B_{k+1}^{(n+1)})}{B_{k+1}^{(n+3)}-B_{k+1}^{(n+2)}-B_{k+1}^{(n+1)}+B_{k+1}^{(n)}}}{B_k^{(n)}-S} \tag{3.78}$$

adding $S-S$ in each parenthesis we obtain

$$\frac{B_{k+2}^{(n)}-S}{B_k^{(n)}-S}=\frac{B_{k+1}^{(n+3)}-S-\frac{[(B_{k+1}^{(n+3)}-S)(B_{k+1}^{(n+2)}-S)][(B_{k+1}^{(n+3)}-S)(B_{k+1}^{(n+1)}-S)]}{(B_{k+1}^{(n+3)}-S)-(B_{k+1}^{(n+2)}-S)-(B_{k+1}^{(n+1)}-S)+(B_{k+1}^{(n)}-S)}}{B_k^{(n)}-S}. \tag{3.79}$$

Taking the limit when n tends to infinity we have

$$\lim_{n\to\infty}\frac{B_{k+2}^{(n)}-S}{B_k^{(n)}-S}=\lim_{n\to\infty}\frac{B_{k+1}^{(n+3)}-S}{B_k^{(n)}-S}-$$
$$-\lim_{n\to\infty}\frac{[(B_{k+1}^{(n+3)}-S)(B_{k+1}^{(n+2)}-S)][(B_{k+1}^{(n+3)}-S)(B_{k+1}^{(n+1)}-S)]}{[(B_{k+1}^{(n+3)}-S)-(B_{k+1}^{(n+2)}-S)-(B_{k+1}^{(n+1)}-S)+(B_{k+1}^{(n)}-S)](B_k^{(n)}-S)}.$$

Again we have to calculate two limits. For the first limit we have:

$$\lim_{n\to\infty}\frac{B_{k+1}^{(n+3)}-S}{B_k^{(n)}-S}=\lim_{n\to\infty}\left(\frac{B_k^{(n+6)}-S}{B_k^{(n)}-S}-\frac{(B_k^{(n+6)}-B_k^{(n+4)})(B_k^{(n+6)}-B_k^{(n+5)})}{(B_k^{(n+6)}-B_k^{(n+5)}-B_k^{(n+4)}+B_k^{(n+3)})(B_k^{(n)}-S)}\right),$$

adding $S-S$ in each parenthesis and multiplying with $\frac{B_k^{(n+3)}-S}{B_k^{(n+3)}-S}$ we obtain

$$\lim_{n\to\infty}\frac{B_{k+1}^{(n+3)}-S}{B_k^{(n)}-S}=\lim_{n\to\infty}\frac{B_k^{(n+6)}-S}{B_k^{(n)}-S}-$$
$$-\lim_{n\to\infty}\frac{\left(\frac{B_k^{(n+6)}-S}{B_k^{(n+3)}-S}-\frac{B_k^{(n+4)}-S}{B_k^{(n+3)}-S}\right)\left(\frac{B_k^{(n+6)}-S}{B_k^{(n+3)}-S}-\frac{B_k^{(n+5)}-S}{B_k^{(n+3)}-S}\right)}{\left(\frac{B_k^{(n+6)}-S}{B_k^{(n+3)}-S}-\frac{B_k^{(n+5)}-S}{B_k^{(n+3)}-S}-\frac{B_k^{(n+4)}-S}{B_k^{(n+3)}-S}+1\right)\left(\frac{B_k^{(n)}-S)}{B_k^{(n+3)}-S}\right)}$$

from where we get

$$\lim_{n\to\infty}\frac{B_{k+1}^{(n+3)}-S}{B_k^{(n)}-S}=b_k^6-\frac{(b_k^3-b_k)\,(b_k^3-b_k^2)}{(b_k^3-b_k^2-b_k+1)\,\frac{1}{b_k^3}} \tag{3.80}$$

making some elementary computations we obtain

$$\lim_{n\to\infty}\frac{B_{k+1}^{(n+3)}-S}{B_k^{(n)}-S}=b_k^6-b_k^6=0. \tag{3.81}$$

For the second limit we add $S-S$ in each parenthesis and then we multiply with $\frac{B_{k+1}^{(n)}-S}{B_{k+1}^{(n)}-S}$ obtaining

$$\lim_{n\to\infty} \frac{[(B_{k+1}^{(n+3)}-S)(B_{k+1}^{(n+2)}-S)][(B_{k+1}^{(n+3)}-S)(B_{k+1}^{(n+1)}-S)]}{[(B_{k+1}^{(n+3)}-S)-(B_{k+1}^{(n+2)}-S)-(B_{k+1}^{(n+1)}-S)+(B_{k+1}^{(n)}-S)](B_k^{(n)}-S)} =$$

$$= \lim_{n\to\infty} \frac{\left(\frac{B_{k+1}^{(n+3)}-S}{B_{k+1}^{(n)}-S} - \frac{B_{k+1}^{(n+2)}-S}{B_{k+1}^{(n)}-S}\right)\left(\frac{B_{k+1}^{(n+3)}-S}{B_{k+1}^{(n)}-S} - \frac{B_{k+1}^{(n+1)}-S}{B_{k+1}^{(n)}-S}\right)}{\left(\frac{B_{k+1}^{(n+3)}-S}{B_{k+1}^{(n)}-S} - \frac{B_{k+1}^{(n+2)}-S}{B_{k+1}^{(n)}-S} - \frac{B_{k+1}^{(n+1)}-S}{B_{k+1}^{(n)}-S} + \frac{B_{k+1}^{(n)}-S}{B_{k+1}^{(n)}-S}\right)\frac{B_k^{(n)}-S}{B_{k+1}^{(n)}-S}}$$

$$= \frac{(b_{k+1}^3-b_{k+1}^2)(b_{k+1}^3-b_{k+1})}{b_{k+1}^3-b_{k+1}^2-b_{k+1}+1} \lim_{n\to\infty} \frac{B_{k+1}^{(n)}-S}{B_k^{(n)}-S}$$

$$= b_{k+1}^3 \lim_{n\to\infty} \left\{ \frac{B_k^{(n+3)}-S}{B_k^{(n)}-S} - \frac{(B_k^{(n+3)}-B_k^{(n+2)})(B_k^{(n+3)}-B_k^{(n+1)})}{(B_k^{(n+3)}-B_k^{(n+2)}-B_k^{(n+1)}+B_k^{(n)})(B_k^{(n)}-S)} \right\}$$

$$= b_{k+1}^3 \left\{ b_k^3 - \frac{(b_k^3-b_k^2)(b_k^3-b_k)}{b_k^3-b_k^2-b_k+1} \right\}$$

$$= b_{k+1}^3 \left\{ b_k^3 - b_k^3 \right\}$$

$$= 0.$$

From (3.81) and the above relation we obtain

$$\lim_{n\to\infty} \frac{B_{k+2}^{(n)}-S}{B_k^{(n)}-S} = 0. \tag{3.82}$$

Theorem 3.13 *(Bumbariu) If the conditions of Theorem 3.11 are satisfied. Then we have the following results for the iterated B-algorithm:*
(i) $\{B_1^{(n+i)}\}$ *converges to S faster than* $\{S_{n+1}\}$*, when n tends to infinity.*
(ii) (a) $\{B_{k-1}^{(n)}\}$ *and* $\{S_{n+k}\}$ *have the same rate of convergence, when n tends to infinity and* $k = 1$.
(b) $\{B_{k-1}^{(n)}\}$ *converges to S faster than* $\{S_{n+k}\}$*, when n tends to infinity and* $k \geq 2$.
(iii) (a) $\{B_k^{(n)}\}$ *and* $\{S_{n+k}\}$ *have the same rate of convergence, when n tends to infinity and* $k = 0$.
(b) $\{B_k^{(n)}\}$ *converges to S faster than* $\{S_{n+k}\}$*, when n tends to infinity and* $k \geq 1$.
(iv) (a) $\{B_k^{(n)}\}$ *and* $\{S_{n+2k}\}$ *have the same rate of convergence, when n tends to infinity and* $k = 0$.
(b) $\{B_k^{(n)}\}$ *converges to S faster than* $\{S_{n+2k}\}$*, when n tends to infinity and* $k \geq 1$.
(v) (a) $\{B_k^{(n)}\}$ *and* $\{S_{n+k+i}\}$ *have the same rate of convergence, when n tends to infinity and* $k = 0$.
(b) $\{B_k^{(n)}\}$ *converges to S faster than* $\{S_{n+k+i}\}$*, when n tends to infinity and* $k \geq 1$.

Proof (i)

$$\lim_{n\to\infty} \frac{B_1^{(n+i)}-S}{S_{n+1}-S} =$$

$$= \lim_{n\to\infty} \left\{ \frac{B_0^{(n+i+3)}-S}{S_{n+1}-S} - \frac{(B_0^{(n+i+3)}-B_0^{(n+i+2)})(B_0^{(n+i+3)}-B_0^{(n+i+1)})}{(B_0^{(n+i+3)}-B_0^{(n+i+2)}-B_0^{(n+i+1)}+B_0^{(n+i)})(S_{n+1}-S)} \right\}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left\{\frac{S_{n+i+3}-S}{S_{n+1}-S} - \frac{(S_{n+i+3}-S_{n+i+2})(S_{n+i+3}-S_{n+i+1})}{(S_{n+i+3}-S_{n+i+2}-S_{n+i+1}+S_{n+i})(S_{n+1}-S)}\right\}\\
&= \lambda^{i+2} - \lambda^2 \lim_{n\to\infty}\frac{S_{n+i+1}-S}{S_{n+1}-S}\\
&= \lambda^{i+2}-\lambda^{i+2}\\
&= 0.
\end{aligned}
$$

(ii) (a) When $k = 1$ we have

$$
\lim_{n\to\infty}\frac{B_0^{(n)}-S}{S_{n+1}-S} = \lim_{n\to\infty}\frac{S_n-S}{S_{n+1}-S} = \frac{1}{\lambda} \neq 0. \tag{3.83}
$$

(b) We prove this part of the theorem by induction.
For $k = 2$ we obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{B_1^{(n)}-S}{S_{n+2}-S} =\\
&= \lim_{n\to\infty}\left\{\frac{B_0^{(n+3)}-S}{S_{n+2}-S} - \frac{(B_0^{(n+3)}-B_0^{(n+2)})(B_0^{(n+3)}-B_0^{(n+1)})}{(B_0^{(n+3)}-B_0^{(n+2)}-B_0^{(n+1)}+B_0^{(n)})(S_{n+2}-S)}\right\}\\
&= \lim_{n\to\infty}\left\{\frac{S_{n+3}-S}{S_{n+2}-S} - \frac{(S_{n+3}-S_{n+2})(S_{n+3}-S_{n+1})}{(S_{n+3}-S_{n+2}-S_{n+1}+S_n)(S_{n+2}-S)}\right\}\\
&= \lambda - \frac{(\lambda^3-\lambda^2)(\lambda^3-\lambda)}{(\lambda^3-\lambda^2-\lambda+1)\lambda^2}\\
&= \lambda-\lambda\\
&= 0.
\end{aligned}
$$

We assume that $\lim_{n\to\infty}\frac{B_{k-1}^{(n)}-S}{S_{n+k}-S} = 0$ holds for every n and for $k \geq 2$ and then we prove that $\lim_{n\to\infty}\frac{B_k^{(n)}-S}{S_{n+k+1}-S} = 0$ holds for every n and for $k \geq 1$.

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{B_k^{(n)}-S}{S_{n+k+1}-S} =\\
&= \lim_{n\to\infty}\left\{\frac{B_{k-1}^{(n+3)}-S}{S_{n+k+1}-S} - \frac{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+2)})(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})}{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+2)}-B_{k-1}^{(n+1)}+B_{k-1}^{(n)})(S_{n+k+1}-S)}\right\}\\
&= \lim_{n\to\infty}\frac{B_{k-1}^{(n+3)}-S}{S_{n+k+3}-S}\frac{S_{n+k+3}-S}{S_{n+k+1}-S} - \lim_{n\to\infty}\frac{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+2)})(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})}{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+2)}-B_{k-1}^{(n+1)}+B_{k-1}^{(n)})(S_{n+k+1}-S)}\\
&= 0\cdot\lambda^2 - \lim_{n\to\infty}\frac{\left(\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n)}-S}-\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n)}-S}\right)\left(\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n)}-S}-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n)}-S}\right)}{\left(\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n)}-S}-\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n)}-S}-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n)}-S}+1\right)\frac{S_{n+k+1}-S}{B_{k-1}^{(n)}-S}}\\
&= -\frac{(b_{k-1}^3-b_{k-1}^2)(b_{k-1}^3-b_{k-1})}{b_{k-1}^3-b_{k-1}^2-b_{k-1}+1}\lim_{n\to\infty}\frac{B_{k-1}^{(n)}-S}{S_{n+k+1}-S}\\
&= -\frac{b_{k-1}^3(b_{k-1}-1)(b_{k-1}^2-1)}{(b_{k-1}-1)(b_{k-1}^2-1)}\lim_{n\to\infty}\frac{B_{k-1}^{(n)}-S}{S_{n+k}-S}\frac{S_{n+k}-S}{S_{n+k+1}-S}
\end{aligned}
$$

$$= -b_{k-1}^3 \cdot 0 \cdot \lim_{n\to\infty} \frac{S_{n+k} - S}{S_{n+k+1} - S}$$
$$= -b_{k-1}^3 \cdot 0 \cdot \frac{1}{\lambda}$$
$$= 0.$$

(iii) (a) When $k = 0$ we have

$$\lim_{n\to\infty} \frac{B_0^{(n)} - S}{S_n - S} = \lim_{n\to\infty} \frac{S_n - S}{S_n - S} = 1. \tag{3.84}$$

(b) We prove this part of the theorem by induction.
For $k = 1$ we obtain

$$\lim_{n\to\infty} \frac{B_1^{(n)} - S}{S_{n+1} - S} =$$
$$= \lim_{n\to\infty} \left\{ \frac{B_0^{(n+3)} - S}{S_{n+1} - S} - \frac{(B_0^{(n+3)} - B_0^{(n+2)})(B_0^{(n+3)} - B_0^{(n+1)})}{(B_0^{(n+3)} - B_0^{(n+2)} - B_0^{(n+1)} + B_0^{(n)})(S_{n+1} - S)} \right\}$$
$$= \lim_{n\to\infty} \left\{ \frac{S_{n+3} - S}{S_{n+1} - S} - \frac{(S_{n+3} - S_{n+2})(S_{n+3} - S_{n+1})}{(S_{n+3} - S_{n+2} - S_{n+1} + S_n)(S_{n+1} - S)} \right\}$$
$$= \lambda^2 - \frac{(\lambda^3 - \lambda^2)(\lambda^3 - \lambda)}{(\lambda^3 - \lambda^2 - \lambda + 1)\lambda}$$
$$= \lambda^2 - \lambda^2$$
$$= 0.$$

We assume that $\lim_{n\to\infty} \frac{B_k^{(n)} - S}{S_{n+k} - S} = 0$ holds for every n and for $k \geq 1$ and then we prove that $\lim_{n\to\infty} \frac{B_{k+1}^{(n)} - S}{S_{n+k+1} - S} = 0$ holds for every n and for $k \geq 0$.

$$\lim_{n\to\infty} \frac{B_{k+1}^{(n)} - S}{S_{n+k+1} - S} =$$
$$= \lim_{n\to\infty} \left\{ \frac{B_k^{(n+3)} - S}{S_{n+k+1} - S} - \frac{(B_k^{(n+3)} - B_k^{(n+2)})(B_k^{(n+3)} - B_k^{(n+1)})}{(B_k^{(n+3)} - B_k^{(n+2)} - B_k^{(n+1)} + B_k^{(n)})(S_{n+k+1} - S)} \right\}$$
$$= \lim_{n\to\infty} \frac{B_k^{(n+3)} - S}{S_{n+k+3} - S} \frac{S_{n+k+3} - S}{S_{n+k+1} - S} - \lim_{n\to\infty} \frac{(B_k^{(n+3)} - B_k^{(n+2)})(B_k^{(n+3)} - B_k^{(n+1)})}{(B_k^{(n+3)} - B_k^{(n+2)} - B_k^{(n+1)} + B_k^{(n)})(S_{n+k+1} - S)}$$
$$= 0 \cdot \lambda^2 - \lim_{n\to\infty} \frac{\left(\frac{B_k^{(n+3)} - S}{B_k^{(n)} - S} - \frac{B_k^{(n+2)} - S}{B_k^{(n)} - S} \right) \left(\frac{B_k^{(n+3)} - S}{B_k^{(n)} - S} - \frac{B_k^{(n+1)} - S}{B_k^{(n)} - S} \right)}{\left(\frac{B_k^{(n+3)} - S}{B_k^{(n)} - S} - \frac{B_k^{(n+2)} - S}{B_k^{(n)} - S} - \frac{B_k^{(n+1)} - S}{B_k^{(n)} - S} + 1 \right) \frac{S_{n+k+1} - S}{B_k^{(n)} - S}}$$
$$= -\frac{(b_k^3 - b_k^2)(b_k^3 - b_k)}{b_k^3 - b_k^2 - b_k + 1} \lim_{n\to\infty} \frac{B_k^{(n)} - S}{S_{n+k+1} - S}$$
$$= -b_k^3 \cdot 0 \cdot \lim_{n\to\infty} \frac{S_{n+k} - S}{S_{n+k+1} - S}$$
$$= -b_k^3 \cdot \lim_{n\to\infty} \frac{B_k^{(n)} - S}{S_{n+k} - S} \frac{S_{n+k} - S}{S_{n+k+1} - S}$$
$$= -b_k^3 \cdot 0 \cdot \frac{1}{\lambda}$$
$$= 0.$$

(iv) (a) When $k = 0$ we have

$$\lim_{n\to\infty} \frac{B_0^{(n)} - S}{S_n - S} = \lim_{n\to\infty} \frac{S_n - S}{S_n - S} = 1. \tag{3.85}$$

(b) We prove this part of the theorem by induction.
For $k = 1$ we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \frac{B_1^{(n)} - S}{S_{n+2} - S} = \\
&= \lim_{n\to\infty} \left\{ \frac{B_0^{(n+3)} - S}{S_{n+2} - S} - \frac{(B_0^{(n+3)} - B_0^{(n+2)})(B_0^{(n+3)} - B_0^{(n+1)})}{(B_0^{(n+3)} - B_0^{(n+2)} - B_0^{(n+1)} + B_0^{(n)})(S_{n+2} - S)} \right\} \\
&= \lim_{n\to\infty} \left\{ \frac{S_{n+3} - S}{S_{n+2} - S} - \frac{(S_{n+3} - S_{n+2})(S_{n+3} - S_{n+1})}{(S_{n+3} - S_{n+2} - S_{n+1} + S_n)(S_{n+2} - S)} \right\} \\
&= \lambda - \frac{(\lambda^3 - \lambda^2)(\lambda^3 - \lambda)}{\lambda^3 - \lambda^2 - \lambda + 1} \\
&= \lambda - \lambda \\
&= 0.
\end{aligned}$$

We assume that $\lim_{n\to\infty} \frac{B_k^{(n)} - S}{S_{n+2k} - S} = 0$ holds for every n and for $k \geq 1$ and then we prove that $\lim_{n\to\infty} \frac{B_{k+1}^{(n)} - S}{S_{n+2k+1} - S} = 0$ holds for every n and for $k \geq 0$.

$$\begin{aligned}
&\lim_{n\to\infty} \frac{B_{k+1}^{(n)} - S}{S_{n+2k+1} - S} = \\
&= \lim_{n\to\infty} \left\{ \frac{B_k^{(n+3)} - S}{S_{n+2k+1} - S} - \frac{(B_k^{(n+3)} - B_k^{(n+2)})(B_k^{(n+3)} - B_k^{(n+1)})}{(B_k^{(n+3)} - B_k^{(n+2)} - B_k^{(n+1)} + B_k^{(n)})(S_{n+2k+1} - S)} \right\} \\
&= \lim_{n\to\infty} \frac{B_k^{(n+3)} - S}{S_{n+2k+3} - S} \frac{S_{n+2k+3} - S}{S_{n+2k+1} - S} - \lim_{n\to\infty} \frac{(B_k^{(n+3)} - B_k^{(n+2)})(B_k^{(n+3)} - B_k^{(n+1)})}{(B_k^{(n+3)} - B_k^{(n+2)} - B_k^{(n+1)} + B_k^{(n)})(S_{n+2k+1} - S)} \\
&= 0 \cdot \lambda^2 - \lim_{n\to\infty} \frac{\left(\frac{B_k^{(n+3)} - S}{B_k^{(n)} - S} - \frac{B_k^{(n+2)} - S}{B_k^{(n)} - S} \right) \left(\frac{B_k^{(n+3)} - S}{B_k^{(n)} - S} - \frac{B_k^{(n+1)} - S}{B_k^{(n)} - S} \right)}{\left(\frac{B_k^{(n+3)} - S}{B_k^{(n)} - S} - \frac{B_k^{(n+2)} - S}{B_k^{(n)} - S} - \frac{B_k^{(n+1)} - S}{B_k^{(n)} - S} + 1 \right) \frac{S_{n+2k+1} - S}{B_k^{(n)} - S}} \\
&= -\frac{(b_k^3 - b_k^2)(b_k^3 - b_k)}{b_k^3 - b_k^2 - b_k + 1} \lim_{n\to\infty} \frac{B_k^{(n)} - S}{S_{n+2k+1} - S} \\
&= -b_k^3 \lim_{n\to\infty} \frac{B_k^{(n)} - S}{S_{n+2k} - S} \frac{S_{n+2k} - S}{S_{n+2k+1} - S} \\
&= -b_k^3 \cdot 0 \cdot \lim_{n\to\infty} \frac{S_{n+2k} - S}{S_{n+2k+1} - S} \\
&= -b_k^3 \cdot 0 \cdot \frac{1}{\lambda} \\
&= 0.
\end{aligned}$$

(v) (a) When $k = 0$ we have

$$\lim_{n\to\infty} \frac{B_0^{(n)} - S}{S_{n+i} - S} = \lim_{n\to\infty} \frac{S_n - S}{S_{n+i} - S} = \frac{1}{\lambda^i} \neq 0. \tag{3.86}$$

(b) We prove this part of the theorem by induction.
For $k = 1$ we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \frac{B_1^{(n)} - S}{S_{n+1+i} - S} = \\
&= \lim_{n\to\infty} \left\{ \frac{B_0^{(n+3)} - S}{S_{n+i+1} - S} - \frac{(B_0^{(n+3)} - B_0^{(n+2)})(B_0^{(n+3)} - B_0^{(n+1)})}{(B_0^{(n+3)} - B_0^{(n+2)} - B_0^{(n+1)} + B_0^{(n)})(S_{n+i+1} - S)} \right\} \\
&= \lim_{n\to\infty} \left\{ \frac{S_{n+3} - S}{S_{n+i+1} - S} - \frac{(S_{n+3} - S_{n+2})(S_{n+3} - S_{n+1})}{(S_{n+3} - S_{n+2} - S_{n+1} + S_n)(S_{n+i+1} - S)} \right\} \\
&= \frac{1}{\lambda^{i-2}} - \frac{(\lambda^3 - \lambda^2)(\lambda^3 - \lambda)}{\lambda^3 - \lambda^2 - \lambda + 1} \lim_{n\to\infty} \frac{S_n - S}{S_{n+1+i} - S} \\
&= \frac{1}{\lambda^{i-2}} - \lambda^3 \frac{1}{\lambda^{i+1}} \\
&= \frac{1}{\lambda^{i-2}} - \frac{1}{\lambda^{i-2}} \\
&= 0.
\end{aligned}$$

We assume that $\lim\limits_{n\to\infty} \frac{B_k^{(n)} - S}{S_{n+k+i} - S} = 0$ holds for every n and for $k \geq 1$ and then we prove that $\lim\limits_{n\to\infty} \frac{B_{k+1}^{(n)} - S}{S_{n+k+i+1} - S} = 0$ holds for every n and for $k \geq 0$.

$$\begin{aligned}
&\lim_{n\to\infty} \frac{B_{k+1}^{(n)} - S}{S_{n+k+i+1} - S} = \\
&= \lim_{n\to\infty} \left\{ \frac{B_k^{(n+3)} - S}{S_{n+k+i+1} - S} - \frac{(B_k^{(n+3)} - B_k^{(n+2)})(B_k^{(n+3)} - B_k^{(n+1)})}{(B_k^{(n+3)} - B_k^{(n+2)} - B_k^{(n+1)} + B_k^{(n)})(S_{n+k+i+1} - S)} \right\} \\
&= \lim_{n\to\infty} \frac{B_k^{(n+3)} - S}{S_{n+k+i+3} - S} \frac{S_{n+k+i+3} - S}{S_{n+k+i+1} - S} - \\
&- \lim_{n\to\infty} \frac{(B_k^{(n+3)} - B_k^{(n+2)})(B_k^{(n+3)} - B_k^{(n+1)})}{(B_k^{(n+3)} - B_k^{(n+2)} - B_k^{(n+1)} + B_k^{(n)})(S_{n+k+i+1} - S)} \\
&= 0 \cdot \lambda^2 - \lim_{n\to\infty} \frac{\left(\frac{B_k^{(n+3)} - S}{B_k^{(n)} - S} - \frac{B_k^{(n+2)} - S}{B_k^{(n)} - S} \right) \left(\frac{B_k^{(n+3)} - S}{B_k^{(n)} - S} - \frac{B_k^{(n+1)} - S}{B_k^{(n)} - S} \right)}{\left(\frac{B_k^{(n+3)} - S}{B_k^{(n)} - S} - \frac{B_k^{(n+2)} - S}{B_k^{(n)} - S} - \frac{B_k^{(n+1)} - S}{B_k^{(n)} - S} + 1 \right) \frac{S_{n+k+i+1} - S}{B_k^{(n)} - S}} \\
&= -\frac{(b_k^3 - b_k^2)(b_k^3 - b_k)}{b_k^3 - b_k^2 - b_k + 1} \lim_{n\to\infty} \frac{B_k^{(n)} - S}{S_{n+k+i+1} - S} \\
&= -b_k^3 \lim_{n\to\infty} \frac{B_k^{(n)} - S}{S_{n+k+i} - S} \frac{S_{n+k+i} - S}{S_{n+k+i+1} - S} \\
&= -b_k^3 \cdot 0 \cdot \lim_{n\to\infty} \frac{S_{n+k+i} - S}{S_{n+k+i+1} - S} \\
&= -b_k^3 \cdot 0 \cdot \frac{1}{\lambda} \\
&= 0.
\end{aligned}$$

Theorem 3.14 *(Bumbariu) If the conditions of Theorem 3.11 are satisfied. Then we have the following results for the iterated B-algorithm:*

(i) (a) $\{B_{k-1}^{(n+3)}\}$ *and* $\{S_{n+k}\}$ *have the same rate of convergence, when* n *tends to infinity and* $k=1$.
(b) $\{B_{k-1}^{(n+3)}\}$ *converges to* S *faster than* $\{S_{n+k}\}$*, when* n *tends to infinity and* $k\geq 2$.
(ii) (a) $\{B_{k-1}^{(n+3+i)}\}$ *and* $\{S_{n+k}\}$ *have the same rate of convergence, when* n *tends to infinity and* $k=1$.
(b) $\{B_{k-1}^{(n+3+i)}\}$ *converges to* S *faster than* $\{S_{n+k}\}$*, when* n *tends to infinity and* $k\geq 2$.

Proof
(i) (a) When $k=1$ we have

$$\lim_{n\to\infty}\frac{B_0^{(n+3)}-S}{S_{n+1}-S}=\lim_{n\to\infty}\frac{S_{n+3}-S}{S_{n+1}-S}=\lambda^2\neq 0. \tag{3.87}$$

(b) We prove this part of the theorem by induction.
For $k=2$ we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\frac{B_1^{(n+3)}-S}{S_{n+2}-S}=\\
&=\lim_{n\to\infty}\left\{\frac{B_0^{(n+6)}-S}{S_{n+2}-S}-\frac{(B_0^{(n+6)}-B_0^{(n+5)})(B_0^{(n+6)}-B_0^{(n+4)})}{(B_0^{(n+6)}-B_0^{(n+5)}-B_0^{(n+4)}+B_0^{(n+3)})(S_{n+2}-S)}\right\}\\
&=\lim_{n\to\infty}\left\{\frac{S_{n+6}-S}{S_{n+2}-S}-\frac{(S_{n+6}-S_{n+5})(S_{n+6}-S_{n+4})}{(S_{n+6}-S_{n+5}-S_{n+4}+S_{n+3})(S_{n+2}-S)}\right\}\\
&=\lambda^4-\frac{(\lambda^3-\lambda^2)(\lambda^3-\lambda)}{(\lambda^3-\lambda^2-\lambda+1)\frac{1}{\lambda}}\\
&=\lambda^4-\lambda^4\\
&=0.
\end{aligned}$$

We assume that $\lim\limits_{n\to\infty}\frac{B_{k-1}^{(n+3)}-S}{S_{n+k}-S}=0$ holds for every n and for $k\geq 2$ and then we prove that $\lim\limits_{n\to\infty}\frac{B_k^{(n+3)}-S}{S_{n+k+1}-S}=0$ holds for every n and for $k\geq 1$.

$$\begin{aligned}
&\lim_{n\to\infty}\frac{B_k^{(n+3)}-S}{S_{n+k+1}-S}=\\
&=\lim_{n\to\infty}\left\{\frac{B_{k-1}^{(n+6)}-S}{S_{n+k+1}-S}-\frac{(B_{k-1}^{(n+6)}-B_{k-1}^{(n+5)})(B_{k-1}^{(n+6)}-B_{k-1}^{(n+4)})}{(B_{k-1}^{(n+6)}-B_{k-1}^{(n+5)}-B_{k-1}^{(n+4)}+B_{k-1}^{(n+3)})(S_{n+k+1}-S)}\right\}\\
&=\lim_{n\to\infty}\frac{B_{k-1}^{(n+i+6)}-S}{S_{n+k+3}-S}\frac{S_{n+k+3}-S}{S_{n+k+1}-S}-\lim_{n\to\infty}\frac{(B_{k-1}^{(n+i+6)}-B_{k-1}^{(n+i+5)})(B_{k-1}^{(n+i+6)}-B_{k-1}^{(n+i+4)})}{(B_{k-1}^{(n+i+6)}-B_{k-1}^{(n+i+5)}-B_{k-1}^{(n+i+4)}+B_{k-1}^{(n+i+3)})(S_{n+k+1}-S)}\\
&=0\cdot\lambda^2-\lim_{n\to\infty}\frac{\left(\frac{B_{k-1}^{(n+6)}-S}{B_{k-1}^{(n+3)}-S}-\frac{B_{k-1}^{(n+5)}-S}{B_{k-1}^{(n+3)}-S}\right)\left(\frac{B_{k-1}^{(n+6)}-S}{B_{k-1}^{(n+3)}-S}-\frac{B_{k-1}^{(n+4)}-S}{B_{k-1}^{(n+3)}-S}\right)}{\left(\frac{B_{k-1}^{(n+6)}-S}{B_{k-1}^{(n+3)}-S}-\frac{B_{k-1}^{(n+5)}-S}{B_{k-1}^{(n+3)}-S}-\frac{B_{k-1}^{(n+4)}-S}{B_{k-1}^{(n+3)}-S}+1\right)\frac{S_{n+k+1}-S}{B_{k-1}^{(n+3)}-S}}\\
&=-\frac{(b_{k-1}^3-b_{k-1}^2)(b_{k-1}^3-b_{k-1})}{b_{k-1}^3-b_{k-1}^2-b_{k-1}+1}\lim_{n\to\infty}\frac{B_{k-1}^{(n+3)}-S}{S_{n+k+1}-S}\\
&=-b_{k-1}^3\lim_{n\to\infty}\frac{B_{k-1}^{(n+3)}-S}{S_{n+k}-S}\frac{S_{n+k}-S}{S_{n+k+1}-S}
\end{aligned}$$

$$
\begin{aligned}
&= -b_{k-1}^3 \cdot 0 \cdot \lim_{n\to\infty} \frac{S_{n+k}-S}{S_{n+k+1}-S} \\
&= -b_{k-1}^3 \cdot 0 \cdot \frac{1}{\lambda} \\
&= 0.
\end{aligned}
$$

(ii) (a) When $k = 1$ we have

$$
\lim_{n\to\infty} \frac{B_0^{(n+3+i)}-S}{S_{n+1}-S} = \lim_{n\to\infty} \frac{S_{n+3+i}-S}{S_{n+1}-S} = \lambda^{i+2} \neq 0. \tag{3.88}
$$

(b) We prove this part of the theorem by induction.
For $k = 2$ we obtain

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{B_1^{(n+i+3)}-S}{S_{n+2}-S} = \\
&= \lim_{n\to\infty} \left\{ \frac{B_0^{(n+i+6)}-S}{S_{n+2}-S} - \frac{(B_0^{(n+i+6)}-B_0^{(n+i+5)})(B_0^{(n+i+6)}-B_0^{(n+i+4)})}{(B_0^{(n+i+6)}-B_0^{(n+i+5)}-B_0^{(n+i+4)}+B_0^{(n+i+3)})(S_{n+2}-S)} \right\} \\
&= \lim_{n\to\infty} \left\{ \frac{S_{n+i+6}-S}{S_{n+2}-S} - \frac{(S_{n+i+6}-S_{n+i+5})(S_{n+i+6}-S_{n+i+4})}{(S_{n+i+6}-S_{n+i+5}-S_{n+i+4}+S_{n+i+3})(S_{n+2}-S)} \right\} \\
&= \lambda^{i+4} - \frac{(\lambda^3-\lambda^2)(\lambda^3-\lambda)}{(\lambda^3-\lambda^2-\lambda+1)\frac{1}{\lambda^{i+1}}} \\
&= \lambda^{i+4} - \lambda^{i+4} \\
&= 0.
\end{aligned}
$$

We assume that $\lim_{n\to\infty} \frac{B_{k-1}^{(n+i+3)}-S}{S_{n+k}-S} = 0$ holds for every n and for $k \geq 2$ and then we prove that $\lim_{n\to\infty} \frac{B_k^{(n+i+3)}-S}{S_{n+k+1}-S} = 0$ holds for every n and for $k \geq 1$.

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{B_k^{(n+i+3)}-S}{S_{n+k+1}-S} = \\
&= \lim_{n\to\infty} \left\{ \frac{B_{k-1}^{(n+i+6)}-S}{S_{n+k+1}-S} - \frac{(B_{k-1}^{(n+i+6)}-B_{k-1}^{(n+i+5)})(B_{k-1}^{(n+i+6)}-B_{k-1}^{(n+i+4)})}{(B_{k-1}^{(n+i+6)}-B_{k-1}^{(n+i+5)}-B_{k-1}^{(n+i+4)}+B_{k-1}^{(n+i+3)})(S_{n+k+1}-S)} \right\} \\
&= \lim_{n\to\infty} \frac{B_{k-1}^{(n+i+6)}-S}{S_{n+k+3}-S} \frac{S_{n+k+3}-S}{S_{n+k+1}-S} - \\
&- \lim_{n\to\infty} \frac{(B_{k-1}^{(n+i+6)}-B_{k-1}^{(n+i+5)})(B_{k-1}^{(n+i+6)}-B_{k-1}^{(n+i+4)})}{(B_{k-1}^{(n+i+6)}-B_{k-1}^{(n+i+5)}-B_{k-1}^{(n+i+4)}+B_{k-1}^{(n+i+3)})(S_{n+k+1}-S)} \\
&= 0 \cdot \lambda^2 - \lim_{n\to\infty} \frac{\left(\frac{B_{k-1}^{(n+i+6)}-S}{B_{k-1}^{(n+i+3)}-S} - \frac{B_{k-1}^{(n+i+5)}-S}{B_{k-1}^{(n+i+3)}-S}\right)\left(\frac{B_{k-1}^{(n+i+6)}-S}{B_{k-1}^{(n+i+3)}-S} - \frac{B_{k-1}^{(n+i+4)}-S}{B_{k-1}^{(n+i+3)}-S}\right)}{\left(\frac{B_{k-1}^{(n+i+6)}-S}{B_{k-1}^{(n+i+3)}-S} - \frac{B_{k-1}^{(n+i+5)}-S}{B_{k-1}^{(n+i+3)}-S} - \frac{B_{k-1}^{(n+i+4)}-S}{B_{k-1}^{(n+i+3)}-S} + 1\right)\frac{S_{n+k+1}-S}{B_{k-1}^{(n+i+3)}-S}} \\
&= -\frac{(b_{k-1}^3-b_{k-1}^2)(b_{k-1}^3-b_{k-1})}{b_{k-1}^3-b_{k-1}^2-b_{k-1}+1} \lim_{n\to\infty} \frac{B_{k-1}^{(n+i+3)}-S}{S_{n+k+1}-S} \\
&= -b_{k-1}^3 \lim_{n\to\infty} \frac{B_{k-1}^{(n+i+3)}-S}{S_{n+k}-S} \frac{S_{n+k}-S}{S_{n+k+1}-S}
\end{aligned}
$$

$$
\begin{aligned}
&= -b_{k-1}^3 \cdot 0 \cdot \lim_{n\to\infty} \frac{S_{n+k} - S}{S_{n+k+1} - S} \\
&= -b_{k-1}^3 \cdot 0 \cdot \frac{1}{\lambda} \\
&= 0.
\end{aligned}
$$

In order to investigate the convergence acceleration of the method, the iterated B-algorithm has been applied to some linear and logarithmically sequences taken from [24].

Example 3.14 *(Bumbariu) Let us consider the sequence defined by*

$$S_n = \frac{1}{n},$$

where $n = 0, 1, 2, \ldots$. *Then*
(i) $B_1^{(n)} = \frac{1}{2n+3}$ *for every* $n = 0, 1, 2, \ldots$;
(ii) $B_k^{(n)} = \frac{1}{2^k(n+k)+2^{k-1}k}$ *for every* $n, k = 0, 1, 2, \ldots$.

Proof (i)

$$
\begin{aligned}
B_1^{(n)} &= S_{n+3} - \frac{(S_{n+3} - S_{n+2})(S_{n+3} - S_{n+1})}{S_{n+3} - S_{n+2} - S_{n+1} - S_n} \\
&= \frac{1}{n+3} - \frac{\left(\frac{1}{n+3} - \frac{1}{n+2}\right)\left(\frac{1}{n+3} - \frac{1}{n+1}\right)}{\frac{1}{n+3} - \frac{1}{n+2} - \frac{1}{n+1} + \frac{1}{n}} \\
&= \frac{1}{2n+5}.
\end{aligned}
$$

(ii) We prove $B_k^{(n)} = \frac{1}{2^k(n+k)+2^{k-1}k}$ by induction.
(a) The first step for induction, $k = 0$, was proved at (i).
(b) We assume that $B_k^{(n)} = \frac{1}{2^k(n+k)+2^{k-1}k}$ holds for every $n, k = 0, 1, 2, \ldots$ and then we prove that $B_{k+1}^{(n)} = \frac{1}{2^{k+1}(n+1+k)+2^k(k+1)}$ also holds for every $n, k = 0, 1, 2, \ldots$.

$$
\begin{aligned}
B_{k+1}^{(n)} &= B_k^{(n+3)} - \frac{(B_k^{(n+3)} - B_k^{(n+2)})(B_k^{(n+3)} - B_k^{(n+1)})}{B_k^{(n+3)} - B_k^{(n+2)} - B_k^{(n+1)} + B_k^{(n)}} \\
&= \frac{1}{2^k(n+3+k) + 2^{k-1}k} - \\
&\quad - \frac{\left(\frac{1}{2^k(n+3+k)+2^{k-1}k} - \frac{1}{2^k(n+2+k)+2^{k-1}k}\right)\left(\frac{1}{2^k(n+3+k)+2^{k-1}k} - \frac{1}{2^k(n+1+k)+2^{k-1}k}\right)}{\frac{1}{2^k(n+3+k)+2^{k-1}k} - \frac{1}{2^k(n+2+k)+2^{k-1}k} - \frac{1}{2^k(n+1+k)+2^{k-1}k} + \frac{1}{2^k(n+k)+2^{k-1}k}} \\
&= \frac{1}{2^k(n+3+k) + 2^{k-1}k} - \frac{1}{2^{k-1}} \frac{\frac{(-2)(-4)}{2(n+3+k)+k}}{\frac{2\cdot 8[2(n+1+k)+(k+1)]}{2(n+k)+k}} \\
&= \frac{2(n+3+k)+k}{2^k[2(n+3+k)+k][2(n+1+k)+(k+1)]} \\
&= \frac{1}{2^{k+1}(n+1+k) + 2^k(k+1)}.
\end{aligned}
$$

Example 3.15 *(Bumbariu, [39]) Let us consider the sequence defined by*

$$S_n = \frac{1}{n+1},$$

where $n = 0, 1, 2, \ldots$. *Then*
(i) $B_1^{(n)} = \frac{1}{2n+5}$ *for every* $n = 0, 1, 2, \ldots$;
(ii) $B_k^{(n)} = \frac{1}{2^k(n+1+k)+2^{k-1}k}$ *for every* $n, k = 0, 1, 2, \ldots$.

Proof (i)

$$\begin{aligned} B_1^{(n)} &= S_{n+3} - \frac{(S_{n+3} - S_{n+2})(S_{n+3} - S_{n+1})}{S_{n+3} - S_{n+2} - S_{n+1} - S_n} \\ &= \frac{1}{n+4} - \frac{\left(\frac{1}{n+4} - \frac{1}{n+3}\right)\left(\frac{1}{n+4} - \frac{1}{n+2}\right)}{\frac{1}{n+4} - \frac{1}{n+3} - \frac{1}{n+2} + \frac{1}{n+1}} \\ &= \frac{1}{2n+5}. \end{aligned}$$

(ii) We prove $B_k^{(n)} = \frac{1}{2^k(n+1+k)+2^{k-1}k}$ by induction.
(a) The first step for induction, $k = 0$, was proved at (i).
(b) We assume that $B_k^{(n)} = \frac{1}{2^k(n+1+k)+2^{k-1}k}$ holds for every $n, k = 0, 1, 2, \ldots$ and then we prove that $B_{k+1}^{(n)} = \frac{1}{2^{k+1}(n+2+k)+2^k(k+1)}$ also holds for every $n, k = 0, 1, 2, \ldots$.

$$\begin{aligned} B_{k+1}^{(n)} &= B_k^{(n+3)} - \frac{(B_k^{(n+3)} - B_k^{(n+2)})(B_k^{(n+3)} - B_k^{(n+1)})}{B_k^{(n+3)} - B_k^{(n+2)} - B_k^{(n+1)} + B_k^{(n)}} \\ &= \frac{1}{2^k(n+4+k)+2^{k-1}k} - \\ &- \frac{\left(\frac{1}{2^k(n+4+k)+2^{k-1}k} - \frac{1}{2^k(n+3+k)+2^{k-1}k}\right)\left(\frac{1}{2^k(n+4+k)+2^{k-1}k} - \frac{1}{2^k(n+2+k)+2^{k-1}k}\right)}{\frac{1}{2^k(n+4+k)+2^{k-1}k} - \frac{1}{2^k(n+3+k)+2^{k-1}k} - \frac{1}{2^k(n+2+k)+2^{k-1}k} + \frac{1}{2^k(n+1+k)+2^{k-1}k}} \\ &= \frac{1}{2^k(n+4+k)+2^{k-1}k} - \frac{1}{2^{k-1}} \frac{\frac{(-2)(-4)}{2(n+4+k)+k}}{\frac{2\cdot 8[2(n+2+k)+(k+1)]}{2(n+1+k)+k}} \\ &= \frac{2(n+4+k)+k}{2^k[2(n+4+k)+k][2(n+2+k)+(k+1)]} \\ &= \frac{1}{2^{k+1}(n+2+k)+2^k(k+1)}. \end{aligned}$$

Example 3.16 *(Bumbariu, [39]) Let us consider the sequence defined by*

$$S_n = \frac{1}{n+\alpha},$$

where $n = 0, 1, 2, \ldots$ *and* α *is an arbitrary constant. Then*
(i) $B_1^{(n)} = \frac{1}{2(n+\alpha)+3}$ *for every* $n = 0, 1, 2, \ldots$;
(ii) $B_k^{(n)} = \frac{1}{2^k(n+\alpha+k)+2^{k-1}k}$ *for every* $n, k = 0, 1, 2, \ldots$.

Proof (i)

$$B_1^{(n)} = S_{n+3} - \frac{(S_{n+3}-S_{n+2})(S_{n+3}-S_{n+1})}{S_{n+3}-S_{n+2}-S_{n+1}-S_n}$$
$$= \frac{1}{n+3+\alpha} - \frac{\left(\frac{1}{n+3+\alpha}-\frac{1}{n+2+\alpha}\right)\left(\frac{1}{n+3+\alpha}-\frac{1}{n+1+\alpha}\right)}{\frac{1}{n+3+\alpha}-\frac{1}{n+2+\alpha}-\frac{1}{n+1+\alpha}+\frac{1}{n+\alpha}}$$
$$= \frac{1}{2(n+\alpha)+3}.$$

(ii) We prove $B_k^{(n)} = \frac{1}{2^k(n+\alpha+k)+2^{k-1}k}$ by induction.
(a) The first step for induction, $k=0$, was proved at (i).
(b) We assume that $B_k^{(n)} = \frac{1}{2^k(n+\alpha+k)+2^{k-1}k}$ holds for every $n, k = 0, 1, 2, \ldots$ and then we prove that $B_{k+1}^{(n)} = \frac{1}{2^{k+1}(n+1+\alpha+k)+2^k(k+1)}$ also holds for every $n, k = 0, 1, 2, \ldots$.

$$B_{k+1}^{(n)} = B_k^{(n+3)} - \frac{(B_k^{(n+3)}-B_k^{(n+2)})(B_k^{(n+3)}-B_k^{(n+1)})}{B_k^{(n+3)}-B_k^{(n+2)}-B_k^{(n+1)}+B_k^{(n)}}$$
$$= \frac{1}{2^k(n+3+\alpha+k)+2^{k-1}k} -$$
$$- \frac{\left(\frac{1}{2^k(n+3+\alpha+k)+2^{k-1}k}-\frac{1}{2^k(n+2+\alpha+k)+2^{k-1}k}\right)\left(\frac{1}{2^k(n+3+\alpha+k)+2^{k-1}k}-\frac{1}{2^k(n+1+\alpha+k)+2^{k-1}k}\right)}{\frac{1}{2^k(n+3+\alpha+k)+2^{k-1}k}-\frac{1}{2^k(n+2+\alpha+k)+2^{k-1}k}-\frac{1}{2^k(n+1+\alpha+k)+2^{k-1}k}+\frac{1}{2^k(n+\alpha+k)+2^{k-1}k}}$$
$$= \frac{1}{2^k(n+3+\alpha+k)+2^{k-1}k} - \frac{1}{2^{k-1}}\frac{\frac{(-2)(-4)}{2(n+3+\alpha+k)+k}}{\frac{2\cdot 8[2(n+1+\alpha+k)+(k+1)]}{2(n+\alpha+k)+k}}$$
$$= \frac{2(n+3+\alpha+k)+k}{2^k[2(n+3+\alpha+k)+k][2(n+1+\alpha+k)+(k+1)]}$$
$$= \frac{1}{2^{k+1}(n+1+\alpha+k)+2^k(k+1)}.$$

Example 3.17 *(Bumbariu, [39]) Let us consider the sequence defined by*

$$S_n = \frac{\beta}{n+\alpha},$$

where $n = 0, 1, 2, \ldots$ and α, β are two arbitrary constants. Then
(i) $B_1^{(n)} = \frac{\beta}{2(n+\alpha)+3}$ for every $n = 0, 1, 2, \ldots$;
(ii) $B_k^{(n)} = \frac{\beta}{2^k(n+\alpha+k)+2^{k-1}k}$ for every $n, k = 0, 1, 2, \ldots$.

Proof (i)

$$B_1^{(n)} = S_{n+3} - \frac{(S_{n+3}-S_{n+2})(S_{n+3}-S_{n+1})}{S_{n+3}-S_{n+2}-S_{n+1}-S_n}$$
$$= \frac{\beta}{n+3+\alpha} - \frac{\left(\frac{\beta}{n+3+\alpha}-\frac{\beta}{n+2+\alpha}\right)\left(\frac{\beta}{n+3+\alpha}-\frac{\beta}{n+1+\alpha}\right)}{\frac{\beta}{n+3+\alpha}-\frac{\beta}{n+2+\alpha}-\frac{\beta}{n+1+\alpha}+\frac{1}{n+\alpha}}$$
$$= \frac{\beta}{2(n+\alpha)+3}.$$

(ii) We prove $B_k^{(n)} = \frac{\beta}{2^k(n+\alpha+k)+2^{k-1}k}$ by induction.
(a) The first step for induction, $k=0$, was proved at (i).
(b) We assume that $B_k^{(n)} = \frac{\beta}{2^k(n+\alpha+k)+2^{k-1}k}$ holds for every $n, k = 0, 1, 2, \ldots$ and then we prove that $B_{k+1}^{(n)} = \frac{\beta}{2^{k+1}(n+1+\alpha+k)+2^k(k+1)}$ also holds for every $n, k = 0, 1, 2, \ldots$.

$$B_{k+1}^{(n)} = B_k^{(n+3)} - \frac{(B_k^{(n+3)} - B_k^{(n+2)})(B_k^{(n+3)} - B_k^{(n+1)})}{B_k^{(n+3)} - B_k^{(n+2)} - B_k^{(n+1)} + B_k^{(n)}}$$
$$= \frac{\beta}{2^k(n+3+\alpha+k)+2^{k-1}k} -$$
$$- \frac{\left(\frac{\beta}{2^k(n+3+\alpha+k)+2^{k-1}k} - \frac{\beta}{2^k(n+2+\alpha+k)+2^{k-1}k}\right)\left(\frac{\beta}{2^k(n+3+\alpha+k)+2^{k-1}k} - \frac{\beta}{2^k(n+1+\alpha+k)+2^{k-1}k}\right)}{\frac{\beta}{2^k(n+3+\alpha+k)+2^{k-1}k} - \frac{\beta}{2^k(n+2+\alpha+k)+2^{k-1}k} - \frac{\beta}{2^k(n+1+\alpha+k)+2^{k-1}k} + \frac{\beta}{2^k(n+\alpha+k)+2^{k-1}k}}$$
$$= \frac{\beta}{2^k(n+3+\alpha+k)+2^{k-1}k} - \frac{\beta}{2^{k-1}} \frac{\frac{(-2)(-4)}{2(n+3+\alpha+k)+k}}{\frac{2\cdot 8[2(n+1+\alpha+k)+(k+1)]}{2(n+\alpha+k)+k}}$$
$$= \frac{\beta[2(n+3+\alpha+k)+k]}{2^k[2(n+3+\alpha+k)+k][2(n+1+\alpha+k)+(k+1)]}$$
$$= \frac{\beta}{2^{k+1}(n+1+\alpha+k)+2^k(k+1)}.$$

Example 3.18 *(Bumbariu) Let us consider the sequence defined by*

$$S_n = \frac{n}{n+1},$$

where $n = 0, 1, 2, \ldots$. *Then*
(i) $B_1^{(n)} = \frac{2n+4}{2n+5}$ *for every* $n = 0, 1, 2, \ldots$;
(ii) $B_k^{(n)} = \frac{2^k(n+1+k)+2^{k-1}(k+2)-1}{2^k(n+1+k)+2^{k-1}(k+2)}$ *for every* $n, k = 0, 1, 2, \ldots$.

Proof (i)

$$B_1^{(n)} = S_{n+3} - \frac{(S_{n+3} - S_{n+2})(S_{n+3} - S_{n+1})}{S_{n+3} - S_{n+2} - S_{n+1} - S_n}$$
$$= \frac{n+3}{n+4} - \frac{\left(\frac{n+3}{n+4} - \frac{n+2}{n+3}\right)\left(\frac{n+3}{n+4} - \frac{n+1}{n+2}\right)}{\frac{n+3}{n+4} - \frac{n+2}{n+3} - \frac{n+1}{n+2} + \frac{n}{n+1}}$$
$$= \frac{2n+4}{2n+5}.$$

(ii) We prove $B_k^{(n)} = \frac{2^k(n+1+k)+2^{k-1}(k+2)-1}{2^k(n+1+k)+2^{k-1}(k+2)}$ by induction.
(a) The first step for induction, $k=0$, was proved at (i).
(b) We assume that $B_k^{(n)} = \frac{2^k(n+1+k)+2^{k-1}(k+2)-1}{2^k(n+1+k)+2^{k-1}(k+2)}$ holds for every $n, k = 0, 1, 2, \ldots$ and

then we prove that $B_{k+1}^{(n)} = \frac{2^{k+1}(n+2+k)+2^k(k+3)-1}{2^{k+1}(n+2+k)+2^k(k+3)}$ also holds for every $n, k = 0, 1, 2, \ldots$.

$$\begin{aligned}
B_{k+1}^{(n)} &= B_k^{(n+3)} - \frac{(B_k^{(n+3)} - B_k^{(n+2)})(B_k^{(n+3)} - B_k^{(n+1)})}{B_k^{(n+3)} - B_k^{(n+2)} - B_k^{(n+1)} + B_k^{(n)}} \\
&= \frac{2^k(n+4+k) + 2^{k-1}(k+2) - 1}{2^k(n+4+k) + 2^{k-1}(k+2)} - \frac{1}{2^{k-1}} \frac{\frac{(-2)(-4)}{2(n+4+k)+(k+2)}}{\frac{-2\cdot 8[2(n+2+k)+(k+3)]}{2(n+1+k)+(k+2)}} \\
&= \frac{[2(n+4+k)+(k+2)][2^{k+1}(n+2+k)+2^k(k+3)-1]}{2^k[2(n+4+k)+(k+2)][2(n+2+k)+(k+3)]} \\
&= \frac{2^{k+1}(n+2+k)+2^k(k+3)-1}{2^{k+1}(n+2+k)+2^k(k+3)}.
\end{aligned}$$

Example 3.19 *(Bumbariu, [39]) Let us consider the sequence defined by*

$$S_n = 1 + \frac{1}{n+1},$$

where $n = 0, 1, 2, \ldots$. *Then*
(i) $B_1^{(n)} = \frac{2n+6}{2n+5}$ *for every* $n = 0, 1, 2, \ldots$;
(ii) $B_k^{(n)} = \frac{2^k(n+1+k)+2^{k-1}(k+2)+1}{2^k(n+1+k)+2^{k-1}(k+2)}$ *for every* $n, k = 0, 1, 2, \ldots$.

Proof (i)

$$\begin{aligned}
B_1^{(n)} &= S_{n+3} - \frac{(S_{n+3} - S_{n+2})(S_{n+3} - S_{n+1})}{S_{n+3} - S_{n+2} - S_{n+1} - S_n} \\
&= 1 + \frac{1}{n+4} - \frac{\left(1 + \frac{1}{n+4} - 1 - \frac{1}{n+3}\right)\left(1 + \frac{1}{n+4} - 1 - \frac{1}{n+2}\right)}{1 + \frac{1}{n+4} - 1 - \frac{1}{n+3} - 1 - \frac{1}{n+2} + 1 + \frac{1}{n+1}} \\
&= \frac{2n+6}{2n+5}.
\end{aligned}$$

(ii) We prove $B_k^{(n)} = \frac{2^k(n+1+k)+2^{k-1}(k+2)+1}{2^k(n+1+k)+2^{k-1}(k+2)}$ by induction.
(a) The first step for induction, $k = 0$, was proved at (i).
(b) We assume that $B_k^{(n)} = \frac{2^k(n+1+k)+2^{k-1}(k+2)+1}{2^k(n+1+k)+2^{k-1}(k+2)}$ holds for every $n, k = 0, 1, 2, \ldots$ and then we prove that $B_{k+1}^{(n)} = \frac{2^{k+1}(n+2+k)+2^k(k+3)+1}{2^{k+1}(n+2+k)+2^k(k+3)}$ also holds for every $n, k = 0, 1, 2, \ldots$.

$$\begin{aligned}
B_{k+1}^{(n)} &= B_k^{(n+3)} - \frac{(B_k^{(n+3)} - B_k^{(n+2)})(B_k^{(n+3)} - B_k^{(n+1)})}{B_k^{(n+3)} - B_k^{(n+2)} - B_k^{(n+1)} + B_k^{(n)}} \\
&= \frac{2^k(n+4+k) + 2^{k-1}(k+2) + 1}{2^k(n+4+k) + 2^{k-1}(k+2)} - \frac{1}{2^{k-1}} \frac{\frac{(-2)(-4)}{2(n+4+k)+(k+2)}}{\frac{2\cdot 8[2(n+2+k)+(k+3)]}{2(n+1+k)+(k+2)}} \\
&= \frac{[2(n+4+k)+(k+2)][2^{k+1}(n+2+k)+2^k(k+3)+1]}{2^k[2(n+4+k)+(k+2)][2(n+2+k)+(k+3)]} \\
&= \frac{2^{k+1}(n+2+k)+2^k(k+3)+1}{2^{k+1}(n+2+k)+2^k(k+3)}.
\end{aligned}$$

In [24], Brezinski considered the sequence, $\{S_n\}$, given at the Example 3.19, consequently it is natural to consider the sequence, $\{S_n\}$, given in the following example.

Example 3.20 *(Bumbariu, [39]) Let us consider the sequence defined by*

$$S_n = \frac{n+1}{n+2},$$

where $n = 0, 1, 2, \ldots$. *Then*
(i) $B_1^{(n)} = \frac{2n+6}{2n+7}$ *for every* $n = 0, 1, 2, \ldots$;
(ii) $B_k^{(n)} = \frac{2^k(n+1+k)+2^{k-1}(k+2)-1}{2^k(n+1+k)+2^{k-1}(k+2)}$ *for every* $n, k = 0, 1, 2, \ldots$.

Proof (i)

$$\begin{aligned} B_1^{(n)} &= S_{n+3} - \frac{(S_{n+3} - S_{n+2})(S_{n+3} - S_{n+1})}{S_{n+3} - S_{n+2} - S_{n+1} - S_n} \\ &= \frac{n+4}{n+5} - \frac{\left(\frac{n+4}{n+5} - \frac{n+3}{n+4}\right)\left(\frac{n+4}{n+5} - \frac{n+2}{n+3}\right)}{\frac{n+4}{n+5} - \frac{n+3}{n+4} - \frac{n+2}{n+3} + \frac{n+1}{n+2}} \\ &= \frac{2n+6}{2n+7}. \end{aligned}$$

(ii) We prove $B_k^{(n)} = \frac{2^k(n+1+k)+2^{k-1}(k+2)-1}{2^k(n+1+k)+2^{k-1}(k+2)}$ by induction.
(a) The first step for induction, $k = 0$, was proved at (i).
(b) We assume that $B_k^{(n)} = \frac{2^k(n+1+k)+2^{k-1}(k+2)-1}{2^k(n+1+k)+2^{k-1}(k+2)}$ holds for every $n, k = 0, 1, 2, \ldots$ and then we prove that $B_{k+1}^{(n)} = \frac{2^{k+1}(n+2+k)+2^k(k+3)-1}{2^{k+1}(n+2+k)+2^k(k+3)}$ also holds for every $n, k = 0, 1, 2, \ldots$.

$$\begin{aligned} B_{k+1}^{(n)} &= B_k^{(n+3)} - \frac{(B_k^{(n+3)} - B_k^{(n+2)})(B_k^{(n+3)} - B_k^{(n+1)})}{B_k^{(n+3)} - B_k^{(n+2)} - B_k^{(n+1)} + B_k^{(n)}} \\ &= \frac{2^k(n+4+k) + 2^{k-1}(k+2) - 1}{2^k(n+4+k) + 2^{k-1}(k+2)} - \frac{1}{2^{k-1}} \frac{\frac{(-2)(-4)}{2(n+4+k)+(k+2)}}{\frac{-2\cdot 8[2(n+2+k)+(k+3)]}{2(n+1+k)+(k+2)}} \\ &= \frac{[2(n+4+k) + (k+2)][2^{k+1}(n+2+k) + 2^k(k+3) - 1]}{2^k[2(n+4+k) + (k+2)][2(n+2+k) + (k+3)]} \\ &= \frac{2^{k+1}(n+2+k) + 2^k(k+3) - 1}{2^{k+1}(n+2+k) + 2^k(k+3)}. \end{aligned}$$

The Examples (3.14)-(3.20) show that the sequences $\{S_n\}$ and $\{B_k^{(n)}$ have the same rate of convergence.
In the sequel we present a numerical example, which will help us to illustrate the efficiency and the performance of the new proposed algorithm.

Example 3.21 *(Bumbariu, [36])[88] Test function* $f(x) = e^x - 3x^2 = 0$, *which has a unique root* $x^* = 3.73307902863281420061995402984351126439 0\ldots$. *This equation can be rewritten into a fixed point problem by* $g(x) = \ln(3x^2)$. *To apply the new acceleration technique we shall take the initial guess* $x_0 \in \{3.3, 3.5, 3.9, 4.0\}$. *The results for Example 3.21 are listed in Table 3.*

Table 3

Initial guess	No. of iterations	1	2	3	4	5	6	7	8
3.3	$B_k^{(n)}$	1	4	8	12	19	25	31	37
	$A_k^{(n)}$	1	3	7	10	13	19	22	25
3.5	$B_k^{(n)}$	2	6	10	14	19	27	34	38
	$A_k^{(n)}$	2	4	7	11	15	20	25	31
3.9	$B_k^{(n)}$	2	5	10	15	19	28	35	39
	$A_k^{(n)}$	2	5	8	11	16	19	29	32
4.0	$B_k^{(n)}$	2	5	9	14	19	27	34	38
	$A_k^{(n)}$	2	5	7	11	15	19	25	30

Table 3 shows that the iterated form of the B-algorithm improves significantly the convergence speed of a scalar sequence. For all twenty test functions, that we have studied, the results were similar with that in case of Example 3.21. In the table are listed the exact number of decimals at the first eight steps for B-algorithm and Aitken's Δ^2 process. All the numerical computations are done with Maple 13, using 39 digits arithmetic.

In what follows we will arrange the elements $B_k^{(n)}$, of the B table, in a rectangular scheme, where the superscript n indicates the row and the subscript k indicates the column of the 2-dimensional array:

$$\begin{array}{cccccc} B_0^{(0)} & B_1^{(0)} & B_2^{(0)} & \ldots & B_n^{(0)} & \ldots \\ B_0^{(1)} & B_1^{(1)} & B_2^{(1)} & \ldots & B_n^{(1)} & \ldots \\ B_0^{(2)} & B_1^{(2)} & B_2^{(2)} & \ldots & B_n^{(2)} & \ldots \\ B_0^{(3)} & B_1^{(3)} & B_2^{(3)} & \ldots & B_n^{(3)} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ B_0^{(n)} & B_1^{(n)} & B_2^{(n)} & \ldots & B_n^{(n)} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array} \tag{3.89}$$

where the entries in the first column of the array are the starting values $B_0^{(n)} = S_n$ of the scheme (3.8) and the others elements of the B table can be computed with the help of the relation (3.8).

Every five terms, which are connected with this nonlinear recursion can be arranged as follows

$$\begin{array}{ll} B_k^{(n)} & B_{k+1}^{(n)} \\ B_k^{(n+1)} & \\ B_k^{(n+2)} & \\ B_k^{(n+3)}. & \end{array} \tag{3.90}$$

3.1.3 Iterated form for the other representations of the iterated B-algorithm for accelerating the Picard iteration

Author's original result in this paragraph are: the iterated forms for the other representation of the B-algorithm, Theorem 3.15, Theorem 3.16 and Example 3.22.
In this paragraph we present the iterated form for the methods (3.46)-(3.51), presented in the Paragraph 3.1.1, we give convergence results for these methods and in the end of the paragraph we give a numerical example to see the practical implication of these acceleration methods.

The iterated forms for the methods (3.46)-(3.51) are:

$$B_{k+1}^{(n)} = \frac{B_k^{(n)} B_k^{(n+3)} - B_k^{(n+1)} B_k^{(n+2)}}{\overline{\Delta} B_k^{(n+1)} - \overline{\Delta} B_k^{(n)}}, \quad n, k \in \mathbb{N} \tag{3.91}$$

$$B_{k+1}^{(n)} = \frac{B_k^{(n)} \overline{\Delta} B_k^{(n+1)} - B_k^{(n+1)} \overline{\Delta} B_k^{(n)}}{\overline{\Delta} B_k^{(n+1)} - \overline{\Delta} B_k^{(n)}}, \quad n, k \in \mathbb{N} \tag{3.92}$$

$$B_{k+1}^{(n)} = \frac{B_k^{(n+2)} \overline{\Delta} B_k^{(n+1)} - B_k^{(n+3)} \overline{\Delta} B_k^{(n)}}{\overline{\Delta} B_k^{(n+1)} - \overline{\Delta} B_k^{(n)}}, \quad n, k \in \mathbb{N} \tag{3.93}$$

$$B_{k+1}^{(n)} = B_k^{(n)} - \frac{\overline{\Delta} B_k^{(n)} - \Delta B_k^{(n)}}{\overline{\Delta} B_k^{(n+1)} - \overline{\Delta} B_k^{(n)}}, \quad n, k \in \mathbb{N} \tag{3.94}$$

$$B_{k+1}^{(n)} = B_k^{(n+1)} - \frac{\overline{\Delta} B_k^{(n+1)} - \Delta B_k^{(n)}}{\overline{\Delta} B_k^{(n+1)} - \overline{\Delta} B_k^{(n)}}, \quad n, k \in \mathbb{N} \tag{3.95}$$

$$B_{k+1}^{(n)} = B_k^{(n+2)} - \frac{\overline{\Delta} B_k^{(n)} - \Delta B_k^{(n+2)}}{\overline{\Delta} B_k^{(n+1)} - \overline{\Delta} B_k^{(n)}}, \quad n, k \in \mathbb{N} \tag{3.96}$$

where $B_0^{(n)} = S_n$, $\{S_n\}$ is the sequence to be accelerated, Δ, the forward difference operator, has been defined before and we denote $\overline{\Delta}$, the forward difference operator with two steps, by $\overline{\Delta} B_k^{(n)} = B_k^{(n+2)} - B_k^{(n)}$, the difference operators, Δ and $\overline{\Delta}$, act only upon superscript n and not upon the subscript k.

In the sequel we will give conditions on the $B_i^{(n)}$'s which insure that $\lim\limits_{n\to\infty} B_k^{(n)} = S$.

Theorem 3.15 *(Bumbariu) If* $\lim\limits_{n\to\infty} S_n = S$, *and if* $\forall i, \exists b_i \neq 1$ *such that* $\lim\limits_{n\to\infty} \frac{B_i^{(n+1)}}{B_i^{(n)}} = b_i$, *for* $\forall i \neq j$, $b_i \neq b_j$. *Then* $\forall k$ *we have* $\lim\limits_{n\to\infty} B_k^{(n)} = S$, *where* $B_k^{(n)}$ *is defined by* (3.91)-(3.96).

Proof In order to prove the relations (3.91)-(3.93) we rewrite them in the same manner as the iterated B-algorithm, (3.57), consequently the first part of the theorem will be exact as for the Theorem 3.10 from the Paragraph 3.1.2, we will not give it here.

The proof for the relation (3.94).
For $k = 0$ we have

$$B_1^{(n)} = B_0^{(n)} - \frac{(B_0^{(n+2)} - B_0^{(n)})(B_0^{(n+1)} - B_0^{(n)})}{(B_0^{(n+3)} - B_0^{(n+1)}) - (B_0^{(n+2)} - B_0^{(n)})},$$

where $B_0^{(n)} = S_n$.

Multiplying with $\frac{B_0^{(n+1)}}{B_0^{(n+1)}}$ we get

$$B_1^{(n)} = B_0^{(n)} - \frac{\frac{B_0^{(n+2)}}{B_0^{(n+1)}} - \frac{B_0^{(n)}}{B_0^{(n+1)}}}{\frac{B_0^{(n+3)}}{B_0^{(n+1)}} - \frac{B_0^{(n+2)}}{B_0^{(n+1)}} - 1 + \frac{B_0^{(n)}}{B_0^{(n+1)}}}(B_0^{(n+1)} - B_0^{(n)}). \tag{3.97}$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n)} - \frac{b_0 - \frac{1}{b_0}}{b_0^2 - b_0 - 1 + \frac{1}{b_0}} \lim_{n\to\infty} (B_0^{(n+1)} - B_0^{(n)}), \tag{3.98}$$

from where it results

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n)} - \frac{1}{b_0 - 1} \lim_{n\to\infty} (B_0^{(n+1)} - B_0^{(n)}), \tag{3.99}$$

where $b_0 \neq 1$ and $B_0^{(n)} = S_n$. Hence

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} S_n - \frac{1}{b_0 - 1} \lim_{n\to\infty} (S_{n+1} - S_n), \tag{3.100}$$

because $\lim\limits_{n\to\infty} S_n = S$ we obtain $\lim\limits_{n\to\infty} B_1^{(n)} = S$.

Suppose that $\lim\limits_{n\to\infty} B_{k-1}^{(n)} = S$ we prove that $\lim\limits_{n\to\infty} B_k^{(n)} = S$.

$$B_k^{(n)} = B_{k-1}^{(n)} - \frac{(B_{k-1}^{(n+2)} - B_{k-1}^{(n)})(B_{k-1}^{(n+1)} - B_{k-1}^{(n)})}{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})},$$

Multiplying with $\frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+1)}}$ we obtain

$$B_k^{(n)} = B_{k-1}^{(n)} - \frac{\frac{B_{k-1}^{(n+2)}}{B_{k-1}^{(n+1)}} - \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+1)}}}{\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+1)}} - \frac{B_{k-1}^{(n+2)}}{B_{k-1}^{(n+1)}} - 1 + \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+1)}}} (B_{k-1}^{(n+1)} - B_{k-1}^{(n)}). \tag{3.101}$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n)} - \frac{b_{k-1} - \frac{1}{b_{k-1}}}{b_{k-1}^2 - b_{k-1} - 1 + \frac{1}{b_{k-1}}} \lim_{n\to\infty} (B_{k-1}^{(n+1)} - B_{k-1}^{(n)}), \tag{3.102}$$

hence

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n)} - \frac{1}{b_{k-1} - 1} \lim_{n\to\infty} (B_{k-1}^{(n+1)} - B_{k-1}^{(n)}), \tag{3.103}$$

where $b_{k-1} \neq 1$ and $B_{k-1}^{(n)} = S_n$ we obtain $\lim\limits_{n\to\infty} B_k^{(n)} = S$.

The proof for the relation (3.95).
For $k = 0$ we have

$$B_1^{(n)} = B_0^{(n+1)} - \frac{(B_0^{(n+3)} - B_0^{(n+1)})(B_0^{(n+1)} - B_0^{(n)})}{(B_0^{(n+3)} - B_0^{(n+1)}) - (B_0^{(n+2)} - B_0^{(n)})},$$

where $B_0^{(n)} = S_n$.

Multiplying with $\frac{B_0^{(n+1)}}{B_0^{(n+1)}}$ we get

$$B_1^{(n)} = B_0^{(n+1)} - \frac{\frac{B_0^{(n+3)}}{B_0^{(n+1)}} - 1}{\frac{B_0^{(n+3)}}{B_0^{(n+1)}} - \frac{B_0^{(n+2)}}{B_0^{(n+1)}} - 1 + \frac{B_0^{(n)}}{B_0^{(n+1)}}}(B_0^{(n+1)} - B_0^{(n)}). \tag{3.104}$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n+1)} - \frac{b_0^2 - 1}{b_0^2 - b_0 - 1 + \frac{1}{b_0}} \lim_{n\to\infty}(B_0^{(n+1)} - B_0^{(n)}), \tag{3.105}$$

from where it results

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n)} - \frac{b_0}{b_0 - 1} \lim_{n\to\infty}(B_0^{(n+1)} - B_0^{(n)}), \tag{3.106}$$

where $b_0 \neq 1$ and $B_0^{(n)} = S_n$. Hence

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} S_{n+1} - \frac{b_0}{b_0 - 1} \lim_{n\to\infty}(S_{n+1} - S_n), \tag{3.107}$$

because $\lim\limits_{n\to\infty} S_n = S$ we obtain $\lim\limits_{n\to\infty} B_1^{(n)} = S$.

Suppose that $\lim\limits_{n\to\infty} B_{k-1}^{(n)} = S$ we prove that $\lim\limits_{n\to\infty} B_k^{(n)} = S$.

$$B_k^{(n)} = B_{k-1}^{(n+1)} - \frac{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})(B_{k-1}^{(n+1)} - B_{k-1}^{(n)})}{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})},$$

Multiplying with $\frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+1)}}$ we obtain

$$B_k^{(n)} = B_{k-1}^{(n+1)} - \frac{\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+1)}} - 1}{\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+1)}} - \frac{B_{k-1}^{(n+2)}}{B_{k-1}^{(n+1)}} - 1 + \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+1)}}}(B_{k-1}^{(n+1)} - B_{k-1}^{(n)}). \tag{3.108}$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+1)} - \frac{b_{k-1}^2 - 1}{b_{k-1}^2 - b_{k-1} - 1 + \frac{1}{b_{k-1}}} \lim_{n\to\infty}(B_{k-1}^{(n+1)} - B_{k-1}^{(n)}), \tag{3.109}$$

hence

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+1)} - \frac{b_{k-1}}{b_{k-1} - 1} \lim_{n\to\infty}(B_{k-1}^{(n+1)} - B_{k-1}^{(n)}), \tag{3.110}$$

where $b_{k-1} \neq 1$ and $B_{k-1}^{(n)} = S_n$ we obtain $\lim\limits_{n\to\infty} B_k^{(n)} = S$.

The proof for the relation (3.96).
For $k = 0$ we have

$$B_1^{(n)} = B_0^{(n+2)} - \frac{(B_0^{(n+2)} - B_0^{(n)})(B_0^{(n+3)} - B_0^{(n+2)})}{(B_0^{(n+3)} - B_0^{(n+1)}) - (B_0^{(n+2)} - B_0^{(n)})},$$

where $B_0^{(n)} = S_n$.

Multiplying with $\frac{B_0^{(n+1)}}{B_0^{(n+1)}}$ we get

$$B_1^{(n)} = B_0^{(n+2)} - \frac{\frac{B_0^{(n+2)}}{B_0^{(n+1)}} - \frac{B_0^{(n)}}{B_0^{(n+1)}}}{\frac{B_0^{(n+3)}}{B_0^{(n+1)}} - \frac{B_0^{(n+2)}}{B_0^{(n+1)}} - 1 + \frac{B_0^{(n)}}{B_0^{(n+1)}}}(B_0^{(n+3)} - B_0^{(n+2)}). \tag{3.111}$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n+2)} - \frac{b_0 - \frac{1}{b_0}}{b_0^2 - b_0 - 1 + \frac{1}{b_0}} \lim_{n\to\infty}(B_0^{(n+3)} - B_0^{(n+2)}), \tag{3.112}$$

from where it results

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n+2)} - \frac{1}{b_0 - 1} \lim_{n\to\infty}(B_0^{(n+3)} - B_0^{(n+2)}), \tag{3.113}$$

where $b_0 \neq 1$ and $B_0^{(n)} = S_n$. Hence

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} S_{n+2} - \frac{1}{b_0 - 1} \lim_{n\to\infty}(S_{n+3} - S_{n+2}), \tag{3.114}$$

because $\lim_{n\to\infty} S_n = S$ we obtain $\lim_{n\to\infty} B_1^{(n)} = S$.

Suppose that $\lim_{n\to\infty} B_{k-1}^{(n)} = S$ we prove that $\lim_{n\to\infty} B_k^{(n)} = S$.

$$B_k^{(n)} = B_{k-1}^{(n+2)} - \frac{(B_{k-1}^{(n+2)} - B_{k-1}^{(n)})(B_{k-1}^{(n+3)} - B_{k-1}^{(n+2)})}{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})},$$

Multiplying with $\frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+1)}}$ we obtain

$$B_k^{(n)} = B_{k-1}^{(n+2)} - \frac{\frac{B_{k-1}^{(n+2)}}{B_{k-1}^{(n+1)}} - \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+1)}}}{\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+1)}} - \frac{B_{k-1}^{(n+2)}}{B_{k-1}^{(n+1)}} - 1 + \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+1)}}}(B_{k-1}^{(n+3)} - B_{k-1}^{(n+2)}). \tag{3.115}$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+2)} - \frac{b_{k-1} - \frac{1}{b_{k-1}}}{b_{k-1}^2 - b_{k-1} - 1 + \frac{1}{b_{k-1}}} \lim_{n\to\infty}(B_{k-1}^{(n+3)} - B_{k-1}^{(n+2)}), \tag{3.116}$$

hence

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+2)} - \frac{1}{b_{k-1} - 1} \lim_{n\to\infty}(B_{k-1}^{(n+3)} - B_{k-1}^{(n+2)}), \tag{3.117}$$

where $b_{k-1} \neq 1$ and $B_{k-1}^{(n)} = S_n$ we obtain $\lim_{n\to\infty} B_k^{(n)} = S$.

Theorem 3.16 *If the conditions of Theorem 3.15 are satisfied and if* $\lim_{n\to\infty} \frac{B_{k-1}^{(n+1)} - S}{B_{k-1}^{(n)} - S} = b_k$. *Then* $\{B_k^{(n)}\}$, *defined by* (3.91)-(3.96), *converges to* S *faster than* $\{B_{k-1}^{(n)}\}$, *defined by* (3.91)-(3.96), *when* n *tends to infinity, that is*

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=0. \tag{3.118}$$

Moreover if $b_k\neq 0$, $\{B_k^{(n)}\}$, *defined by* (3.91)-(3.96), *converges to* S *faster than* $\{B_{k-1}^{(n+1)}\}$, *defined by* (3.91)-(3.96).

Proof The proof for the acceleration methods (3.91)-(3.93) is exact as the proof for the iterated B-algorithm, (3.57), Theorem 3.11, Paragraph 3.1.2, we will not give it here.

We prove the first part of theorem for the acceleration method (3.94).

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=\frac{B_{k-1}^{(n)}-S-\frac{(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})(B_{k-1}^{(n+1)}-B_{k-1}^{(n)})}{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})-(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})}}{B_{k-1}^{(n)}-S}, \tag{3.119}$$

adding $S-S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n)}-S}-$$
$$-\frac{[(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n)}-S)][(B_{k-1}^{(n+1)}-S)-(B_{k-1}^{(n)}-S)]}{\{[(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+1)}-S)]-[(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n)}-S)]\}(B_{k-1}^{(n)}-S)}.$$

Multiplying with $\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+1)}-S}$ it follows that

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n)}-S}-\frac{\left[\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+1)}-S}-\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+1)}-S}\right]\left[1-\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+1)}-S}\right]}{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+1)}-S}-\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+1)}-S}-1+\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+1)}-S}\right]\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+1)}-S}}. \tag{3.120}$$

Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=1-\frac{\left(b_{k-1}-\frac{1}{b_{k-1}}\right)\left(1-\frac{1}{b_{k-1}}\right)}{\left(b_{k-1}^2-b_{k-1}-1+\frac{1}{b_{k-1}}\right)\frac{1}{b_{k-1}}}, \tag{3.121}$$

from where it follows

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=1-1=0. \tag{3.122}$$

The proof of the second part of the theorem for the acceleration method (3.94)

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=\frac{B_{k-1}^{(n)}-S-\frac{(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})(B_{k-1}^{(n+1)}-B_{k-1}^{(n)})}{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})-(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})}}{B_{k-1}^{(n+1)}-S}, \tag{3.123}$$

adding $S-S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = \frac{B_{k-1}^{(n)} - S}{B_{k-1}^{(n+1)} - S} -$$
$$- \frac{[(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n)} - S)][(B_{k-1}^{(n+1)} - S) - (B_{k-1}^{(n)} - S)]}{\{[(B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+1)} - S)] - [(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n+1)} - S)]\}(B_{k-1}^{(n+1)} - S)}.$$

Multiplying with $\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+1)}-S}$

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = \frac{B_{k-1}^{(n)} - S}{B_{k-1}^{(n+1)} - S} -$$
$$- \frac{\left[\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+1)}-S} - \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+1)}-S}\right]\left[1 - \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+1)}-S} - 1\right]}{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+1)}-S} - \frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+1)}-S} - 1 + \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+1)}-S}\right]\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+1)}-S}}.$$

Taking the limit when n tends to infinity

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = \frac{1}{b_{k-1}} - \frac{\left(b_{k-1} - \frac{1}{b_{k-1}}\right)\left(1 - \frac{1}{b_{k-1}}\right)}{\left(b_{k-1}^2 - b_{k-1} - 1 + \frac{1}{b_{k-1}}\right) \cdot 1}, \tag{3.124}$$

from where it results

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = \frac{1}{b_{k-1}} - \frac{1}{b_{k-1}} = 0. \tag{3.125}$$

Which completes this part of the proof.

We prove the first part of theorem for the acceleration method (3.95).

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = \frac{B_{k-1}^{(n+1)} - S - \frac{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})(B_{k-1}^{(n+1)} - B_{k-1}^{(n)})}{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})}}{B_{k-1}^{(n)} - S}, \tag{3.126}$$

adding $S - S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = \frac{B_{k-1}^{(n+1)} - S}{B_{k-1}^{(n)} - S} -$$
$$- \frac{[(B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+1)} - S)][(B_{k-1}^{(n+1)} - S) - (B_{k-1}^{(n)} - S)]}{\{[(B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+1)} - S)] - [(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n)} - S)]\}(B_{k-1}^{(n)} - S)}.$$

Multiplying with $\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+1)}-S}$ it follows that

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = \frac{B_{k-1}^{(n+1)} - S}{B_{k-1}^{(n)} - S} - \frac{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+1)}-S} - 1\right]\left[1 - \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+1)}-S}\right]}{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+1)}-S} - \frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+1)}-S} - 1 + \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+1)}-S}\right]\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+1)}-S}}. \tag{3.127}$$

Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = b_{k-1} - \frac{\left(b_{k-1}^2 - 1\right)\left(1 - \frac{1}{b_{k-1}}\right)}{\left(b_{k-1}^2 - b_{k-1} - 1 + \frac{1}{b_{k-1}}\right)\frac{1}{b_{k-1}}}, \tag{3.128}$$

from where it follows

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = b_{k-1} - b_{k-1} = 0. \tag{3.129}$$

The proof of the second part of the theorem for the acceleration method (3.95)

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = \frac{B_{k-1}^{(n+1)} - S - \frac{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})(B_{k-1}^{(n+1)} - B_{k-1}^{(n)})}{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})}}{B_{k-1}^{(n+1)} - S}, \tag{3.130}$$

adding $S - S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = \frac{B_{k-1}^{(n+1)} - S}{B_{k-1}^{(n+1)} - S} - \\ - \frac{[(B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+1)} - S)][(B_{k-1}^{(n+1)} - S) - (B_{k-1}^{(n)} - S)]}{\{[(B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+1)} - S)] - [(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n+1)} - S)]\}(B_{k-1}^{(n+1)} - S)}.$$

Multiplying with $\frac{B_{k-1}^{(n+1)} - S}{B_{k-1}^{(n+1)} - S}$

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = \frac{B_{k-1}^{(n+1)} - S}{B_{k-1}^{(n+1)} - S} - \\ - \frac{\left[\frac{B_{k-1}^{(n+3)} - S}{B_{k-1}^{(n+1)} - S} - 1\right]\left[1 - \frac{B_{k-1}^{(n)} - S}{B_{k-1}^{(n+1)} - S} - 1\right]}{\left[\frac{B_{k-1}^{(n+3)} - S}{B_{k-1}^{(n+1)} - S} - \frac{B_{k-1}^{(n+2)} - S}{B_{k-1}^{(n+1)} - S} - 1 + \frac{B_{k-1}^{(n)} - S}{B_{k-1}^{(n+1)} - S}\right]\frac{B_{k-1}^{(n+1)} - S}{B_{k-1}^{(n+1)} - S}}.$$

Taking the limit when n tends to infinity

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = 1 - \frac{\left(b_{k-1}^2 - 1\right)\left(1 - \frac{1}{b_{k-1}}\right)}{\left(b_{k-1}^2 - b_{k-1} - 1 + \frac{1}{b_{k-1}}\right)\cdot 1}, \tag{3.131}$$

from where it results

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = 1 - 1 = 0. \tag{3.132}$$

Which completes this part of the proof.

We prove the first part of theorem for the acceleration method (3.96).

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=\frac{B_{k-1}^{(n+2)}-S-\frac{(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})(B_{k-1}^{(n+3)}-B_{k-1}^{(n+2)})}{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})-(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})}}{B_{k-1}^{(n)}-S}, \tag{3.133}$$

adding $S-S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n)}-S}-$$
$$-\frac{[(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n)}-S)][(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+2)}-S)]}{\{[(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+1)}-S)]-[(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n)}-S)]\}(B_{k-1}^{(n)}-S)}.$$

Multiplying with $\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+1)}-S}$ it follows that

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n)}-S}-\frac{\left[\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+1)}-S}-\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+1)}-S}\right]\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+1)}-S}-\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+1)}-S}\right]}{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+1)}-S}-\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+1)}-S}-1+\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+1)}-S}\right]\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+1)}-S}}. \tag{3.134}$$

Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=b_{k-1}^2-\frac{\left(b_{k-1}-\frac{1}{b_{k-1}}\right)\left(b_{k-1}^2-b_{k-1}\right)}{\left(b_{k-1}^2-b_{k-1}-1+\frac{1}{b_{k-1}}\right)\frac{1}{b_{k-1}}}, \tag{3.135}$$

from where it follows

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=b_{k-1}^2-b_{k-1}^2=0. \tag{3.136}$$

The proof of the second part of the theorem for the acceleration method (3.96)

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=\frac{B_{k-1}^{(n+2)}-S-\frac{(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})(B_{k-1}^{(n+3)}-B_{k-1}^{(n+2)})}{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})-(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})}}{B_{k-1}^{(n+1)}-S}, \tag{3.137}$$

adding $S-S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+1)}-S}-$$
$$-\frac{[(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n)}-S)][(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+2)}-S)]}{\{[(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+1)}-S)]-[(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n+1)}-S)]\}(B_{k-1}^{(n+1)}-S)}.$$

Multiplying with $\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+1)}-S}$

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = \frac{B_{k-1}^{(n+2)} - S}{B_{k-1}^{(n+1)} - S} - \frac{\left[\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+1)}-S} - \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+1)}-S}\right]\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+1)}-S} - \frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+1)}-S}\right]}{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+1)}-S} - \frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+1)}-S} - 1 + \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+1)}-S}\right]\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+1)}-S}}.$$

Taking the limit when n tends to infinity

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = b_{k-1} - \frac{\left(b_{k-1} - \frac{1}{b_{k-1}}\right)\left(b_{k-1}^2 - b_{k-1}\right)}{\left(b_{k-1}^2 - b_{k-1} - 1 + \frac{1}{b_{k-1}}\right)\cdot 1}, \tag{3.138}$$

from where it results

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = b_{k-1} - b_{k-1} = 0. \tag{3.139}$$

In the sequel we present a numerical example, which will help us to illustrate the efficiency and the performance of the iterate representations, (3.91)-(3.96), for the iterated B-algorithm.

Example 3.22 *(Bumbariu)[16] Test function $f(x) = x - 3^{-x} = 0$, which has a unique root $x^* = 0.54780862165409744645\ldots$. This equation can be rewritten into a fixed point problem by $g(x) = 3^{-x}$. To apply the iterated representations for the iterated B-algorithm we shall take the initial guess $x_0 \in \{0.3, 0.4, 0.5, 0.6\}$. The results for Example 3.22, starting from the initial guess $x_0 = 0.5$, are listed in Table 4.*

Table 4

$B_k^{(n)}$	$B_k^{(n)}(3.91)$	$B_k^{(n)}(3.92)$	$B_k^{(n)}(3.93)$	$B_k^{(n)}(3.94)$	$B_k^{(n)}(3.95)$	$B_k^{(n)}(3.96)$	$B_k^{(n)}(3.57)$
$B_0^{(0)}$	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$B_1^{(0)}$	3	3	3	3	3	3	3
$B_2^{(0)}$	6	6	6	6	6	6	6
$B_3^{(0)}$	10	10	10	10	10	10	10
$B_4^{(0)}$	15	15	15	15	15	15	15
$B_5^{(0)}$	20	20	20	20	20	20	20
$B_6^{(0)}$	20	27	27	27	27	27	27
$B_7^{(0)}$	21	32	32	32	32	32	32
$B_8^{(0)}$	19	37	37	37	37	37	37

Table 4 shows that the iterated representations are given the same number of exact digits as the iterated B-algorithm, excepting the iterated representation (3.91), which gives, in all cases, less number of exact digits. We have done tests on twenty test functions, the results were similar with that in case of Example 3.22. In the table are listed the exact number of decimals at the first eight steps, all the numerical computations are done with Maple 13, using 39 digits arithmetic.

3.1.4 Some extensions for the iterated B-algorithm for accelerating the Picard iteration

Author's original result in this paragraph are: some extensions for the iterated B-algorithm, Theorem 3.17, Theorem 3.18, Observation 3.4 and Example 3.23.
Motivated by the efficiency and the performance of iterated B-algorithm we were able to give some extensions for it. The new acceleration methods are given by:

$$B_{k+1,1}^{(n)} = \frac{[\overline{\Delta} B_k^{(n+1)}] B_k^{(n+1)} - [\overline{\Delta} B_k^{(n)}] B_k^{(n+2)}}{\overline{\Delta} B_k^{(n+1)} - \overline{\Delta} B_k^{(n)}}, \quad n, k \in \mathbb{N} \tag{3.140}$$

$$B_{k+1,2}^{(n)} = B_k^{(n+2)} - \frac{[\overline{\Delta} B_k^{(n+1)}][\Delta B_k^{(n+1)}]}{\overline{\Delta} B_k^{(n+1)} - \overline{\Delta} B_k^{(n)}}, \quad n, k \in \mathbb{N} \tag{3.141}$$

$$B_{k+1,3}^{(n)} = B_k^{(n+1)} - \frac{[\overline{\Delta} B_k^{(n)}][\Delta B_k^{(n+1)}]}{\overline{\Delta} B_k^{(n+1)} - \overline{\Delta} B_k^{(n)}}, \quad n, k \in \mathbb{N} \tag{3.142}$$

$$B_{k+1,4}^{(n)} = B_k^{(n+3)} - \frac{[\overline{\Delta} B_k^{(n+1)}][\Delta B_k^{(n+2)}][\overline{\Delta} B_k^{(n+1)} - \overline{\Delta} B_k^{(n)}]}{[\overline{\Delta} B_k^{(n+1)}][\Delta^2 B_k^{(n+1)}] - [\overline{\Delta} B_k^{(n)}][\Delta^2 B_k^{(n)}]}, \quad n, k \in \mathbb{N} \tag{3.143}$$

$$B_{k+1,5}^{(n)} = B_k^{(n+3)} + \frac{[\overline{\Delta} B_k^{(n+2)} - \overline{\Delta} B_k^{(n+1)}][\overline{\Delta} B_k^{(n+2)}][\overline{\Delta} B_k^{(n)}]}{[\overline{\Delta} B_k^{(n)}][\overline{\Delta} B_k^{(n+1)} - \overline{\Delta} B_k^{(n)}] - [\overline{\Delta} B_k^{(n)}][\overline{\Delta} B_k^{(n+2)} - \overline{\Delta} B_k^{(n+1)}]}, \quad n, k \in \mathbb{N} \tag{3.144}$$

where Δ and $\overline{\Delta}$ act like in case of iterated B-algorithm.

Remark 3.4 *(Bumbariu) The first three extensions,* (3.140)-(3.142)*, for iterated B-algorithm are equivalent.*

In the sequel we will give conditions on the $B_i^{(n)}$'s which insure that $\lim_{n\to\infty} B_k^{(n)} = S$, where $B_k^{(n)}$ is defined by the relations (3.140)-(3.144).

Theorem 3.17 *(Bumbariu, [38]) If $\lim_{n\to\infty} S_n = S$, and if $\forall i, \exists b_i \neq 1$ such that $\lim_{n\to\infty} \frac{B_i^{(n+1)}}{B_i^{(n)}} = b_i$, for $\forall i \neq j$, $b_i \neq b_j$. Then $\forall k$ we have $\lim_{n\to\infty} B_k^{(n)} = S$, where $B_k^{(n)}$ is defined by the relations* (3.140)-(3.144).

Proof We prove that $\lim_{n\to\infty} B_k^{(n)} = S$, where $B_k^{(n)}$ is defined by the relation (3.140).
For $k = 0$ we have

$$B_1^{(n)} = \frac{(B_0^{(n+3)} - B_0^{(n+1)}) B_0^{(n+1)} - B_0^{(n+2)} (B_0^{(n+2)} - B_0^{(n)})}{(B_0^{(n+3)} - B_0^{(n+1)}) - (B_0^{(n+2)} - B_0^{(n)})},$$

where $B_0^{(n)} = S_n$.

Adding and decreasing $B_0^{(n+1)}(B_0^{(n+2)} - B_0^{(n)})$ we obtain

$$B_1^{(n)} = B_0^{(n+1)} - \frac{(B_0^{(n+2)} - B_0^{(n)})(B_0^{(n+2)} - B_0^{(n+1)})}{(B_0)^{(n+3)} - B_0^{(n+1)}) - (B_0^{(n+2)} - B_0^{(n)})}.$$

Multiplying with $\frac{B_0^{(n+1)}}{B_0^{(n+1)}}$ we get

$$B_1^{(n)} = B_0^{(n+1)} - \frac{\frac{B_0^{(n+2)}}{B_0^{(n+1)}} - \frac{B_0^{(n)}}{B_0^{(n+1)}}}{\frac{B_0^{(n+3)}}{B_0^{(n+1)}} - 1 - \frac{B_0^{(n+2)}}{B_0^{(n+1)}} + \frac{B_0^{(n)}}{B_0^{(n+1)}}}(B_0^{(n+2)} - B_0^{(n+1)}). \tag{3.145}$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n+1)} - \frac{b_0 - \frac{1}{b_0}}{b_0^2 - 1 - b_0 + \frac{1}{b_0}} \lim_{n\to\infty}(B_0^{(n+2)} - B_0^{(n+1)}), \tag{3.146}$$

from where it results

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n+1)} - \frac{1}{b_0 - 1} \lim_{n\to\infty}(B_0^{(n+2)} - B_0^{(n+1)}), \tag{3.147}$$

where $b_0 \neq 1$ and $B_0^{(n)} = S_n$. Hence

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} S_{n+1} - \frac{1}{b_0 - 1} \lim_{n\to\infty}(S_{n+2} - S_{n+1}), \tag{3.148}$$

because $\lim_{n\to\infty} S_n = S$ we obtain $\lim_{n\to\infty} B_1^{(n)} = S$.

Suppose that $\lim_{n\to\infty} B_{k-1}^{(n)} = S$ we prove that $\lim_{n\to\infty} B_k^{(n)} = S$.

$$\begin{aligned} B_k^{(n)} &= \frac{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})B_{k-1}^{(n+1)} - B_{k-1}^{(n+2)}(B_{k-1}^{(n+2)} - B_{k-1}^{(n)})}{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})} \\ &= B_{k-1}^{(n+1)} - \frac{(B_{k-1}^{(n+2)} - B_{k-1}^{(n)})(B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)})}{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})}. \end{aligned}$$

Multiplying with $\frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+1)}}$ we obtain

$$B_k^{(n)} = B_{k-1}^{(n+1)} - \frac{\frac{B_{k-1}^{(n+2)}}{B_{k-1}^{(n+1)}} - \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+1)}}}{\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+1)}} - 1 - \frac{B_{k-1}^{(n+2)}}{B_{k-1}^{(n+1)}} + \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+1)}}}(B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)}). \tag{3.149}$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+1)} - \frac{b_{k-1} - \frac{1}{b_{k-1}}}{b_{k-1}^2 - 1 - b_{k-1} + \frac{1}{b_{k-1}}} \lim_{n\to\infty}(B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)}), \tag{3.150}$$

hence

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+1)} - (b_{k-1} - 1) \lim_{n\to\infty}(B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)}), \tag{3.151}$$

where $b_{k-1} \neq 1$ and $B_{k-1}^{(n)} = S_n$ we obtain $\lim_{n\to\infty} B_k^{(n)} = S$.

We prove that $\lim_{n\to\infty} B_k^{(n)} = S$, where $B_k^{(n)}$ is defined by the relation (3.141). For $k = 0$ we have

$$B_1^{(n)} = B_0^{(n+2)} - \frac{(B_0^{(n+3)} - B_0^{(n+1)})(B_0^{(n+2)} - B_0^{(n+1)})}{(B_0^{(n+3)} - B_0^{(n+1)}) - (B_0^{(n+2)} - B_0^{(n)})},$$

where $B_0^{(n)} = S_n$.

Multiplying with $\frac{B_0^{(n+2)}}{B_0^{(n+2)}}$ we get

$$B_1^{(n)} = B_0^{(n+2)} - \frac{\frac{B_0^{(n+3)}}{B_0^{(n+2)}} - \frac{B_0^{(n+1)}}{B_0^{(n+2)}}}{\frac{B_0^{(n+3)}}{B_0^{(n+1)}} - \frac{B_0^{(n+1)}}{B_0^{(n+2)}} - 1 + \frac{B_0^{(n)}}{B_0^{(n+1)}}}(B_0^{(n+2)} - B_0^{(n+1)}). \tag{3.152}$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n+2)} - \frac{b_0 - \frac{1}{b_0}}{b_0 - \frac{1}{b_0} - 1 + \frac{1}{b_0^2}} \lim_{n\to\infty}(B_0^{(n+2)} - B_0^{(n+1)}), \tag{3.153}$$

from where it results

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n+2)} - \frac{b_0}{b_0 - 1} \lim_{n\to\infty}(B_0^{(n+2)} - B_0^{(n+1)}), \tag{3.154}$$

where $b_0 \neq 1$ and $B_0^{(n)} = S_n$. Hence

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} S_{n+2} - \frac{b_0}{b_0 - 1} \lim_{n\to\infty}(S_{n+2} - S_{n+1}), \tag{3.155}$$

because $\lim_{n\to\infty} S_n = S$ we obtain $\lim_{n\to\infty} B_1^{(n)} = S$.

Suppose that $\lim_{n\to\infty} B_{k-1}^{(n)} = S$ we prove that $\lim_{n\to\infty} B_k^{(n)} = S$.

$$B_k^{(n)} = B_{k-1}^{(n+2)} - \frac{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})}{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})}(B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)}). \tag{3.156}$$

Multiplying with $\frac{B_{k-1}^{(n+2)}}{B_{k-1}^{(n+2)}}$ we obtain

$$B_k^{(n)} = B_{k-1}^{(n+2)} - \frac{\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+2)}} - \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}}{\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+2)}} - \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}} - 1 + \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+2)}}}(B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)}). \tag{3.157}$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+2)} - \frac{b_{k-1} - \frac{1}{b_{k-1}}}{b_{k-1} - \frac{1}{b_{k-1}} - 1 + \frac{1}{b_{k-1}^2}} \lim_{n\to\infty}(B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)}), \tag{3.158}$$

hence

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+2)} - \frac{b_{k-1}}{b_{k-1}-1} \lim_{n\to\infty} (B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)}), \tag{3.159}$$

where $b_{k-1} \neq 1$ and $B_{k-1}^{(n)} = S_n$ we obtain $\lim\limits_{n\to\infty} B_k^{(n)} = S$.

We prove that $\lim\limits_{n\to\infty} B_k^{(n)} = S$, where $B_k^{(n)}$ is defined by the relation (3.142). For $k = 0$ we have

$$B_1^{(n)} = B_0^{(n+1)} - \frac{(B_0^{(n+2)} - B_0^{(n)})(B_0^{(n+2)} - B_0^{(n+1)})}{(B_0^{(n+3)} - B_0^{(n+1)}) - (B_0^{(n+2)} - B_0^{(n)})},$$

where $B_0^{(n)} = S_n$.

Multiplying with $\frac{B_0^{(n+2)}}{B_0^{(n+2)}}$ we get

$$B_1^{(n)} = B_0^{(n+1)} - \frac{1 - \frac{B_0^{(n)}}{B_0^{(n+2)}}}{\frac{B_0^{(n+3)}}{B_0^{(n+2)}} - \frac{B_0^{(n+1)}}{B_0^{(n+2)}} - 1 + \frac{B_0^{(n)}}{B_0^{(n+2)}}} (B_0^{(n+2)} - B_0^{(n+1)}). \tag{3.160}$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n+1)} - \frac{1 - \frac{1}{b_0^2}}{b_0 - \frac{1}{b_0} - 1 + \frac{1}{b_0^2}} \lim_{n\to\infty} (B_0^{(n+2)} - B_0^{(n+1)}), \tag{3.161}$$

from where it results

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n+1)} - \frac{1}{b_0 - 1} \lim_{n\to\infty} (B_0^{(n+2)} - B_0^{(n+1)}), \tag{3.162}$$

where $b_0 \neq 1$ and $B_0^{(n)} = S_n$. Hence

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} S_{n+1} - \frac{1}{b_0 - 1} \lim_{n\to\infty} (S_{n+2} - S_{n+1}), \tag{3.163}$$

because $\lim\limits_{n\to\infty} S_n = S$ we obtain $\lim\limits_{n\to\infty} B_1^{(n)} = S$.

Suppose that $\lim\limits_{n\to\infty} B_{k-1}^{(n)} = S$ we prove that $\lim\limits_{n\to\infty} B_k^{(n)} = S$.

$$B_k^{(n)} = B_{k-1}^{(n+1)} - \frac{(B_{k-1}^{(n+2)} - B_{k-1}^{(n)})}{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})} (B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)}). \tag{3.164}$$

Multiplying with $\frac{B_{k-1}^{(n+2)}}{B_{k-1}^{(n+2)}}$ we obtain

$$B_k^{(n)} = B_{k-1}^{(n+1)} - \frac{1 - \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+2)}}}{\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+2)}} - \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}} - 1 + \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+2)}}} (B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)}). \tag{3.165}$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+1)} - \frac{1-\frac{1}{b_{k-1}^2}}{b_{k-1}-\frac{1}{b_{k-1}}-1+\frac{1}{b_{k-1}^2}} \lim_{n\to\infty}(B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)}), \tag{3.166}$$

hence

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+1)} - \frac{1}{b_{k-1}-1} \lim_{n\to\infty}(B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)}), \tag{3.167}$$

where $b_{k-1} \neq 1$ and $B_{k-1}^{(n)} = S_n$ we obtain $\lim\limits_{n\to\infty} B_k^{(n)} = S$.

We prove that $\lim\limits_{n\to\infty} B_k^{(n)} = S$, where $B_k^{(n)}$ is defined by the relation (3.143). For $k = 0$ we have

$$B_1^{(n)} = B_0^{(n+3)} - \frac{M_1}{N_1}$$

where $B_0^{(n)} = S_n$ and we denote by

$$M_1 = (B_0^{(n+3)} - B_0^{(n+1)})(B_0^{(n+3)} - B_0^{(n+2)})[(B_0^{(n+3)} - B_0^{(n+1)}) - (B_0^{(n+2)} - B_0^{(n)})],$$

$$N_1 = (B_0^{(n+3)}-B_0^{(n+1)})(B_0^{(n+3)}-2B_0^{(n+2)}+B_0^{(n+1)})-(B_0^{(n+2)}-B_0^{(n)})(B_0^{(n+2)}-2B_0^{(n+1)}+B_0^{(n)}).$$

Multiplying with $\frac{B_0^{(n+2)}}{B_0^{(n+2)}}$ we get

$$B_1^{(n)} = B_0^{(n+3)} - \frac{\left(\frac{B_0^{(n+3)}}{B_0^{(n+2)}} - \frac{B_0^{(n+1)}}{B_0^{(n+2)}}\right)\left(\frac{B_0^{(n+3)}}{B_0^{(n+2)}} - \frac{B_0^{(n+1)}}{B_0^{(n+2)}} - 1 + \frac{B_0^{(n)}}{B_0^{(n+2)}}\right)(B_0^{(n+3)} - B_0^{(n+2)})}{\left(\frac{B_0^{(n+3)}}{B_0^{(n+2)}} - \frac{B_0^{(n+1)}}{B_0^{(n+2)}}\right)\left(\frac{B_0^{(n+3)}}{B_0^{(n+2)}} - 2 + \frac{B_0^{(n+1)}}{B_0^{(n+2)}}\right) - \left(1 - \frac{B_0^{(n)}}{B_0^{(n+2)}}\right)\left(1 - 2\frac{B_0^{(n+1)}}{B_0^{(n+2)}} + \frac{B_0^{(n)}}{B_0^{(n+2)}}\right)}.$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n+3)} - \frac{\left(b_0 - \frac{1}{b_0}\right)\left(b_0 - \frac{1}{b_0} - 1 + \frac{1}{b_0^2}\right)}{\left(b_0 - \frac{1}{b_0}\right)\left(b_0 - 2 + \frac{1}{b_0}\right) - \left(1 - \frac{1}{b_0^2}\right)\left(1 - \frac{2}{b_0} + \frac{1}{b_0^2}\right)} \lim_{n\to\infty}(B_0^{(n+3)} - B_0^{(n+2)}),$$

from where it results

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} B_0^{(n+3)} - \frac{b_0}{b_0-1} \lim_{n\to\infty}(B_0^{(n+3)} - B_0^{(n+2)}), \tag{3.168}$$

where $b_0 \neq 1$ and $B_0^{(n)} = S_n$. Hence

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} S_{n+3} - \frac{b_0}{b_0-1} \lim_{n\to\infty}(S_{n+3} - S_{n+2}), \tag{3.169}$$

because $\lim_{n\to\infty} S_n = S$ we obtain $\lim_{n\to\infty} B_1^{(n)} = S$.
Suppose that $\lim_{n\to\infty} B_{k-1}^{(n)} = S$ we prove that $\lim_{n\to\infty} B_k^{(n)} = S$.

$$B_k^{(n)} = B_{k-1}^{(n+3)} - \frac{M_2}{N_2},$$

where

$$M_2 = (B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})(B_{k-1}^{(n+3)} - B_{k-1}^{(n+2)})[(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})],$$

$$N_2 = (B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})(B_{k-1}^{(n+3)} - 2B_{k-1}^{(n+2)} + B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})(B_{k-1}^{(n+2)} - 2B_{k-1}^{(n+1)} + B_{k-1}^{(n)}).$$

Multiplying with $\frac{B_{k-1}^{(n+2)}}{B_{k-1}^{(n+2)}}$ we obtain

$$B_k^{(n)} = B_{k-1}^{(n+3)} - $$
$$- \frac{\left(\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+2)}} - \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}\right)\left(\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+2)}} - \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}} - 1 + \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+2)}}\right)(B_{k-1}^{(n+3)} - B_{k-1}^{(n+2)})}{\left(\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+2)}} - \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}\right)\left(\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+2)}} - 2 + \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}\right) - \left(1 - \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+2)}}\right)\left(1 - 2\frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}} + \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+2)}}\right)}.$$

Taking the limit when n tends to infinity we obtain

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+3)} - $$
$$- \frac{\left(b_{k-1} - \frac{1}{b_{k-1}}\right)\left(b_{k-1} - \frac{1}{b_{k-1}} - 1 + \frac{1}{b_{k-1}^2}\right)}{\left(b_{k-1} - \frac{1}{b_{k-1}}\right)\left(b_{k-1} - 2 + \frac{1}{b_{k-1}}\right) - \left(1 - \frac{1}{b_{k-1}^2}\right)\left(1 - \frac{2}{b_{k-1}} + \frac{1}{b_{k-1}^2}\right)}.$$
$$\cdot \lim_{n\to\infty}(B_0^{(n+3)} - B_0^{(n+2)}),$$

hence there

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+3)} - \frac{b_{k-1}}{b_{k-1} - 1} \lim_{n\to\infty}(B_{k-1}^{(n+3)} - B_{k-1}^{(n+2)}), \tag{3.170}$$

where $b_{k-1} \neq 1$ and $B_{k-1}^{(n)} = S_n$ we obtain $\lim_{n\to\infty} B_k^{(n)} = S$.

We prove that $\lim_{n\to\infty} B_k^{(n)} = S$, where $B_k^{(n)}$ is defined by the relation (3.144).
For $k = 0$ we have

$$B_1^{(n)} = B_0^{(n+3)} + \frac{M_3}{N_3},$$

where $B_0^{(n)} = S_n$ and we denote by

$$M_3 = (B_0^{(n+4)} - B_0^{(n+2)})(B_0^{(n+2)} - B_0^{(n)})[(B_0^{(n+4)} - B_0^{(n+2)}) - (B_0^{(n+3)} - B_0^{(n+1)})],$$

$$N_3 = (B_0^{(n+2)} - B_0^{(n)})[(B_0^{(n+3)} - B_0^{(n+1)}) - (B_0^{(n+2)} - B_0^{(n)})] - (B_0^{(n+3)} - B_0^{(n+1)})[(B_0^{(n+4)} - B_0^{(n+2)}) -$$

$$-(B_0^{(n+3)} - B_0^{(n+1)})].$$

Grouping $(B_0^{(n+4)}-B_0^{(n+3)})$ and $(B_0^{(n+2)}-B_0^{(n+1)})$ and separating into two fractions we obtain

$$\begin{aligned}B_1^{(n)} &= B_0^{(n+3)}+\\ &+\frac{(B_0^{(n+4)}-B_0^{(n+2)})(B_0^{(n+2)}-B_0^{(n)})(B_0^{(n+4)}-B_0^{(n+3)})}{M}-\\ &-\frac{(B_0^{(n+4)}-B_0^{(n+2)})(B_0^{(n+2)}-B_0^{(n)})(B_0^{(n+2)}-B_0^{(n+1)})}{M},\end{aligned}$$

where
$M=(B_0^{(n+2)}-B_0^{(n)})[(B_0^{(n+3)}-B_0^{(n+1)})-(B_0^{(n+2)}-B_0^{(n)})]-(B_0^{(n+3)}-B_0^{(n+1)})[(B_0^{(n+4)}-B_0^{(n+2)})-(B_0^{(n+3)}-B_0^{(n+1)})]$.

Multiplying with $\frac{B_0^{(n+2)}}{B_0^{(n+2)}}$ we get

$$\begin{aligned}B_1^{(n)} &= B_0^{(n+3)}+\\ &+\frac{\left(1-\frac{B_0^{(n)}}{B_0^{(n+2)}}\right)\left(\frac{B_0^{(n+4)}}{B_0^{(n+2)}}-1\right)\left(B_0^{(n+4)}-B_0^{(n+3)}\right)}{N}-\\ &-\frac{\left(1-\frac{B_0^{(n)}}{B_0^{(n+2)}}\right)\left(\frac{B_0^{(n+4)}}{B_0^{(n+2)}}-1\right)\left(B_0^{(n+2)}-B_0^{(n+1)}\right)}{N},\end{aligned}$$

where
$N=\left(1-\frac{B_0^{(n)}}{B_0^{(n+2)}}\right)\left[\left(\frac{B_0^{(n+3)}}{B_0^{(n+2)}}-\frac{B_0^{(n+1)}}{B_0^{(n+2)}}\right)-\left(1-\frac{B_0^{(n+1)}}{B_0^{(n+2)}}\right)\right]-\left(\frac{B_0^{(n+3)}}{B_0^{(n+2)}}-\frac{B_0^{(n+1)}}{B_0^{(n+2)}}\right)\cdot \left[\left(\frac{B_0^{(n+4)}}{B_0^{(n+2)}}-1\right)-\left(\frac{B_0^{(n+3)}}{B_0^{(n+2)}}-\frac{B_0^{(n+1)}}{B_0^{(n+2)}}\right)\right]$.

Taking the limit when n tends to infinity we obtain

$$\begin{aligned}\lim_{n\to\infty}B_1^{(n)} &= \lim_{n\to\infty}B_0^{(n+3)}+\\ &+\frac{\left(1-\frac{1}{b_0^2}\right)(b_0^2-1)}{\left(1-\frac{1}{b_0^2}\right)\left(b_0-\frac{1}{b_0}-1+\frac{1}{b_0^2}\right)-\left(b_0-\frac{1}{b_0}\right)\left(b_0^2-1-b_0+\frac{1}{b_0}\right)}\lim_{n\to\infty}(B_0^{(n+4)}-B_0^{(n+3)})-\\ &-\frac{\left(1-\frac{1}{b_0^2}\right)(b_0^2-1)}{\left(1-\frac{1}{b_0^2}\right)\left(b_0-\frac{1}{b_0}-1+\frac{1}{b_0^2}\right)-\left(b_0-\frac{1}{b_0}\right)\left(b_0^2-1-b_0+\frac{1}{b_0}\right)}\lim_{n\to\infty}(B_0^{(n+2)}-B_0^{(n+1)}),\end{aligned}$$

from where it results

$$\begin{aligned}\lim_{n\to\infty}B_1^{(n)} &= \lim_{n\to\infty}B_0^{(n+3)}+\frac{b_0(b_0+1)}{b_0-1}\lim_{n\to\infty}(B_0^{(n+4)}-B_0^{(n+3)})-\frac{b_0(b_0+1)}{b_0-1}\cdot\\ &\cdot\lim_{n\to\infty}(B_0^{(n+2)}-B_0^{(n+1)}),\end{aligned}$$

where $b_0 \neq 1$ and $B_0^{(n)} = S_n$. Hence

$$\lim_{n\to\infty} B_1^{(n)} = \lim_{n\to\infty} S_{n+3} + \frac{b_0(b_0+1)}{b_0-1}\lim_{n\to\infty}(S_{n+4}-S_{n+3}) - \frac{b_0(b_0+1)}{b_0-1}\lim_{n\to\infty}(S_{n+2}-S_{n+1}), \tag{3.171}$$

because $\lim_{n\to\infty} S_n = S$ we obtain $\lim_{n\to\infty} B_1^{(n)} = S$.

Suppose that $\lim_{n\to\infty} B_{k-1}^{(n)} = S$ we prove that $\lim_{n\to\infty} B_k^{(n)} = S$.

$$\begin{aligned} B_k^{(n)} &= B_{k-1}^{(n+3)} + \\ &+ \frac{(B_{k-1}^{(n+4)} - B_{k-1}^{(n+2)})(B_{k-1}^{(n+2)} - B_{k-1}^{(n)})(B_{k-1}^{(n+4)} - B_{k-1}^{(n+3)})}{P} - \\ &- \frac{(B_{k-1}^{(n+4)} - B_{k-1}^{(n+2)})(B_{k-1}^{(n+2)} - B_{k-1}^{(n)})(B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)})}{P}, \end{aligned}$$

where
$P = (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})[(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})] - (B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})\cdot$
$\cdot[(B_{k-1}^{(n+4)} - B_{k-1}^{(n+2)}) - (B_{k-1}^{(n+3)} + B_{k-1}^{(n+1)})]$.

Multiplying with $\frac{B_{k-1}^{(n+2)}}{B_{k-1}^{(n+2)}}$ we obtain

$$\begin{aligned} B_k^{(n)} &= B_{k-1}^{(n+3)} + \\ &+ \frac{\left(1 - \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+2)}}\right)\left(\frac{B_{k-1}^{(n+4)}}{B_{k-1}^{(n+2)}} - 1\right)\left(B_{k-1}^{(n+4)} - B_{k-1}^{(n+3)}\right)}{Q} - \\ &- \frac{\left(1 - \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+2)}}\right)\left(\frac{B_{k-1}^{(n+4)}}{B_{k-1}^{(n+2)}} - 1\right)\left(B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)}\right)}{Q}, \end{aligned}$$

where
$Q = \left(1 - \frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+2)}}\right)\left[\left(\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+2)}} - \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}\right) - \left(1 - \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}\right)\right] - \left(\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+2)}} - \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}\right)\cdot$
$\cdot\left[\left(\frac{B_{k-1}^{(n+4)}}{B_{k-1}^{(n+2)}} - 1\right) - \left(\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+2)}} - \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}\right)\right]$.

Taking the limit when n tends to infinity we obtain

$$\begin{aligned} &\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+3)} + \\ &+ \frac{\left(1 - \frac{1}{b_{k-1}^2}\right)\left(b_{k-1}^2 - 1\right)}{\left(1 - \frac{1}{b_{k-1}^2}\right)\left(b_{k-1} - \frac{1}{b_{k-1}} - 1 + \frac{1}{b_{k-1}^2}\right) - \left(b_{k-1} - \frac{1}{b_{k-1}}\right)\left(b_{k-1}^2 - 1 - b_{k-1} + \frac{1}{b_{k-1}}\right)}\cdot \\ &\cdot \lim_{n\to\infty}(B_{k-1}^{(n+4)} - B_{k-1}^{(n+3)}) - \\ &- \frac{\left(1 - \frac{1}{b_{k-1}^2}\right)\left(b_{k-1}^2 - 1\right)}{\left(1 - \frac{1}{b_{k-1}^2}\right)\left(b_{k-1} - \frac{1}{b_{k-1}} - 1 + \frac{1}{b_{k-1}^2}\right) - \left(b_{k-1} - \frac{1}{b_{k-1}}\right)\left(b_{k-1}^2 - 1 - b_{k-1} + \frac{1}{b_{k-1}}\right)}\cdot \\ &\cdot \lim_{n\to\infty}(B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)}), \end{aligned}$$

hence

$$\lim_{n\to\infty} B_k^{(n)} = \lim_{n\to\infty} B_{k-1}^{(n+3)} + \frac{b_{k-1}(b_{k-1}+1)}{b_{k-1}-1} \lim_{n\to\infty}[B_{k-1}^{(n+4)} - B_{k-1}(n+3)] - \\ - \frac{b_{k-1}(b_{k-1}+1)}{b_{k-1}-1} \lim_{n\to\infty}[B_{k-1}^{(n+2)} - B_{k-1}(n+1)],$$

where $b_{k-1} \neq 1$ and $B_{k-1}^{(n)} = S_n$ we obtain $\lim_{n\to\infty} B_k^{(n)} = S$.

In what follows we give convergence results for the acceleration methods (3.140)-(3.144).

Theorem 3.18 *(Bumbariu, [38]) If the conditions of Theorem 3.17 are satisfied and if* $\lim_{n\to\infty} \frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n)}-S} = b_k$*. Then* $\{B_k^{(n)}\}$*, defined by the relations* (3.140)-(3.144)*, converges to* S *faster than* $\{B_{k-1}^{(n)}\}$*, defined by the relations* (3.140)-(3.144)*, when* n *tends to infinity, that is*

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = 0. \tag{3.172}$$

Moreover if $b_k \neq 0$, $\{B_k^{(n)}\}$*, defined by the relations* (3.140)-(3.144)*, converges to* S *faster than* $\{B_{k-1}^{(n+1)}\}$*, defined by the relations* (3.140)-(3.144).

Proof We prove the first part of theorem for each method (3.140)-(3.144).

The first part of the proof for the acceleration technique (3.140).

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = \frac{B_{k-1}^{(n+1)} - S - \frac{(B_{k-1}^{(n+2)} - B_{k-1}^{(n)})(B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)})}{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})}}{B_{k-1}^{(n)} - S}, \tag{3.173}$$

adding $S - S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = \frac{B_{k-1}^{(n+1)} - S}{B_{k-1}^{(n)} - S} - \\ - \frac{[(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n)} - S)][(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n+1)} - S)]}{\{[(B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+1)} - S)] - [(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n)} - S)]\}(B_{k-1}^{(n)} - S)}.$$

Multiplying with $\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+2)}-S}$ it follows that

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = \frac{B_{k-1}^{(n+1)} - S}{B_{k-1}^{(n+2)} - S} \cdot \frac{B_{k-1}^{(n+2)} - S}{B_{k-1}^{(n)} - S} - \frac{\left(1 - \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right)\left(1 - \frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}\right)}{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S} - \frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S} - 1 + \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right] \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}}. \tag{3.174}$$

Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = \frac{b_{k-1}^2}{b_{k-1}} - \frac{\left(1 - \frac{1}{b_{k-1}^2}\right)\left(1 - \frac{1}{b_{k-1}}\right)}{\left(b_{k-1} - \frac{1}{b_{k-1}} - 1 + \frac{1}{b_{k-1}^2}\right)\frac{1}{b_{k-1}^2}}, \tag{3.175}$$

from where it follows

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = b_{k-1} - b_{k-1} = 0. \tag{3.176}$$

The first part of the proof for the acceleration technique (3.141).

$$\begin{aligned}\frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = & \frac{B_{k-1}^{(n+2)} - S}{B_{k-1}^{(n)} - S} - \\ & - \frac{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})(B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)})}{[(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})](B_{k-1}^{(n)} - S)},\end{aligned}$$

adding $S - S$ in each parenthesis and separating into two fractions we obtain

$$\begin{aligned}\frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = & \frac{B_{k-1}^{(n+2)} - S}{B_{k-1}^{(n)} - S} - \\ & - \frac{[(B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+1)} - S)][(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n+1)} - S)]}{\{[(B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+1)} - S)] - [(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n)} - S)]\}(B_{k-1}^{(n)} - S)}.\end{aligned}$$

Multiplying with $\frac{B_{k-1}^{(n+1)} - S}{B_{k-1}^{(n+1)} - S}$ it follows that

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = \frac{B_{k-1}^{(n+2)} - S}{B_{k-1}^{(n+1)} - S} \cdot \frac{B_{k-1}^{(n+1)} - S}{B_{k-1}^{(n)} - S} - \frac{\left(\frac{B_{k-1}^{(n+3)} - S}{B_{k-1}^{(n+1)} - S} - 1\right)\left(\frac{B_{k-1}^{(n+2)} - S}{B_{k-1}^{(n+1)} - S} - 1\right)}{\left[\frac{B_{k-1}^{(n+3)} - S}{B_{k-1}^{(n+1)} - S} - 1 - \frac{B_{k-1}^{(n+2)} - S}{B_{k-1}^{(n+1)} - S} + \frac{B_{k-1}^{(n)} - S}{B_{k-1}^{(n+1)} - S}\right]\frac{B_{k-1}^{(n)} - S}{B_{k-1}^{(n+1)} - S}}. \tag{3.177}$$

Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = b_{k-1}^2 - \frac{\left(b_{k-1}^2 - 1\right)(b_{k-1} - 1)}{\left(b_{k-1}^2 - 1 + \frac{1}{b_{k-1}} - b_{k-1}\right)\frac{1}{b_{k-1}}}, \tag{3.178}$$

from where it follows

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = b_{k-1}^2 - b_{k-1}^2 = 0. \tag{3.179}$$

The first part of the proof for the acceleration technique (3.142).

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=\frac{B_{k-1}^{(n+1)}-S-\frac{(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})(B_{k-1}^{(n+2)}-B_{k-1}^{(n+1)})}{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})-(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})}}{B_{k-1}^{(n)}-S}, \tag{3.180}$$

adding $S-S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n)}-S}-$$
$$-\frac{[(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n)}-S)][(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n+1)}-S)]}{\{[(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+1)}-S)]-[(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n)}-S)]\}(B_{k-1}^{(n)}-S)}.$$

Multiplying with $\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+2)}-S}$ it follows that

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}\cdot\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n)}-S}-\frac{\left(1-\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right)\left(1-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}\right)}{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}-1+\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right]\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}}. \tag{3.181}$$

Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=\frac{b_{k-1}^2}{b_{k-1}}-\frac{\left(1-\frac{1}{b_{k-1}^2}\right)\left(1-\frac{1}{b_{k-1}}\right)}{\left(b_{k-1}-\frac{1}{b_{k-1}}-1+\frac{1}{b_{k-1}^2}\right)\frac{1}{b_{k-1}^2}}, \tag{3.182}$$

from where it follows

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=b_{k-1}-b_{k-1}=0. \tag{3.183}$$

The first part of the proof for the acceleration technique (3.143).

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n)}-S}-$$
$$-\frac{\frac{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})(B_{k-1}^{(n+3)}-B_{k-1}^{(n+2)})[(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})-(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})]}{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})(B_{k-1}^{(n+3)}-2B_{k-1}^{(n+2)}+B_{k-1}^{(n+1)})-(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})(B_{k-1}^{(n+2)}-2B_{k-1}^{(n+1)}+B_{k-1}^{n})}}{B_{k-1}^{(n)}-S},$$

adding $S-S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n)}-S}-\frac{M_4}{N_4},$$

where
$M_4=[(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+1)}-S)][(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+2)}-S)][(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+1)}-S)-(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n)}-S)]$,

$N_4 = (B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+1)} - S)][(B_{k-1}^{(n+3)} - S) - 2(B_{k-1}^{(n+2)} - S) + (B_{k-1}^{(n+1)} - S)] -$
$-[(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n)} - S)][(B_{k-1}^{(n+2)} - S) - 2(B_{k-1}^{(n+1)} - S) + (B_{k-1}^{n} - S)](B_{k-1}^{(n)} - S).$

Multiplying with $\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+2)}-S}$ it follows that

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = \frac{B_{k-1}^{(n+3)} - S}{B_{k-1}^{(n+2)} - S} \cdot \frac{B_{k-1}^{(n+2)} - S}{B_{k-1}^{(n)} - S} - \frac{M_5}{N_5},$$

where
$M_5 = \left(\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S} - \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}\right)\left(\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S} - 1\right)\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S} - \frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S} - 1 + \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right],$
$N_5 = \left(\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S} - \frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}\right)\left(\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S} - 2 + \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right) - \left(1 - \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right) \cdot$
$\cdot \left(1 - 2\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S} + \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right)\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}.$

Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = b_{k-1}^3 - $$
$$- \frac{\left(b_{k-1} - \frac{1}{b_{k-1}}\right)(b_{k-1} - 1)\left(b_{k-1} - \frac{1}{b_{k-1}} - 1 + \frac{1}{b_{k-1}^2}\right)}{\left[\left(b_{k-1} - \frac{1}{b_{k-1}}\right)\left(b_{k-1} - 2 + \frac{1}{b_{k-1}^2}\right) - \left(1 - \frac{1}{b_{k-1}^2}\right)\left(1 - \frac{2}{b_{k-1}} + \frac{1}{b_{k-1}^2}\right)\right]\frac{1}{b_{k-1}^2}},$$

from where it follows

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = b_{k-1}^3 - b_{k-1}^3 = 0. \tag{3.184}$$

The first part of the proof for the acceleration technique (3.144).

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = \frac{B_{k-1}^{(n+3)} - S}{B_{k-1}^{(n)} - S} +$$
$$+ \frac{\frac{[(B_{k-1}^{(n+4)} - B_{k-1}^{(n+2)}) - (B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})](B_{k-1}^{(n+4)} - B_{k-1}^{(n+2)})(B_{k-1}^{(n+2)} - B_{k-1}^{(n)})}{(B_{k-1}^{(n+2)} - B_{k-1}^{(n)})[(B_{k-1}^{(n+3)} - B_{k-1}^{(n+2)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})] - (B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})[(B_{k-1}^{(n+4)} - B_{k-1}^{(n+2)}) - (B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})}}{B_{k-1}^{(n)} - S},$$

adding $S - S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n)} - S} = \frac{B_{k-1}^{(n+3)} - S}{B_{k-1}^{(n)} - S} + \frac{M_6}{N_6},$$

where
$M_6 = [(B_{k-1}^{(n+4)} - S) - (B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n+3)} - S) + (B_{k-1}^{(n+1)} - S)][(B_{k-1}^{(n+4)} - S) - (B_{k-1}^{(n+2)} - S)] \cdot$
$\cdot [(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n)} - S)]$
$N_6 = [(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n)} - S)][(B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+1)} - S) - (B_{k-1}^{(n+2)} - S) + (B_{k-1}^{(n)} - S)] -$
$-[(B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+1)} - S)][(B_{k-1}^{(n+4)} - S) - (B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n+3)} - S) + (B_{k-1}^{(n+1)} -$
$S)]\frac{1}{B_{k-1}^{(n)} - S}.$

Multiplying with $\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+2)}-S}$ it follows that

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}\cdot\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n)}-S}-\frac{M_7}{N_7}$$

where

$M_7=\left(\frac{B_{k-1}^{(n+4)}-S}{B_{k-1}^{(n+2)}-S}-1-\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+2)}}+\frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}\right)\left(\frac{B_{k-1}^{(n+4)}-S}{B_{k-1}^{(n+2)}-S}-1\right)\left(1-\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right)$

$N_7=\left(1-\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right)\left(\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}-1+\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right)-\left(\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}\right)\cdot$

$\cdot\left(\frac{B_{k-1}^{(n+4)}-S}{B_{k-1}^{(n+2)}-S}-1-\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}+\frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}\right)\frac{B_{k-1}^{(n)}}{B_{k-1}^{(n+2)}}$

Taking the limit when n tends to infinity we get

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=b_{k-1}^3+$$

$$+\frac{\left(b_{k-1}^2-1-b_{k-1}+\frac{1}{b_{k-1}}\right)\left(b_{k-1}^2-1\right)\left(1-\frac{1}{b_{k-1}^2}\right)}{\left[\left(1-\frac{1}{b_{k-1}^2}\right)\left(b_{k-1}-\frac{1}{b_{k-1}}-1+\frac{1}{b_{k-1}^2}\right)-\left(b_{k-1}-\frac{1}{b_{k-1}}\right)\left(b_{k-1}^2-1-b_{k-1}+\frac{1}{b_{k-1}}\right)\right]\frac{1}{b_{k-1}^2}},$$

from where it follows

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n)}-S}=b_{k-1}^3-b_{k-1}^3=0. \tag{3.185}$$

We prove the second part of theorem for each method (3.140)-(3.144). The second part of the proof for the acceleration technique (3.140).

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=\frac{B_{k-1}^{(n+1)}-S-\frac{(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})(B_{k-1}^{(n+2)}-B_{k-1}^{(n+1)})}{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})-(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})}}{B_{k-1}^{(n+1)}-S}, \tag{3.186}$$

adding $S-S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=1-$$

$$-\frac{[(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n)}-S)][(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n+1)}-S)]}{\{[(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+1)}-S)]-[(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n)}-S)]\}(B_{k-1}^{(n+1)}-S)}.$$

Multiplying with $\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+2)}-S}$ it follows that

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=1-\frac{\left(1-\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right)\left(1-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}\right)}{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}-1+\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right]\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}}. \tag{3.187}$$

Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = 1 - \frac{\left(1 - \frac{1}{b_{k-1}^2}\right)\left(1 - \frac{1}{b_{k-1}}\right)}{\left(b_{k-1} - \frac{1}{b_{k-1}} - 1 + \frac{1}{b_{k-1}^2}\right)\frac{1}{b_{k-1}}}, \tag{3.188}$$

from where it follows

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = 1 - 1 = 0. \tag{3.189}$$

The second part of the proof for the acceleration technique (3.141).

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = \frac{B_{k-1}^{(n+2)} - S - \frac{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})(B_{k-1}^{(n+2)} - B_{k-1}^{(n+1)})}{(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})}}{B_{k-1}^{(n+1)} - S}, \tag{3.190}$$

adding $S - S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = \frac{B_{k-1}^{(n+2)} - S}{B_{k-1}^{(n+1)} - S} -$$
$$- \frac{[(B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+1)} - S)][(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n+1)} - S)]}{\{[(B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+1)} - S)] - [(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n)} - S)]\}(B_{k-1}^{(n+1)} - S)},$$

Multiplying with $\frac{B_{k-1}^{(n+3)} - S}{B_{k-1}^{(n+3)} - S}$ it follows that

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = \frac{B_{k-1}^{(n+2)} - S}{B_{k-1}^{(n+1)} - S} - \frac{\left(1 - \frac{B_{k-1}^{(n+1)} - S}{B_{k-1}^{(n+3)} - S}\right)\left(\frac{B_{k-1}^{(n+2)} - S}{B_{k-1}^{(n+3)} - S} - \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+3)}}\right)}{\left[1 - \frac{B_{k-1}^{(n+1)} - S}{B_{k-1}^{(n+3)} - S} - \frac{B_{k-1}^{(n+2)} - S}{B_{k-1}^{(n+3)} - S} + \frac{B_{k-1}^{(n)} - S}{B_{k-1}^{(n+3)} - S}\right]\frac{B_{k-1}^{(n+1)} - S}{B_{k-1}^{(n+3)} - S}}. \tag{3.191}$$

Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = b_{k-1} - \frac{\left(1 - \frac{1}{b_{k-1}^2}\right)\left(\frac{1}{b_{k-1}} - \frac{1}{b_{k-1}^2}\right)}{\left(1 - \frac{1}{b_{k-1}^2} - \frac{1}{b_{k-1}} + \frac{1}{b_{k-1}^3}\right)\frac{1}{b_{k-1}^2}}, \tag{3.192}$$

from where it follows

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = b_{k-1} - b_{k-1} = 0. \tag{3.193}$$

The second part of the proof for the acceleration technique (3.142).

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=\frac{B_{k-1}^{(n+1)}-S-\frac{(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})(B_{k-1}^{(n+2)}-B_{k-1}^{(n+1)})}{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})-(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})}}{B_{k-1}^{(n+1)}-S}, \tag{3.194}$$

adding $S-S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=1-$$

$$-\frac{[(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n)}-S)][(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n+1)}-S)]}{\{[(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+1)}-S)]-[(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n)}-S)]\}(B_{k-1}^{(n+1)}-S)}.$$

Multiplying with $\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+2)}-S}$ it follows that

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=1-\frac{\left(1-\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right)\left(1-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}\right)}{\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}-1+\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right]\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}}. \tag{3.195}$$

Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=1-\frac{\left(1-\frac{1}{b_{k-1}^2}\right)\left(1-\frac{1}{b_{k-1}}\right)}{\left(b_{k-1}-\frac{1}{b_{k-1}}-1+\frac{1}{b_{k-1}^2}\right)\frac{1}{b_{k-1}}}, \tag{3.196}$$

from where it follows

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=1-1=0. \tag{3.197}$$

The second part of the proof for the acceleration technique (3.143).

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=$$

$$=\frac{B_{k-1}^{(n+3)}-S-\frac{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})(B_{k-1}^{(n+3)}-B_{k-1}^{(n+2)})[(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})-(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})]}{(B_{k-1}^{(n+3)}-B_{k-1}^{(n+1)})(B_{k-1}^{(n+3)}-2B_{k-1}^{(n+2)}+B_{k-1}^{(n+1)})-(B_{k-1}^{(n+2)}-B_{k-1}^{(n)})(B_{k-1}^{(n+2)}-2B_{k-1}^{(n+1)}+B_{k-1}^{n})}}{B_{k-1}^{(n+1)}-S},$$

adding $S-S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+1)}-S}-\frac{M_8}{N_8},$$

where
$M_8=[(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+1)}-S)][(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+2)}-S)][(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+1)}-$

$S) - (B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n)} - S)]$
$N_8 = [(B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+1)} - S)][(B_{k-1}^{(n+3)} - S) - 2(B_{k-1}^{(n+2)} - S) + (B_{k-1}^{(n+1)} - S)] - [(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n)} - S)][(B_{k-1}^{(n+2)} - S) - 2(B_{k-1}^{(n+1)} - S) + (B_{k-1}^{n} - S)](B_{k-1}^{(n+1)} - S).$
Multiplying with $\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+2)}-S}$ it follows that

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = \frac{B_{k-1}^{(n+3)} - S}{B_{k-1}^{(n+2)} - S} \cdot \frac{B_{k-1}^{(n+2)} - S}{B_{k-1}^{(n+1)} - S} - \frac{M_9}{N_9},$$

where
$M_9 = \left(\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S} - \frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}\right)\left(\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S} - 1\right)\left[\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S} - \frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S} - 1 + \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right]$
$N_9 = \left(\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S} - \frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}\right)\left(\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S} - 2 + \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right)\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S} - \left(1 - \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right) \cdot$
$\cdot \left(1 - 2\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S} + \frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right)\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}$. Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = b_{k-1}^2 -$$
$$- \frac{\left(b_{k-1} - \frac{1}{b_{k-1}}\right)(b_{k-1} - 1)\left(b_{k-1} - \frac{1}{b_{k-1}} - 1 + \frac{1}{b_{k-1}^2}\right)}{\left[\left(b_{k-1} - \frac{1}{b_{k-1}}\right)\left(b_{k-1} - 2 + \frac{1}{b_{k-1}^2}\right) - \left(1 - \frac{1}{b_{k-1}^2}\right)\left(1 - \frac{2}{b_{k-1}} + \frac{1}{b_{k-1}^2}\right)\right]\frac{1}{b_{k-1}}},$$

from where it follows

$$\lim_{n\to\infty} \frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = b_{k-1}^2 - b_{k-1}^2 = 0. \tag{3.198}$$

The second part of the proof for the acceleration technique (3.144).

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = \frac{R}{B_{k-1}^{(n+1)} - S},$$

where
$R = B_{k-1}^{(n+3)} - S +$
$+ \frac{[(B_{k-1}^{(n+4)} - B_{k-1}^{(n+2)}) - (B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})](B_{k-1}^{(n+4)} - B_{k-1}^{(n+2)})(B_{k-1}^{(n+2)} - B_{k-1}^{(n)})}{(B_{k-1}^{(n+2)} - B_{k-1}^{(n)})[(B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)}) - (B_{k-1}^{(n+2)} - B_{k-1}^{(n)})] - (B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})[(B_{k-1}^{(n+4)} - B_{k-1}^{(n+2)}) - (B_{k-1}^{(n+3)} - B_{k-1}^{(n+1)})]}$
adding $S - S$ in each parenthesis and separating into two fractions we obtain

$$\frac{B_k^{(n)} - S}{B_{k-1}^{(n+1)} - S} = \frac{B_{k-1}^{(n+3)} - S}{B_{k-1}^{(n+1)}} + \frac{M_{10}}{N_{10}} \cdot \frac{1}{(B_{k-1}^{(n+1)} - S)},$$

where
$M_{10} = [(B_{k-1}^{(n+4)} - S) - (B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n+3)} - S) + (B_{k-1}^{(n+1)} - S)][(B_{k-1}^{(n+4)} - S) - (B_{k-1}^{(n+2)} - S)][(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n)} - S)]$
$N_{10} = [(B_{k-1}^{(n+2)} - S) - (B_{k-1}^{(n)} - S)][(B_{k-1}^{(n+3)} - S) - (B_{k-1}^{(n+1)} - S) - (B_{k-1}^{(n+2)} - S) + (B_{k-1}^{(n)} - S)] -$

$-[(B_{k-1}^{(n+3)}-S)-(B_{k-1}^{(n+1)}-S)][(B_{k-1}^{(n+4)}-S)-(B_{k-1}^{(n+2)}-S)-(B_{k-1}^{(n+3)}-S)+(B_{k-1}^{(n+1)}-S)].$
Multiplying with $\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+2)}-S}$ it follows that

$$\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}\cdot\frac{B_{k-1}^{(n+2)}-S}{B_{k-1}^{(n+1)}-S}-\frac{M_{11}}{N_{11}},$$

where

$M_{11}=\left(\frac{B_{k-1}^{(n+4)}-S}{B_{k-1}^{(n+2)}-S}-1-\frac{B_{k-1}^{(n+3)}}{B_{k-1}^{(n+2)}}+\frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}\right)\left(\frac{B_{k-1}^{(n+4)}-S}{B_{k-1}^{(n+2)}-S}-1\right)\left(1-\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right)$

$N_{11}=\left(1-\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right)\left(\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}-1+\frac{B_{k-1}^{(n)}-S}{B_{k-1}^{(n+2)}-S}\right)\cdot\frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}-$
$-\left(\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}-\frac{B_{k-1}^{(n+1)}-S}{B_{k-1}^{(n+2)}-S}\right)\left(\frac{B_{k-1}^{(n+4)}-S}{B_{k-1}^{(n+2)}-S}-1-\frac{B_{k-1}^{(n+3)}-S}{B_{k-1}^{(n+2)}-S}+\frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}\right)\cdot\frac{B_{k-1}^{(n+1)}}{B_{k-1}^{(n+2)}}.$

Taking the limit when n tends to infinity we get

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=b_{k-1}^2+$$
$$+\frac{\left(b_{k-1}^2-1-b_{k-1}+\frac{1}{b_{k-1}}\right)\left(b_{k-1}^2-1\right)\left(1-\frac{1}{b_{k-1}^2}\right)}{\left[\left(1-\frac{1}{b_{k-1}^2}\right)\left(b_{k-1}-\frac{1}{b_{k-1}}-1+\frac{1}{b_{k-1}^2}\right)-\left(b_{k-1}-\frac{1}{b_{k-1}}\right)\left(b_{k-1}^2-1-b_{k-1}+\frac{1}{b_{k-1}}\right)\right]\frac{1}{b_{k-1}}},$$

from where it follows

$$\lim_{n\to\infty}\frac{B_k^{(n)}-S}{B_{k-1}^{(n+1)}-S}=b_{k-1}^2-b_{k-1}^2=0. \tag{3.199}$$

Which completes the proof.

We end this paragraph with one example which will show the practical implication of the extensions for the iterated B-algorithm.

Example 3.23 *(Bumbariu, [38])[35], [16] Test function: $f(x)=\cos x-x$, which has a unique root $x^*=0.73908513321516064165531208767387340401 34\ldots$. This equation can be rewritten into a fixed point problem by $g(x)=\cos x$. To apply the new acceleration technique we shall take the initial guess $x_0\in\{0.5,0.7,0.8,0.9\}$. The results for Example 3.23 with the initial value $x_0=0.8$ are listed in Table 5, for the other initial values the results are the same.*

Table 5

$B_k^{(n)}$	$B_k^{(n)}$(3.140)	$B_k^{(n)}$(3.141)	$B_k^{(n)}$(3.142)	$B_k^{(n)}$(3.143)	$B_k^{(n)}$(3.144)	$B_k^{(n)}$(3.57)
$B_0^{(0)}$	0.8	0.8	0.8	0.8	0.8	0.8
$B_1^{(0)}$	2	2	2	1	1	3
$B_2^{(0)}$	3	3	3	3	4	4
$B_3^{(0)}$	6	6	6	6	7	7
$B_4^{(0)}$	9	9	9	9	10	11
$B_5^{(0)}$	12	12	12	12	13	13
$B_6^{(0)}$	15	15	15	16	17	18
$B_7^{(0)}$	18	18	18	22	21	22
$B_8^{(0)}$	23	23	23	25	26	28

From Table 5 we can conclude that when comparing the five extensions for the iterated B-algorithm, the last extension, (3.144) has a better convergence speed than the other four extensions, we remind the fact that the first three extensions, (3.140)-(3.142), are equivalent, consequently they have the exact number of digits at each step. But when comparing (3.144) with the iterated B-algorithm the last one has a better convergence speed; the same results were obtained for all twenty test functions that we have studied. In the table are listed the exact number of decimals at the first eight steps, all the numerical computations are done with Maple 13, using 39 digits arithmetic.

3.2 Accelerating the Krasnoselskij, Mann and Ishikawa iterations with the acceleration techniques presented in the previous paragraph

In this paragraph we propose an empirical study for accelerating the Krasnoselskij, Mann and Ishikawa iterations with the help of the B-algorithm and its iterated form, with the help of the other representations for the B-algorithm and their iterated forms and with the help of the extensions of the iterated B-algorithm.

In order to see the practical implication of the new acceleration techniques we will apply them to some sequences arising in application of the Krasnoselskij, Mann and Ishikawa iterations to some operators taken from the literature.

3.2.1 Accelerating the Krasnoselskij, Mann and Ishikawa iterations with the B-algorithm

Author's original contribution in this paragraph is Example 3.24.
In this paragraph we apply the B-algorithm to some sequences arising when we implement the Krasnoselskij, Mann and Ishikawa iterations to the operator T, defined in Example 3.24.

Example 3.24 $Tx = \frac{1}{x}$, $T : \left[\frac{1}{2}, 2\right] \to \left[\frac{1}{2}, 2\right]$ *which has a unique fixed point* $F_T = \{1\}$.

In our experiments we took the initial guesses $x_0 = 1.25$ and $x_0 = 1.5$, for the operator T, and different values for the constant μ that appears in the definition of the Krasnoselskij iteration and for the strings α_n and β_n that occur in the definition of the Mann and Ishikawa iterations. We here present the results only for the for the initial value $x_0 = 1.25$. The details for the Krasnoselskij iteration, with the constants $\mu \in \{\frac{1}{4}, \frac{1}{5}, \frac{2}{3}\}$ are listed in Table 1, for the Mann iteration, with the strings $\alpha_n \in \{\frac{n}{4n+1}, \frac{n}{5n+1}, \frac{2n}{3n+1}\}$, are listed in Table 2 and for the Ishikawa iteration, with the strings $(\alpha_n, \beta_n) \in \{(\frac{n}{4n+1}, \frac{3n}{4n+1}), (\frac{2n}{3n+1}, \frac{4n}{5n+1}), (\frac{2n}{5n+1}, \frac{2n}{3n+1})\}$, are listed in Table 3.

Table 1

n	$\mu = \frac{1}{4}, B-alg$	$\mu = \frac{1}{4}, Kr.it$	$\mu = \frac{1}{5}, B-alg$	$\mu = \frac{1}{5}, Kr.it.$	$\mu = \frac{2}{3}, B-alg$	$\mu = \frac{2}{3}, Kr.it.$
0	1.25	1.25	1.25	1.25	1.25	1.25
1	0	0	0	0	0	0
2	1	1	0	0	1	1
3	1	1	1	1	0	0
4	0	1	0	1	0	2
5	0	2	0	1	3	0
...	...	...	...	...	...	...
20	39		39		38	39
21					39	
...	...	...	...	...	...	...
83						39
...	...	...	...	...	...	...
100		30		22		

Table 2

n	$\alpha_n = \frac{n}{4n+1}, B$	$\alpha_n = \frac{n}{4n+1}, M$	$\alpha_n = \frac{n}{5n+1}, B$	$\alpha_n = \frac{n}{5n+1}, M$	$\alpha_n = \frac{2n}{3n+1}, B$	$\alpha_n = \frac{2n}{3n+1}, M$
0	1.25	1.25	1.25	1.25	1.25	1.25
1	0	0	0	0	1	1
2	1	1	0	0	1	1
3	1	1	1	1	3	3
4	0	1	0	1	2	2
5	0	1	0	1	2	4
...	...	...	...	...	...	...
43	33	12	30	9	39	21
...	...	...	...	...	...	...
50	39	14	38	11	39	26
...	...	...	...	...	...	...
53	39	15	39	11	39	28

Table 3

n	$\alpha_n = \frac{n}{4n+1}, \beta_n = \frac{3n}{4n+1}, B$	$it.Is.$	$\alpha_n = \frac{2n}{3n+1}, \beta_n = \frac{4n}{5n+1}, B$	$it.Is.$	$\alpha_n = \frac{2n}{5n+1}, \beta_n = \frac{2n}{3n+1} B$	$it.Is.$
0	1.25	1.25	1.25	1.25	1.25	1.25
1	0	0	0	0	0	0
2	0	0	1	1	0	0
3	0	0	1	1	0	0
4	1	0	1	1	1	1
5	1	0	1	1	1	1
...	...	...	...	...	...	...
57	31	4	38	7	38	8
...	...	...	...	...	...	...
60	34	4	39	7	38	8
...	...	...	...	...	...	...
68	39	4	39	9	38	9

From the results listed in the tables we can conclude that the B-algorithm accelerates the convergence speed for the Krasnoselskij, Mann and Ishikawa iterations, exception are some cases for the Krasnoselskij iteration, when the number of iterations that differ are very little. When comparing the results for the Krasnoselskij, Mann and Ishikawa iterations we can say that the best convergence speed was obtained in the case of the Krasnoselkij iteration and than for Mann iteration. The same results where obtained when taking the initial value $x_0 = 1.5$ and for the other values of the constant $\mu \in \{\frac{1}{2}, \frac{2}{5}\}$, and for the strings $\alpha_n \in \{\frac{n}{2n+1}, \frac{2n}{5n+1}\}$ and for $(\alpha_n, \beta_n) \in \{(\frac{n}{2n+1}, \frac{n}{2n+1}), (\frac{2n}{3n+1}, \frac{3n}{4n+1})\}$. The computations were done with Maple 13, using 39 digit floating point arithmetics. In the above tables are listed the exact number of digits at each step.

3.2.2 Accelerating the Krasnoselskij, Mann and Ishikawa iterations with the representations of the B-algorithm

Author's original contribution in this paragraph is Example 3.25.
As in the previous paragraph, to see the practical implication of the other representations of the B-algorithm, we will apply the techniques to the sequences arising in the application of the Krasnoselskij, Mann and Ishikawa iterations to the operator T, defined in the Example 3.25.

Example 3.25 $Tx = 1-x^2$ *which has a unique fixed point* $F_T = \{0.618033988749894\ldots\}$.

In our experiments we took the initial guesses $x_0 = 0.5$ and $x_0 = 0.7$, for the operator T and different values for the constant μ that appears in the definition of the Krasnoselskij iteration and for the strings α_n and β_n that occur in the definition of the Mann and Ishikawa iterations. We here present the results for the initial value $x_0 = 0.7$, the details for the Krasnoselskij iteration, with the constant $\mu = \frac{2}{3}$ are listed in Table 4, for the Mann iteration, with the string $\alpha_n = \frac{2n}{3n+1}$, are listed in Table 5 and for the Ishikawa iteration, with the strings $\alpha_n = \frac{2n}{3n+1}, \beta_n = \frac{4n}{5n+1}$, are listed in Table 6.

Table 4

n	$B_1^{(n)}(3.46)$	$B_1^{(n)}(3.47)$	$B_1^{(n)}(3.48)$	$B_1^{(n)}(3.49)$	$B_1^{(n)}(3.50)$	$B_1^{(n)}(3.51)$	$B_1^{(n)}(3.8)$	$it.K.$
0	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7
1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1
4	3	3	3	3	3	3	3	1
5	2	2	2	2	2	2	2	2
6	3	3	3	3	3	3	3	2
7	4	4	4	4	4	4	4	2
8	8	8	8	8	8	8	8	2
9	8	8	8	8	8	8	8	2
10	8	8	8	8	8	8	8	3
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
20	16	38	38	39	39	39	39	5
21	17	38	38					7
22	17	39	38					6
23	17		39					8
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
50	22							16
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
100	20							30

Table 5

n	$B_1^{(n)}(3.46)$	$B_1^{(n)}(3.47)$	$B_1^{(n)}(3.48)$	$B_1^{(n)}(3.49)$	$B_1^{(n)}(3.50)$	$B_1^{(n)}(3.51)$	$B_1^{(n)}(3.8)$	$it.M.$
0	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7
1	1	1	1	1	1	1	1	0
2	1	1	1	1	1	1	1	0
3	2	2	2	2	2	2	2	2
4	2	2	2	2	2	2	2	3
5	3	3	3	3	3	3	3	2
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
50	16	36	36	36	36	36	36	1

Table 6

n	$B_1^{(n)}(3.46)$	$B_1^{(n)}(3.47)$	$B_1^{(n)}(3.48)$	$B_1^{(n)}(3.49)$	$B_1^{(n)}(3.50)$	$B_1^{(n)}(3.51)$	$B_1^{(n)}(3.8)$	$it.I.$
0	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7
1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1
4	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
50	6	6	6	6	6	6	6	1

From the results listed in the above tables we can conclude that the representations of the B-algorithm are accelerating the convergence speed for the Krasnoselskij, Mann and Ishikawa iterations, as a observation we can say that the first representation, (3.46) is numerically unstable, and when comparing the other five representations, (3.47)-(3.51) with the B-algorithm, (3.8), we obtained, in most cases, the same number of iterations and when the number of iterations is different, the difference is not significant. And when comparing the results for the Krasnoselskij, Mann and Ishikawa iterations we can say that the best convergence speed was obtained in the case of the Krasnoselkij iteration and than for Mann iteration. The same results where obtained when taking the initial value $x_0 = 0.5$ and for the other values of the constant $\mu \in \{\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}\}$, and for the strings $\alpha_n \in \{\frac{n}{2n+1}, \frac{n}{4n+1}, \frac{n}{5n+1}, \frac{2n}{5n+1}\}$ and for $(\alpha_n, \beta_n) \in \{(\frac{n}{2n+1}, \frac{n}{2n+1}), (\frac{n}{4n+1}, \frac{3n}{4n+1}), (\frac{2n}{5n+1}, \frac{2n}{3n+1}), (\frac{2n}{3n+1}, \frac{3n}{4n+1})\}$. The computations were done with Maple 13, using 39 digit floating point arithmetics. In the above tables are listed the exact number of digits at each step.

3.2.3 Accelerating the Krasnoselskij, Mann and Ishikawa iterations with the iterated B-algorithm

Author's original contribution in this paragraph is Example 3.26.
In order to see the practical implication of the iterated B-algorithm we will apply the technique to the sequences arising in application of the Krasnoselskij, Mann and Ishikawa iterations to the operator T, defined in the Example 3.26.

Example 3.26 *$Tx = \sqrt{x}$ which has a unique fixed point $F_T = \{1\}$.*

In our experiments we took the initial guesses $x_0 = 0.8$ and $x_0 = 1.2$, for the operator T and different values for the constant μ that appears in defining the Krasnoselskij iteration and for the strings α_n and β_n that occur in defining the Mann and Ishikawa iterations. We here present the results for the initial value $x_0 = 1.25$, the details for the Krasnoselskij iteration, with the constants $\mu \in \{\frac{1}{4}, \frac{1}{5}, \frac{2}{3}\}$ are listed in Table 7, for the Mann iteration, with the strings $\alpha_n \in \{\frac{n}{4n+1}, \frac{n}{5n+1}, \frac{2n}{3n+1}\}$, are listed in Table 8 and for the Ishikawa iteration, with the strings $(\alpha_n, \beta_n) \in \{(\frac{n}{4n+1}, \frac{3n}{4n+1}), (\frac{2n}{3n+1}, \frac{4n}{5n+1}), (\frac{2n}{5n+1}, \frac{2n}{3n+1})\}$, are listed in Table 9.

Table 7

$B_k^{(n)}$	$\mu = \frac{1}{4}, B - alg$	$\mu = \frac{1}{4}, Kr.it$	$\mu = \frac{1}{5}, B - alg$	$\mu = \frac{1}{5}, Kr.it.$	$\mu = \frac{2}{3}, B - alg$	$\mu = \frac{2}{3}, Kr.it.$
$B_0^{(0)}$	0.8	0.8	0.8	0.8	0.8	0.8
$B_1^{(0)}$	1	0	1	0	2	0
$B_2^{(0)}$	3	0	3	0	4	0
$B_3^{(0)}$	5	0	5	0	7	0
$B_4^{(0)}$	8	0	8	0	12	0
$B_5^{(0)}$	11	0	10	0	17	0
$B_6^{(0)}$	14	0	13	0	22	0
$B_7^{(0)}$	17	0	15	0	27	0
$B_8^{(0)}$	20	0	19	0	32	0

Table 8

$B_k^{(n)}$	$\alpha_n = \frac{n}{4n+1}, B$	$\alpha_n = \frac{n}{4n+1}, M$	$\alpha_n = \frac{n}{5n+1}, B$	$\alpha_n = \frac{n}{5n+1}, M$	$\alpha_n = \frac{2n}{3n+1}, B$	$\alpha_n = \frac{2n}{3n+1}, M$
$B_0^{(0)}$	0.8	0.8	0.8	0.8	0.8	0.8
$B_1^{(0)}$	0	0	0	0	1	0
$B_2^{(0)}$	1	0	1	0	2	0
$B_3^{(0)}$	3	0	2	0	4	0
$B_4^{(0)}$	4	0	4	0	8	0
$B_5^{(0)}$	4	0	4	0	8	0
$B_6^{(0)}$	6	0	4	0	9	0
$B_7^{(0)}$	7	0	6	0	9	0
$B_8^{(0)}$	7	0	6	0	9	0

Table 9

$B_k^{(n)}$	$\alpha_n = \frac{n}{4n+1}, \beta_n = \frac{3n}{4n+1}, B$	$it.Is.$	$\alpha_n = \frac{2n}{3n+1}, \beta_n = \frac{4n}{5n+1}, B$	$it.Is.$	$\alpha_n = \frac{2n}{5n+1}, \beta_n = \frac{2n}{3n+1} B$	$it.Is.$
$B_0^{(0)}$	0.8	0.8	0.8	0.8	0.8	0.8
$B_1^{(0)}$	0	0	1	0	1	0
$B_2^{(0)}$	2	0	3	0	2	0
$B_3^{(0)}$	3	0	5	0	4	0
$B_4^{(0)}$	4	0	6	0	5	0
$B_5^{(0)}$	5	0	9	0	6	0
$B_6^{(0)}$	5	0	9	1	5	0
$B_7^{(0)}$	6	1	9	1	9	0
$B_8^{(0)}$	6	1	9	1	9	0

From the results listed in the above tables we can conclude that the iterated B-algorithm accelerates the convergence speed for the Krasnoselskij, Mann and Ishikawa iterations. When comparing the results for the Krasnoselskij, Mann and Ishikawa iterations we can say that the best convergence speed was obtained in the case of the

Krasnoselkij iteration and than for Mann iteration. The same results where obtained when taking the initial value $x_0 = 1.2$ and for the other values of the constant $\mu \in \{\frac{1}{2}, \frac{2}{5}\}$, and for the strings $\alpha_n \in \{\frac{n}{2n+1}, \frac{2n}{5n+1}\}$ and for $(\alpha_n, \beta_n) \in \{(\frac{n}{2n+1}, \frac{n}{2n+1}), (\frac{2n}{3n+1}, \frac{3n}{4n+1})\}$. The computations were done with Maple 13, using 39 digit floating point arithmetics. In the above tables are listed the exact number of digits at each step.

3.2.4 Accelerating the Krasnoselskij, Mann and Ishikawa iterations with the iterated forms of the other representations for the iterated B-algorithm

Author's original contribution in this paragraph is Example 3.27.
To see the practical implication of the iterated representations of the iterated B-algorithm, we will apply the techniques to the sequences arising in the application of the Krasnoselskij, Mann and Ishikawa iterations to the operator T, defined in the Example 3.27.

Example 3.27 *$Tx = 1-x^2$ which has a unique fixed point $F_T = \{0.618033988749894\ldots\}$.*

In our experiments we took the initial guesses $x_0 = 0.5$ and $x_0 = 0.7$, for the operator T and different values for the constant μ that appears in defining the Krasnoselskij iteration and for the strings α_n and β_n that occur in defining the Mann and Ishikawa iterations. We here present the results for the initial value $x_0 = 0.5$, the details for the Krasnoselskij iteration, with the constant $\mu = \frac{2}{3}$ are listed in Table 10, for the Mann iteration, with the string $\alpha_n = \frac{2n}{3n+1}$, are listed in Table 11 and for the Ishikawa iteration, with the strings $\alpha_n = \frac{2n}{3n+1}, \beta_n = \frac{4n}{5n+1}$, are listed in Table 12.
Table 10

$B_k^{(n)}$	$B_k^{(n)}$, (3.91)	$B_k^{(n)}$, (3.92)	$B_k^{(n)}$, (3.93)	$B_k^{(n)}$, (3.94)	$B_k^{(n)}$, (3.95)	$B_k^{(n)}$, (3.96)	$B_k^{(n)}$, (3.57)	*it.K.*
$B_0^{(0)}$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$B_1^{(0)}$	2	2	2	2	2	2	2	0
$B_2^{(0)}$	5	5	5	5	5	5	5	0
$B_3^{(0)}$	9	9	9	9	9	9	9	1
$B_4^{(0)}$	13	13	13	13	13	13	13	2
$B_5^{(0)}$	18	18	18	18	18	18	18	1
$B_6^{(0)}$	21	25	25	25	25	25	25	2
$B_7^{(0)}$	20	32	32	32	32	32	32	3
$B_8^{(0)}$	19	37	37	37	37	37	37	2

Table 11

$B_k^{(n)}$	$B_k^{(n)}$, (3.91)	$B_k^{(n)}$, (3.92)	$B_k^{(n)}$, (3.93)	$B_k^{(n)}$, (3.94)	$B_k^{(n)}$, (3.95)	$B_k^{(n)}$, (3.96)	$B_k^{(n)}$, (3.57)	*it.M.*
$B_0^{(0)}$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$B_1^{(0)}$	3	3	3	3	3	3	3	0
$B_2^{(0)}$	5	5	5	5	5	5	5	2
$B_3^{(0)}$	8	8	8	8	8	8	8	2
$B_4^{(0)}$	12	12	12	12	12	12	12	2
$B_5^{(0)}$	12	12	12	12	12	12	12	4
$B_6^{(0)}$	13	13	13	13	13	13	13	5
$B_7^{(0)}$	16	16	16	16	16	16	16	6
$B_8^{(0)}$	18	18	18	18	18	18	18	6

Table 12

$B_k^{(n)}$	$B_k^{(n)}$, (3.91)	$B_k^{(n)}$, (3.92)	$B_k^{(n)}$, (3.93)	$B_k^{(n)}$, (3.94)	$B_k^{(n)}$, (3.95)	$B_k^{(n)}$, (3.96)	$B_k^{(n)}$, (3.57)	$it.I.$
$B_0^{(0)}$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$B_1^{(0)}$	0	0	0	0	0	0	0	0
$B_2^{(0)}$	0	0	0	0	0	0	0	0
$B_3^{(0)}$	2	2	2	2	2	2	2	0
$B_4^{(0)}$	2	2	2	2	2	2	2	0
$B_5^{(0)}$	2	2	2	2	2	2	2	0
$B_6^{(0)}$	2	2	2	2	2	2	2	0
$B_7^{(0)}$	2	2	2	2	2	2	2	0
$B_8^{(0)}$	2	2	2	2	2	2	2	0

From the results listed in the above tables we can conclude that the iterated representations of the iterated B-algorithm are accelerating the convergence speed for the Krasnoselskij, Mann and Ishikawa iterations. If we compare the first representation, (3.91), in case of Krasnoselski iteration, with the other five representations, (3.92)-(3.96), and with iterated B-algorithm, (3.57), we obtain less number of iterations than the other methods, in case of Mann and Ishikawa iterations we obtain the same number of iterations for all seven techniques. When we compare the representations, (3.92)-(3.96) with the iterated B-algorithm, (3.57), we obtained, in most cases, the same number of iterations at each step and when the number of iterations is different, the difference is very little, only a few iterations. When we compare the results for the Krasnoselskij, Mann and Ishikawa iterations we can say that the best convergence speed was obtained in the case of the Krasnoselkij iteration and then for Mann iteration. The same results where obtained when taking the initial value $x_0 = 0.7$ and for the other values of the constant $\mu \in \{\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}\}$, and for the strings $\alpha_n \in \{\frac{n}{2n+1}, \frac{n}{4n+1}, \frac{n}{5n+1}, \frac{2n}{5n+1}\}$ and for $(\alpha_n, \beta_n) \in \{(\frac{n}{2n+1}, \frac{n}{2n+1}), (\frac{n}{4n+1}, \frac{3n}{4n+1}), (\frac{2n}{5n+1}, \frac{2n}{3n+1}), (\frac{2n}{3n+1}, \frac{3n}{4n+1})\}$. The computations were done with Maple 13, using 39 digit floating point arithmetics. In the above tables are listed the exact number of digits at each step.

3.2.5 Accelerating the Krasnoselskij, Mann and Ishikawa iterations with the extensions of the iterated B-algorithm

Author's original contribution in this paragraph is Example 3.28.
In this paragraph we apply the extensions of the iterated B-algorithm to some sequences arising when we implement the Krasnoselskij, Mann and Ishikawa iterations to the operator T, defined in the Example 3.28.

Example 3.28 $Tx = \frac{1}{x}$, $T : \left[\frac{1}{2}, 2\right] \to \left[\frac{1}{2}, 2\right]$ *which has a unique fixed point* $F_T = \{1\}$.

In our experiments we took the initial guesses $x_0 = 1.25$ and $x_0 = 1.5$, for the operator T and different values for the constant μ that appears in defining the Krasnoselskij iteration and for the strings α_n and β_n that occur in defining the Mann and Ishikawa iterations. We here present the results for the initial value $x_0 = 1.5$, the details for the Krasnoselskij iteration, with the constant $\mu = \frac{1}{4}$ are listed in Table 13, for the Mann iteration, with the strings $\alpha_n = \frac{2n}{5n+1}$, are listed in Table 14 and for the Ishikawa iteration, with the strings $\alpha_n = \frac{2n}{5n+1}, \beta_n = \frac{2n}{3n+1}$, are listed in Table 15.

Table 13

$B_k^{(n)}$	$B_k^{(n)}$, (3.140)	$B_k^{(n)}$, (3.141)	$B_k^{(n)}$, (3.142)	$B_k^{(n)}$, (3.143)	$B_k^{(n)}$, (3.144)	$B_k^{(n)}$, (3.57)	*it.K.*
$B_0^{(0)}$	1.5	1.5	1.5	1.5	1.5	1.5	1.5
$B_1^{(0)}$	0	0	0	0	0	0	0
$B_2^{(0)}$	2	2	2	3	3	3	1
$B_3^{(0)}$	4	4	4	5	5	5	1
$B_4^{(0)}$	7	7	7	9	9	10	2
$B_5^{(0)}$	11	11	11	16	15	14	2
$B_6^{(0)}$	15	15	15	19	20	20	3
$B_7^{(0)}$	18	18	18	26	27	27	3
$B_8^{(0)}$	24	24	24	34	35	34	4

Table 14

$B_k^{(n)}$	$B_k^{(n)}$, (3.140)	$B_k^{(n)}$, (3.141)	$B_k^{(n)}$, (3.142)	$B_k^{(n)}$, (3.143)	$B_k^{(n)}$, (3.144)	$B_k^{(n)}$, (3.57)	*it.M.*
$B_0^{(0)}$	1.5	1.5	1.5	1.5	1.5	1.5	1.5
$B_1^{(0)}$	0	0	0	0	0	0	0
$B_2^{(0)}$	3	3	3	4	6	5	1
$B_3^{(0)}$	4	4	4	8	8	7	1
$B_4^{(0)}$	6	6	6	10	10	10	2
$B_5^{(0)}$	9	9	9	13	13	13	2
$B_6^{(0)}$	11	11	11	15	14	15	3
$B_7^{(0)}$	11	11	11	19	19	19	4
$B_8^{(0)}$	11	11	11	21	20	21	4

Table 15

$B_k^{(n)}$	$B_k^{(n)}$, (3.140)	$B_k^{(n)}$, (3.141)	$B_k^{(n)}$, (3.142)	$B_k^{(n)}$, (3.143)	$B_k^{(n)}$, (3.144)	$B_k^{(n)}$, (3.57)	*it.I.*
$B_0^{(0)}$	1.5	1.5	1.5	1.5	1.5	1.5	1.5
$B_1^{(0)}$	1	1	1	1	1	1	1
$B_2^{(0)}$	0	0	0	1	1	1	1
$B_3^{(0)}$	4	4	4	3	4	4	1
$B_4^{(0)}$	4	4	4	4	4	4	1
$B_5^{(0)}$	0	0	0	4	2	6	1
$B_6^{(0)}$	6	6	6	6	5	6	1
$B_7^{(0)}$	4	4	4	6	7	7	1
$B_8^{(0)}$	6	6	6	8	7	6	1

From the results listed in the tables we can conclude that the extensions of the iterated B-algorithm are accelerating the convergence speed for the Krasnoselskij, Mann and Ishikawa iterations. Because the first three extensions, (3.140)-(3.142), can be equivalently written they give exact number of digits at each step. When comparing the five extensions, (3.140)-(3.144), with the iterated B-algorithm, (3.57), we can say that the extensions (3.143) and (3.144) give almost the same number of digits as in case of iterated B-algorithm, (3.57), so we can conclude that among these last three algorithms it is not one that has a better convergence speed than the others. And when comparing the results for the Krasnoselskij, Mann and Ishikawa iterations we can say that the best convergence speed was obtained in the case of the Krasnoselkij iteration and than for Mann iteration. The same results where obtained when taking the initial value $x_0 = 1.25$ and for the other values of the constant $\mu \in \{\frac{1}{2}, \frac{1}{5}, \frac{2}{3}, \frac{2}{5}\}$, and for the strings $\alpha_n \in \{\frac{n}{2n+1}, \frac{n}{4n+1}, \frac{n}{5n+1}, \frac{2n}{3n+1}\}$

and for $(\alpha_n, \beta_n) \in \{(\frac{n}{2n+1}, \frac{n}{2n+1}), (\frac{n}{4n+1}, \frac{3n}{4n+1}), (\frac{2n}{3n+1}, \frac{4n}{5n+1}), (\frac{2n}{3n+1}, \frac{3n}{4n+1})\}$. The computations were done with Maple 13, using 39 digit floating point arithmetics. In the above tables are listed the exact number of digits at each step.

4 Conclusions

In the last decades much interest has been shown to the development and to the improvement of the existing nonlinear transformations. The problem with the nonlinear transformations, is that, they are usually nonregular, that means that there is no guarantee that the transformed sequence $\{S'_n\}$ will converge at the same limit as the initial one, $\{S_n\}$, unless we have additional information about the sequence to be transformed. But this inconvenience is compensated by the empirical fact, if a nonlinear transformation works, then it works in general. Consequently a sequence transformation is able to accelerate only a class of scalar sequences. Our aim in this thesis was to accelerate only the sequences that converge linearly.

The oldest and the most well known nonlinear transformation, for a sequence that converges linearly, is Aitken's Δ^2 process

$$A_1^{(n)} = S_n - \frac{(S_{n+1} - S_n)^2}{S_{n+2} - 2S_{n+1} + S_n} = S_n - \frac{[\Delta S_n]^2}{\Delta^2 S_n}, \quad n = 0, 1, \dots, \tag{4.1}$$

where $\{S_n\}$ is the sequence to be accelerated and Δ denotes the forward difference operator, $\Delta S_n = S_{n+1} - S_n$ and $\Delta^2 S_n = \Delta S_{n+1} - \Delta S_n = S_{n+2} - 2S_{n+1} + S_n$.

Other important sequence transformations are: Richardson extrapolation [98], (1.24), Shanks transformation [108], (1.25), Romberg transformation [102], (1.24), Wynn's ϵ-algorithm [126], (1.31), Wynn's ρ-algorithm [127], (1.45), Brezinski's θ-algorithm [21], (1.52) and Levin's transformation [75], (1.78).

Starting from Aitken's Δ^2 process, (4.1) we constructed a new acceleration method, that we call the B-algorithm, defined by

$$B_1^{(n)} = S_{n+3} - \frac{[\overline{\Delta} S_{n+1}][\Delta S_{n+2}]}{\overline{\Delta} S_{n+1} - \overline{\Delta} S_n}, \quad n = 0, 1, \dots \tag{4.2}$$

where $\{S_n\}$ is the sequence to be accelerated, Δ denotes the forward difference operator, defined by $\Delta S_n = S_{n+1} - S_n$ and we denote $\overline{\Delta}$, the forward difference operator with two steps, defined by $\overline{\Delta} S_n = S_{n+2} - S_n$.

For the new algorithm, (4.2), we gave some other representations (3.46)-(3.51), an iterate form, (3.57) and the iterate forms for the other representations, (3.91)-(3.96), and some extensions (3.140)-(3.144).

Further research regarding the B-algorithm and the related topics :

1. Prove if the B-algorithm is exact for the model sequence:

a) $S_n = S + (a + bx_n)\lambda^n$;

b)$S_n = +\frac{\lambda^n}{a+bx_n}$;

where x_n is a given known sequence.

2. Apply the B-algorithm to some logarithmically sequences.

3. Apply the B-algorithm to some initial value problems.

4. Extend the theory for the B-algorithm from the scalar sequences to the vector sequences. In particular, study the acceleration method for logarithmically convergent sequences.

References

[1] Aitken A. C., *On Bernoulli's numerical solution of algebraic equations*, Proc. Roy. Soc. Edinburgh 46 (1926) 289-305

[2] Arai M., Okamoto K., Kametaka Y., *Aitken acceleration and Fibonacci numbers*, Japan J. Appl. Math. 5 (1988) 145-152

[3] Babu G.V.R., Vara Prasad K.N.V.V., *Comparison of fastness of the convergence among Krasnoselskij, Mann, and Ishikawa iterations in arbitrary real Banach spaces*, Fixed Point Theory Appl., Vol. (2006), Article ID 35706, 12 pages

[4] Babu G.V.R., Vara Prasad K.N.V.V., *Mann iteration converges faster than Ishikawa iteration for the class of Zamfirescu operators*, Fixed Point Theory Appl., Vol. (2007), Article ID 97986, 2 pages

[5] Babu G.V.R., Vara Prasad, K.N.V.V. *Mann iteration converges faster than Ishikawa iteration for the class of Zamfirescu operators*, Fixed Point Theory Appl., Vol. (2006), Article ID 49615, 6 pages

[6] G.A. Baker Jr., P. Graves-Morris, *Padé Approximants*, 2nd edition, in: Encyclopedia of Mathematics and its Applications, Vol. 59, Cambridge University Press, Cambridge, 1996

[7] Baker Jr. G.A., Graves-Morris P., *Padé Approximants*, 2nd Edition, Cambridge University Press, Cambridge, 1996

[8] Banach S., *Sur les operations dans les ensembles abstraits et leur applications aux equations integrales* Fund. Math., 3 (1922) 133181

[9] Bauer F.L., *La méthode d'intégration numérique de Romberg*, in Colloque sur l'Analyse Numérique, Librairie Universitaire, Louvain, (1961) 119-129

[10] Bell G.E., Phillips G.M., *Aitken acceleration of some alternating series*, BIT 24 (1984) 70-77

[11] Benazzous A., *Quasilinear sequence transformations*, Numer. Algorithms, 15 (1997) 275-285

[12] Berinde V., *Iterative Approximation of Fixed Points*, 2nd Ed., Springer Verlag, Berlin Heidelberg New York, 2007

[13] Berinde V., *On the convergence of the Ishikawa iteration in the class of quasi contractive operators*, Acta Math. Univ.Comenian., Vol. 73, 1 (2004) 119-126

[14] Berinde V., *Picard iteration converges faster than the Mann iteration in the class of quasi-contractive operators*, Fixed Point Theory Appl., 2 (2004) 97-105

[15] Berinde V., Berinde M., *The fastest Krasnoselskij iteration for approximating fixed points of strictly pseudo-contractive mappings*, Carpath. J. Math. 21 (2005) No. 1-2, 13 - 20

[16] Biazar J. and Amirteimoori A., *An improvement to the fixed point iterative method*, Appl. Math. Comput. 182 (2006) 567-571

[17] Bjørstad P., Dahlquist G., Grosse E., *Extrapolations of asymptotic expansions by a modified Aitken δ^2-formula*, BIT 21 (1981) 56-65

[18] Böhning D., *Acceleration techniques in fixed-point methods for finding percentage points*, Statist. Comput. 3 (1) (1993) 1-5

[19] Brezinski C., *Convergence acceleration during the 20th century*, J. Comput. Appl. Math. 122, No. 1-2 (2000) 1-21

[20] Brezinski C., *Accélération de la Convergence en Analyse Numérique*, Springer, Berlin, 1977

[21] Brezinski C., *Accélération de suites à convergence logarithmique*, C. R. Acad. Sci. Paris 273 A (1971) 727-730

[22] Brezinski C., *Algorithmes d'ccélération de la Convergence*, Étude Numérique, Editions Technip, Paris, 1978

[23] Brezinski C., *Etude sur les ϵ et ρ-algorithmes*, Numer. Math. 17 (1971) 153-162

[24] Brezinski C., Redivo Zaglia M., *Extrapolation methods. Theory and Practice*, North Holland, Amsterdam, 1991

[25] Brezinski, C., Redivo Zaglia, M., *Generalizations of Aitkens process for accelerating the convergence of sequences*, Comput. Appl. Math. 26, No. 2 (2007) 171-189

[26] Brezinski C., *L' ϵ-algorithme et les suites totalement monotones et oscillantes*, C.R. Acad. Sci. Paris, 276 A (1973) 305-308

[27] Brezinski C., Chehab J.P. *Nonlinear hybrid procedures and fixed points iterations*, Numer. Funct. Anal. Optim. 19 (5-6) (1998) 465-487

[28] Brezinski C., *Padé-Type Approximation and General Orthogonal Polynomials*, Birklzäuser, Basel 1980

[29] Brezinski C., *Some new convergence acceleration methods*, Math. Comput. 39 (1982) 133-145

[30] Brezinski C. ,Bultheel A., Cools R., *Some pioneers of extrapolation methods*, The Birth of Numerical Analysis, World Scientific Publ. Co., Singapore, (2009) 1-22

[31] Bogolyubov N., Kylov N., *On Rayleigh's principle in the theory of differential equations of mathematical physics and upon Euler's method in the calculus of variation* (in Russian), Acad. Sci. Ukraine (Phys. Math.) 3 (1926) 3-22

[32] Bumbariu O., *A new Aitken type method for accelerating iterative sequences*, Appl. Math. Comput., 219 (2012) 78-82

[33] Bumbariu O., *A convergence result for the B-algorithm*, Appl. Math. Sci., 6, No 77, (2012) 3821-3826

[34] Bumbariu O., Berinde V., *An empirical study of the ϵ-algorithm for accelerating numerical sequences*, Appl. Math. Sci., 6, No 24, (2012) 1181-1190

[35] Bumbariu O., Berinde V., *Empirical study of a Padé type accelerating method of Picard iteration*, Creative Math. Inform. 19, No. 2, (2010) 149-159

[36] Bumbariu O., *An acceleration technique for slowly convergent fixed point iterative methods* Miskolc Math. Notes, 13, No. 2, (2012) 271-281

[37] Bumbariu O., *A Padé type acceleration of some fixed point iterative methods* I.C.A.M.7, September 1-4, 2010, Baia Mare, Romania

[38] Bumbariu O., *Some extensions for the B-algorithm for accelerating iterative methods*, I.C.A.M.8, October 27-30, 2011, Baia Mare, Romania

[39] Bumbariu O., *Some convergent theorems for the B-algorithm*, S.Y.N.A.S.C., September 26-30, 2012, Timişoara, Romania

[40] Bumbariu O., *A Padé-type acceleration technique for solving ordinary differential equations*, Appl. Math. Sci. (submitted)

[41] Bumbariu O., *Other representations for the B-algorithm for accelerating the Picard iteration*, Rev. Anal. Numr. Thor. Approx (submitted)

[42] Chidume C.E., Mutangadura S. A., *An example on the Mann iteration method for Lipshitz pseudocontractions*, Proc. Amer. Math. Soc., 129, 8 (2001) 2359-2363

[43] Chidume C.E., *Geometric properties of Banach spaces and nonlinear iterations*, Springer-Verlag London, 2009

[44] Chun C. Ham Y., *A fifth-order iterative method for solving nonlinear equations*, Appl. Math. Comput. (2007) 287-290

[45] Clark W.D., Gray H.L., Adams J.E., *A note on the T-transformation of Lubkin*, J. Res. Natl. Bur. Stand. B 73 (1969) 25-29

[46] Collatz L., *Numerische Behandlung von Differentialgleichungen*, Springer-Verlag, Berlin, 1951

[47] Cordellier F., Démonstration algébrique de l'extension de l'identité de Wynn aux tables de Padé non normales, in *Padé Approximation and its Applications*, L. Wuytack ed., LNM vol. 765, Springer Verlag, Berlin, (1979), 36-60

[48] Cordellier F., *Sur la régularité des procédés Δ^2 d'Aitken et W de Lubkin, in: L. Wuytack (Ed.), Padé Approximation and its Applications*, Springer, Berlin, (1979), 20-35

[49] Crouzeix M. Mignot A.L., *Analyse Numérique des Equations Différentielles*, 2nd ed., Masson, Paris, 1989

[50] Dellwo D.R., *accelerated projection and iterated projection methods with application to nonlinear integral equations*, SIAM J. Numer. Anal. 28 (1) (1991) 236-250

[51] Drummond J.E., *Summing a common type of slowly convergent series of positive terms*, J. Austral. Math. Soc. B 19 (1976) 416-421

[52] Dutka J., *Richardson-extrapolation and Romberg-integration*, Historia Math., 11 (1984) 3-21

[53] Fdil A., *A new method for solving nonlinear equation with error estimations*, Appl. Numer. Math. 21 (1996) 417-429

[54] Feng X., He Y. *Parametric iterative method of second-order for solving nonlinear equation*, Appl. Math. Comput. 173 (2006) 1060-1067

[55] Fessler T., Ford W.F., Smith D.A., *HURRY: An acceleration algorithm for scalar sequences and series*, AMC Trans. Math. Software 9 (1983) 346-354

[56] Flannery B.P., Teukolsky S.A., Vetterling W.T., *Numerical recipes*, Cambridge University Press, Cambridge, 1986

[57] Graça M.M., *Acceleration of nonhyperbolic sequences of mann*, Int. J. Math. Math. Sci. 32 (9) (2002) 565-572

[58] Graves-Morris P.R., *Stabilised vector Padé approximation with application to the linear systems*, Numer. Algorithms 33 (1-4) (2003) 293-304

[59] Grotendorst J., Weniger E.J., Steinborn E.O., *Efficient evaluation of infinite series representation for overlap, two-center nuclear attraction, and Coulomb integrals using nonlinear convergence accelerators*, Phys. Rev. A 33 (1986) 3706-3726

[60] Grotendorst J., *On calculating the rate of linear convergence of non-linear transformed sequences*, SNC'11 Proceedings of the 2011 International Workshop on Symbolic-Numeric Computation 24-33

[61] Hankel H., *Ueber eine besondere Classe der symmetrieschen Determinanten*, Inaugural Diss., Universität Göttingen, 1861

[62] Håvie T., *On a modification of Romberg's algorithm*, BIT, 6 (1966) 24-30

[63] Henrici P., *Elements of numerical analysis*, Wiley, New York, 1964

[64] Hillion P. *Méthode d'Aitken itérée pour les suites oscillantes d'approximations*, C.R. Acad. Sc. Paris A 280 (1975) 1701-1704

[65] Hirayama A., Shimodaira K., Hirose H., *Takakazu Seki's Collected Works Edited with Explanations*, Osaka Kyoiku Tosho, Osaka, 1974

[66] http://en.wikipedia.org

[67] Ionescu D. V., *Ecuaţii diferenţiale şi integrale*, Editura Didactică şi Pedagogică, Bucureşti, 1964

[68] Isaacson E., Keller H.B., *Analysis of numerical methods*, Wiley, New York, 1966

[69] Ishikawa, S., *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., 44, No. 1, (1974) 147150

[70] Jamieson M.J., O'Beirne T.H.., *A note on a generalization of Aitken's* Δ^2 *transformation*, J. Phys. B 11 (1978) L31-L35

[71] Joyce D.C., *Survey of extrapolation processes in numerical analysis*, SIAM Rev., 13 (1971) 435-490

[72] Kommerell K., *Das Grenzgebiet der Elementaren und Höheren Mathematik*, Verlag Köhler, Leipzig, 1936

[73] Krasnoselskij M. A., *Two observations about the method of successive approximations*, Uspehi. Math. Nauk. 10 (1955) 123127

[74] Laurent P.J., *Un théoreme de convergence pour le procédé d'extrapolation de Richardson*, C.R. Acad. Sci. Paris, 256 (1963) 1435-1437

[75] Levin D., *Development of non-linear transformations for improving convergence of sequences*, Int. J. Comput. Math. 3 (1973) 371-388

[76] Longman I.M., *Difficulties of convergence acceleration*, in M.G. de Bruin and H. van Rossum (eds.), *Padé approximation and its applications*, Amsterdam 1980 (Springer-Verlag, Berlin, 1981) 273-289

[77] Lubkin S., *A method of summing infinite series*, J. Res. Nat. Bur. Stand. 48 (1952) 228-254

[78] Maclaurin C., *Treatise of Fluxions*, Edinburgh, 1742

[79] MacLeod A.J., *Acceleration of vector sequences by multi-dimensional Δ^2-methods*, Comm. Appl. Numer. Methods 2 (1986) 385-392

[80] McCabe J.H., Phillips G.M., *Aitken sequences and generalized Fibonacci numbers*, Math. Comp. 45 (1985) 553-558

[81] Mathews J.H., Fink D.K., *Numerical methods using mathlab*, Fourth Edition, Prentice-Hall Inc., Upper Saddle River, New Jersey, U.S.A., 2004

[82] Maxwell J.C., *A Treatise on Electricity and Magnetism*, Oxford University Press, Oxford, 1973

[83] Măruşter Şt., *Metode numerice în rezolvarea ecuaţiilor neliniare*, Editura Tehnica, Bucuresti, 1981

[84] Murray D.A., *A first course in infinitesimal calculus*, Classic reprint series, New York, 1903

[85] Milne R.M., *Extension of Huygen's approximation to a circular arc*, Math. Gaz., 2 (1903) 309-311

[86] Naegelsback H., *Studien zu F urstenau's neuer Methode der Darstellung und Darstellung und Berechnung der Wurzeln algebraischer Gleichungen durch Determinanten der Coefficienten*, Arch. Math. Phys. 59 (1876) 147-192; 61 (1877) 19-85

[87] Nievergelt Y. *Aitken's and Steffensen's accelerations in several variables*, Numer. Math. 51 (1) (1991) 295-310

[88] Noor M.A., Noor K.I., Khan W.A., Ahmad F. *On iterative methods for nonlinear equations*, Appl. Math. Comput. 183 (2006) 128-133

[89] Noor M.A., Ahmad F., Javeed S. *Two-step iterative methods for nonlinear equations*, Appl. Math. Comput. 181 (2006) 1068-1075

[90] Ortega J.M., Rheinboldt W.C., *Iterative Solution of Nonlinear Equation in Several Variables*, Academic Press, New York, 1970

[91] Ortloff H., *Asymtotisches Verhalten und Konvergenz-beschleunigung von Iterations folgen (Asymptotic behaviour and acceleration of iterative sequences)*, Numer. Math. 49 (5) (1986) 545-559

[92] Petersen B.E. *Functional analysis*, available at: http://oregonstate.edu/'peterseb

[93] Petković M.S. *Efficient halley-like methods for the inclusion of multiple zeros of polynomials*, Comput. Methods Appl. Math. 12 (3), (2012) 351-366

[94] Petković M.S. *Higher order methods for the inclusion of the multiple zeros of polynomials*, Comput. Methods Appl. Math. 15 (2), (2011) 91-108

[95] Ramos J.I., *Iterative and non-iterative methods for non-linear Volerra integro-differential equations*, App. Math. Comput., 214 (1) (2009) 287-296

[96] Ramos J.I., Garcia-López C. M., *Linearized θ-methods, I. Ordinary differential equations*, Comp. Meth. Appl. Mech. Eng., 129 (1996) 255-269

[97] Richardson L.F., *The approximate arithmetical solution by finite difference of physical problems involving differential equations, with an application to the stress in a masonry dam*, Philos. Trans. Roy. Soc. London, ser. A, 210 (1910) 307-357

[98] Richardson L.F., *The deferred approach to the limit, I: Single lattice*, Philos. Trans. Roy. Soc. London, ser. A, 226 (1927) 299-349

[99] Rhoades, B.E., *A biased discussion on fixed point theory*, Carpathian J. Math., 23 (2007) No. 1-2, 11-26

[100] Rhoads B.E., *Fixed point iterations using infinite matrices*, Trans. Amer. Math. Soc. 196 (1997) 161-176

[101] Rhoades, B.E., *Some fixed point iteration procedures*, Int. J. Math. Math. Sci., 14, No. 1, (1991) 116

[102] Romberg W., *Vereinfachte numerische Integration*, Kgl. Norske Vid. Selsk. Forsk, 28 (1955) 30-36

[103] Sablonnière P., *Comparison of four algorithm accelerating convergence of a subset of logarithmic fixed point sequences*, Numer. Algorithms, 1 (2) (1991) 177-197

[104] Sablonnière P., *Comparison of four nonlinear transforms one some classes of logarithmic fixed point sequences*, J. Comput. Appl. Math., 62 (1) (1955) 103-127

[105] Sablonnière P., *Convergence acceleration of logarithmic fixed point sequences*, J. Comput. Appl. Math., 19 (1) (1987) 55-60

[106] Schmidt J.R., *On the numerical solution of linear simultaneous equations by an iterative method*, Phil. Mag., 7 (1941) 369-383

[107] Shaefer H., *Uber die Methode Sukzessiver Approximationen*, Jber. Deutsch. Math. Verein., 59 (1957) 131-140

[108] Shanks D., *Non-linear transformations of divergent and slowly convergent sequences*, J. Math. and Phys. (Cambridge, MA) 34 (1955) 1-42

[109] Sheppard W.F., *central difference formulae*, Proc. London Math. Soc., 31 (1899) 449-488

[110] Sidi A., *Efficient implementation of minimal polynomial and reduced rank extrapolation methods*, J. Comput. Appl. Math. 36 (3) (1991) 305-337

[111] Sidi A., *Convergence properties of some nonlinear sequence transformations*, Math. Comput. 33 (1979) 315-326

[112] Sister M.C. Fasenmyer, *Some generalized hypergeometric polynomials*, Bull. Amer. Math. Soc. 53 (1947) 806-812, A good discussion of Sister Celine's technique can also be found in chapter 14 of Rainville E.D., *Special functions* (Chelsea, New York, 1960)

[113] Smith D.A., Ford W.F. *Acceleration of linear and logarithmic convergence*, SIAM J. Numer. Anal. 16 (1979) 223-240

[114] Smith D.A., Ford W.F. *Numerical comparisons of nonlinear convergence accelerators*, Math. Comp. 38 (1982) 481-499

[115] Stiefel E., *Altes und neues über numerische Quadratur*, Z. Angew Math. Mech., 41 (1961) 408-413

[116] Telyakovskiy A.S. *Regarding polynomial approximation for ordinary differential equations*, Comput. Math. Appl. 55 (2008) 1122-1128

[117] Thakkar A.J., *A technique for increasing the utility of the Wigner-Kirkwood expansion for the second virial coefficient*, Mol. Phys. 36 (1978) 887-892

[118] Tucker R.R. *The Δ^2 process and related topics*, Pacific J. Math. 22 (1967) 349-359

[119] Tucker R.R. *The Δ^2 process and related topics*, Pacific J. Math. 28 (1969) 455-463

[120] Walz G., *Asymptotics and Extrapolation*, Akademie Verlag, Berlin, 1996

[121] Walz G., *The history of extrapolation methods in numerical analysis*, Report Nr. 130, Universität Mannheim, Fakultät für Mathematik und Informatik, 1991

[122] Weniger E.J.., *Nonlinear sequence transformations for the acceleration of convergence and the summation of divergent series*, Comput. Phys. Rep. 10 (1989) 189-371

[123] Williams H.C., Shanks D., (1917-1996) Math. Comp., 66 (1997) 929-934 and *Notices of the AMS*, 44 (1997) 813-816

[124] Wimp J., *Sequence Transformations and their Applications*, Academic Press, New York, 1981

[125] Wynn P., *Acceleration technique for iterated vector and matrix problems*, Math. Comp. 16 (1962) 301-322

[126] Wynn P., *On a device for computing the em(Sn) transformation*, MTAC 10 (1956) 91-96

[127] Wynn P., *On a Procrustean technique for the numerical transformation of slowly convergent sequences and series*, Proc. Camb. Phil. Soc. 52 (1956) 663 - 671

[128] Wynn P., *On the convergence and stability of the epsilon algorithm*, SIAM J. Numer. Anal., 3 (1966) 91-122

[129] Wynn P., *Singular rules for certain non-linear algorithms*, BIT 3 (1963) 175-195

[130] Wynn P., *Upon systems of recursions which obtain among the quotients of the Padé table*, Numer. Math. 8 (1966) 264-269

[131] www.math.lsa.umich.edu/courses/216/LinkTargets/Lab4.pdf

[132] Xue Zhiqun, *Comparison of the Convergence Speed between Picard, Mann, Krasnoselskij and Ishikawa Iterations in Banach Spaces*, Fixed Point Theory Appl., Vol. 2008, ID 387056, 5 pages

[133] Zamfirescu T., *Fixed point theorems in metric spaces*, Archiv. der Mathamatick (Basel), 23 (1972) 292-298