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FACULTATEA DE ȘTIINȚE

**Ioana DĂRĂBAN (TIMIȘ)**

# PhD Thesis

# STABILITY OF FIXED POINT ITERATION PROCEDURES

**PhD Supervisor**

**Professor Vasile BERINDE, PhD**

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# TEZĂ DE DOCTORAT

# STABILITY OF FIXED POINT ITERATION PROCEDURES

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# Contents

# Introduction

The fixed point theory is a very reaching domain of nonlinear analysis, with an expansive evolution in the last decades. There are many scientific papers in the literature based on this important researching area.

The basic result from metrical fixed point theory is the Contraction Principle of Picard-Banach-Caccioppoli [**14**] and it followed an important research on fixed point theory and applications of this theory to functional equations, differential equations, integral equations etc.

The problem of solving a nonlinear equation involves approximating fixed points of a corresponding contractive type mapping. There exists several methods for approximating fixed points: Picard iteration which is the most used for strict contractive type operators, Krasnoselskij, Mann and Ishikawa iterations etc.

In practical applications, it is important to establish if these methods are numerically stable or not. A fixed point iteration is numerically stable if small modifications due to approximation during computations, will produce small modifications on the approximate value of the fixed point computed by means of this method.

The concept of stability is fundamental in various mathematical domains, such as Differential Equations, Difference Equations, Dynamical Systems, Numerical Analysis etc. Our interest is for stability theory in Discrete Dynamical Systems.

In this context, one of the concepts of stability that we use in the paper is the one considered by Harder [**60**], Harder and Hicks [**61**], [**62**], who has been systematically studied this problem.

Other stability results for several fixed point iteration procedures and for various classes of nonlinear operators were obtained by Berinde [**26**], [**27**], [**28**], [**29**], Imoru and Olatinwo [**69**], Imoru, Olatinwo and Owojori [**70**], [**104**], Olatinwo [**99**], Osilike [**110**], [**111**], Osilike and Udomene [**114**], Rhoades [**132**], [**133**] and many others.

The subject of this paper treats the problem of stability of fixed point, common fixed point, coincidence point and tripled fixed point iteration procedures, for certain class of mappings. The study material has been organized on six chapters, not including an introduction and a list of bibliographic resources, as follows:

The first chapter, **Preliminaries**, provides the terminology, basic concepts and notations from fixed point theory used in this paper. Most of the material in this chapter is taken from the monography named "Iterative Approximation of Fixed Points" of Professor Berinde [**27**]. In writing of this chapter, I also used the following bibliographical references [**1**], [**7**], [**67**], [**77**], [**79**], [**146**], [**150**].

The second chapter, **Stability of fixed point, common fixed point and coincidence point iterative procedures for mappings satisfying an explicit contractive condition**, presents the concept of stability of fixed point iteration procedures and surveys the most significant contributions to this area.

One of them was made by Berinde [**27**] who introduced a weaker and more natural notion of stability, called *weak stability*, by adopting approximate sequences instead of arbitrary sequences in the definition of stability. Following this concept, we continued to study the problem of weak stability of common fixed point iterative procedures for some classes of contractive type mappings.

The author's original contributions in this chapter are: Definition 5.19, Theorem 4.8, Theorem 4.9, Examples 6.8-6.10, Example 6.11, Example 6.12, Definition 7.22 and Theorem 7.12.

Most of them were published in [**158**] (Timiş, I., *On the weak stability of fixed point iterative methods*, presented at ICAM7, Baia Mare, 1-4 Sept. 2010), [**159**] (Timiş, I., *On the weak stability of Picard iteration for some contractive type mappings*, An. Univ. Craiova Ser. Mat. Inform. 37 (2) (2010), 106-114), [**160**] (Timiş, I., *On the weak stability of Picard iteration for some contractive type mappings and coincidence theorems*, International Journal of Computer Applications 37 (4) (2012), 9-13) and [**169**] (Timiş, I. and Berinde, V., *Weak stability of iterative procedures for some coincidence theorems*, Creative Math. Inform. 19 (2010), 85-95).

In the third chapter, **Stability of fixed point, common fixed point and coincidence point iterative procedures for contractive mappings defined**

**by implicit relations**, we study the stability of Picard iterative procedure and also of Jungck iterative procedure for common fixed points and coincidence points, for contractive mappings satisfying various implicit relations, with different number of parameters.

Several classical fixed point theorems and common fixed point theorems have been recently unified by considering general contractive conditions expressed by an implicit relation. This development has been initiated by Popa [**119**], [**120**], [**121**] and following this approach, a consistent part of the literature on fixed point, common fixed point and coincidence theorems, both for single valued and multi-valued mappings, in various ambient spaces have been accomplished.

For these new fixed point theorems did not exist corresponding stability results and Berinde [**19**], [**30**] filled this gap and established corresponding stability results for fixed point iterative procedures associated to contractive mappings defined by an implicit relation.

We continue the study of stability and the results obtained in this chapter are generalizations of fixed point theorems and stability theorems for Picard iteration existing in literature: see Berinde [**20**], [**24**], [**26**] [**27**], [**29**], [**31**], Chatterjea [**45**], Harder and Hicks [**61**], [**62**], Hardy and Rogers [**63**], Imoru and Olatinwo [**69**], Jungck [**78**], Kannan [**81**], Olatinwo [**100**], Osilike [**111**], [**110**], Ostrowski [**115**], Popa [**120**], Reich [**127**], Reich and Rus [**154**], Rhoades [**130**], [**132**], [**133**], Rus [**138**], [**139**], Zamfirescu [**173**] and most of their references.

The author's original contributions in this chapter are: Example 1.15, Theorem 1.14, Corollary 1.1, Corollary 1.2, Theorem 2.15, Examples 3.23-3.25, Examples 3.27-3.29, Theorem 3.16, Corollary 3.3 and Corollary 3.4.

Most of them were published in [**161**] (Timiş, I., *Stability of Jungck-type iterative procedure for some contractive type mappings via implicit relations*, Miskolc Math. Notes 13 (2) (2012), 555-567), [**163**] (Timiş, I., *Stability of Jungck-type iterative procedure for common fixed points and contractive mappings via implicit relations*, presented at ICAM8, Baia Mare, 27-30 Oct. 2011) and [**164**] (Timiş, I., *Stability of the Picard iterative procedure for mappings which satisfy implicit relations*, Comm. Appl. Nonlinear Anal. 19 (2012), no. 4, 37-44).

The idea of the fourth chapter, **A new point of view on the stability of fixed point iterative procedures**, is due to Professor I. A. Rus [**136**], who unified the notions of stability in difference equations, dynamical systems, differential equations, operator theory and numerical analysis by new ones.

We consider these new notions in this chapter and study the stability of Picard iteration for mappings which satisfy certain contractive conditions. We also give some illustrative examples.

The author's original contributions in this chapter are: Theorem 1.17, Proposition 1.2, Corollary 1.5, Corollary 1.6, Corollary 1.7, Example 1.33, Corollary 1.8, Theorem 2.18, Corollary 2.9, Example 2.34, Theorem 2.19, Corollary 2.10, Examples 3.35 - 3.42, Definition 4.26, Definition 4.27, Proposition 4.3, Theorem 4.20, Theorem 4.21, Theorem 5.22.

Some of them are included in [**156**] (Timiş, I., *New stability results of Picard iteration for common fixed points and contractive type mappings*, presented at SYNASC 2012, Timişoara, 26-29 Sept. 2012).

In the fifth chapter, **Stability of tripled fixed point iteration procedures**, following the results of Berinde and Borcut [**32**], [**38**] who introduced the concept of tripled fixed points, we introduce the notion of stability for tripled fixed point iterative procedures and also establish stability results for mixed monotone mappings and monotone mappings, satisfying various contractive conditions. An illustrative example is also given.

The author's original contributions in this chapter are: Definition 2.30, Theorem 2.23, Corollary 2.11, Theorem 2.24, Theorem 2.25, Lemma 3.4, Definition 3.33, Theorem 3.26, Corollary 3.12, Theorem 3.27, Theorem 3.28, Example 4.43 and the contractive conditions (2.35)-(2.40), (3.46)-(3.51).

Most of them were published in [**166**] (Timiş, I., *Stability of tripled fixed point iteration procedures for monotone mappings*, Ann. Univ. Ferrara (2012) DOI 10.1007/s11565-012-0171-7).

In the sixth chapter, **Conclusions**, we surveyed the original contributions from this thesis and we mentioned the possible research directions by following our results.

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First of all, I want to thank our Good Lord, for watching over me and enlightening me, in order to understand the useful precepts.

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I reserve for the end the most beautiful gratitude, to my husband Ilie, for his loyalty, for his support and for his love, with which he always surrounds me.

CHAPTER 1

# Preliminaries

The purpose of this chapter is to provide the terminology, basic concepts and notations from fixed point theory used in this paper.

Most of the material in this chapter is taken from the monography named "Iterative Approximation of Fixed Points" of Professor Berinde [**27**].

In writing of this chapter, I also used the following bibliographical references [**1**], [**7**], [**67**], [**77**], [**79**], [**146**], [**150**].

## 1. The background of metrical fixed point theory

Let $X$ be a nonempty set and $T : X \to X$ be a selfmap. We say that $x \in X$ is a fixed point of $T$ if

$$T(x) = x$$

and denote by $F_T$ or $Fix(T)$ the set of all fixed points of $T$.

For any given $x \in X$, we define $T^n(x)$ inductively by

$$T^0(x) = x, \quad T^{n+1}(x) = T\left(T^n(x)\right),$$

and we call it the $n^{th}$ *iterate of* $x$ *under* $T$. In order to simplify the notations, we will often use $Tx$ instead of $T(x)$.

For any $x_0 \in X$, the sequence $\{x_n\}_{n\geq 0} \subset X$ given by

$$x_n = Tx_{n-1} = T^n x_0, \quad n = 1, 2, ...$$

is called *the sequence of successive approximations with the initial value* $x_0$. It is also known as the *Picard iteration* starting at $x_0$.

For a given selfmap, the following properties obviously hold:

(1) $F_T \subset F_{T^n}$, for each $n \in \mathbb{N}^*$;

(2) $F_{T^n} = \{x\}$, for some $n \in \mathbb{N}^* \Rightarrow F_T = \{x\}$.

The fixed point theory is concerned with finding conditions on the structure that the set $X$ must be endowed as well as on the properties of the operator $T : X \to X$, in order to obtain results on:

(1) the existence and uniqueness of fixed points;
(2) the data dependence of fixed points;
(3) the construction of fixed points.

The ambient spaces $X$ involved in fixed point theory cover a variety of spaces: lattice, metric space, normed linear space, generalized metric space, uniform space, linear topological space etc., while the conditions imposed on the operator $T$ are generally metrical or compactness type conditions.

*Metric spaces*

**Definition 1.1.** *Let $X$ be a nonempty set. A mapping $d : X \times X \to \mathbb{R}_+$ is called a metric or a distance on $X$ provided that*

*(1) $d(x, y) = 0 \Leftrightarrow x = y$ ("separation axiom")*
*(2) $d(x, y) = d(y, x)$, for all $x, y \in X$ ("symmetry")*
*(3) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$ ("the triangle inequality").*

A set endowed with a metric $d$ is called *metric space* and is denoted by $(X, d)$.

**Example 1.1.**

Let $X = \mathbb{R}$. Then $d(x, y) = |x - y|$, $\forall x, y \in \mathbb{R}$, where $|\cdot|$ denotes the absolute value, is a metric (a distance) on $\mathbb{R}$.

**Example 1.2.**

(1) Let $X = \mathbb{R}^n$. Then $d(x, y) = \left[\sum_{i=1}^n (x_i - y_i)^2\right]^{\frac{1}{2}}$, $\forall x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, is a metric on $\mathbb{R}^n$, called the *euclidean metric.* The next two mappings:

$$\delta(x, y) = \sum_{i=1}^{n} |x_i - y_i|, \quad x, y \in \mathbb{R}^n,$$

$$\rho(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|, \quad x, y \in \mathbb{R}^n,$$

are also metrics on $\mathbb{R}^n$;

(2) Let $X = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$. We define $d : X \times X \to \mathbb{R}_+$ by

$$d(f,g) = \max_{x \in [a,b]} |f(x) - g(x)|, \quad \forall f, g \in X.$$

Then, $d$ is a metric on $X$, called the *Chebyshev metric* and the metric space $(X,d)$ is usually denoted by $C[a,b]$;

(3) Let $X = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$ and $\rho : X \times X \to \mathbb{R}_+$ be given by

$$\rho(f,g) = \max_{x \in [a,b]} \left( |f(x) - g(x)| \, e^{-\tau|x-x_0|} \right), \quad \forall f, g \in X,$$

where $\tau > 0$ is a constant and $x_0 \in [a,b]$ is fixed.

Then, $\rho$ is a metric on $X$, called the *Bielecki metric* and the metric space $(X,\rho)$ is usually denoted by $B[a,b]$.

**Definition 1.2.** *Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in a metric space $(X,d)$. We say that the sequence $\{x_n\}_{n=0}^{\infty}$ is convergent to $a \in X$ if, for any $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that*

$$d(x_n, a) < \epsilon, \quad \forall n \in \mathbb{N}, \ n \geq n_0.$$

**Definition 1.3.** *Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in a metric space $(X,d)$. We say that the sequence $\{x_n\}_{n=0}^{\infty}$ is fundamental or Cauchy sequence if, for any $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that*

$$d(x_n, x_{n+p}) < \epsilon, \quad \forall n \in \mathbb{N}, \ n \geq n_0, \ \forall p \in \mathbb{N}^*.$$

**Remark 1.1.** *In a metric space, any convergent sequence is a Cauchy sequence too, but the reverse is not generally true.*

**Definition 1.4.** *A metric space $(X,d)$ is called complete if any Cauchy sequence in $X$ is convergent.*

Using the metrics given in Example 1.2, the following are complete metric spaces: $(\mathbb{R}, |\cdot|)$; $(\mathbb{R}^n, d)$; $(\mathbb{R}^n, \delta)$; $(\mathbb{R}^n, \rho)$; $C[a,b]$; $B[a,b]$. On the other hand, $(\mathbb{Q}, |\cdot|)$ is not a complete metric space.

**Definition 1.5.** *Let $(X,d)$ be a metric space. A mapping $T : X \to X$ is called*

*(1) Lipschitzian if there exists $L > 0$ such that*

$$d(Tx, Ty) \leq L \cdot d(x,y), \quad \forall x, y \in X;$$

*(2) (strict) contraction (or a-contraction) if $T$ is a-Lipschitzian, with $a \in [0,1)$;*

(3) *nonexpansive, if $T$ is 1-Lipschitzian;*
(4) *contractive, if* $d(Tx, Ty) < d(x, y), \quad \forall x, y \in X, \ x \neq y;$
(5) *isometry, if* $d(Tx, Ty) = d(x, y), \quad \forall x, y \in X.$

**Example 1.3.** *(1)* $T : \mathbb{R} \to \mathbb{R}$, $T(x) = \frac{x}{2} + 3$, $x \in \mathbb{R}$, *is a strict contraction and* $F_T = \{6\}$;
*(2) The function* $T : \left[\frac{1}{2}, 2\right] \to \left[\frac{1}{2}, 2\right]$, $T(x) = \frac{1}{x}$, *is 4-Lipschitzian with* $F_T = \{1\}$;
*(3)* $T : [1, \infty] \to [1, \infty]$, $T(x) = x + \frac{1}{x}$, *is contractive and* $F_T = \emptyset$.

The following theorem is the classical method of successive approximations and is of fundamental importance in the metrical fixed point theory. It is called *contraction mapping theorem* or *Banach's theorem* or *theorem of Picard-Banach* or *theorem of Picard-Banach-Caccioppoli.*

**Theorem 1.1.** ***(Contraction mapping principle)*** *Let $(X, d)$ be a complete metric space and $T : X \to X$ be a given contraction. Then $T$ has an unique fixed point $p$, and*

$$T^n(x) \to p \ (\text{ as } \ n \to \infty \ ), \quad \forall x \in X.$$

There are various generalizations of the contraction mapping principle, roughly obtained in two ways:

(1) by weakening the contractive properties of the map and, possibly, by simultaneously giving to the space a sufficiently rich structure, in order to compensate the relaxation of the contractiveness assumptions;
(2) by extending the structure of the ambient space.

Several fixed point theorems have been also obtained by combining the two ways previously described or by adding supplementary conditions.

**Remark 1.2.** *The conclusion of Theorem 1.1 is not valid if we consider "T contractive" instead of "T strict contraction" but if we ask that $(X, d)$ is a compact metric space, then the conclusion still holds.*

*Normed spaces*

**Definition 1.6.** *Let $E$ be a real (complex) vector space. A norm on $E$ is a mapping $\|\cdot\| : E \times E \to \mathbb{R}_+$ having the following properties:*

*(1) $\|x\| = 0 \;\Leftrightarrow\; x = 0$, the null element of $E$;*

*(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$, for any $x \in E$ and any scalar $\lambda$;*

*(3) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in E$ ("the triangle inequality").*

*The pair $(E, \|\cdot\|)$ is called normed (linear) space.*

**Remark 1.3.** *If $\|\cdot\|$ is a norm on the (linear) vector space $E$, then $d : E \times E \to \mathbb{R}_+$ given by*

$$d(x, y) = \|x - y\| \,, \quad x, y \in E,$$

*is a distance on $E$. This shows that any normed space can be always regarded as a metric space with respect to the distance induced by the norm.*

**Remark 1.4.** *A Banach space is a normed space which is complete (as a metric space).*

Therefore, we deduce that all concepts related to the norm in a normed space could be adapted from the metric space setting, including the contraction mapping principle and all contractive type conditions.

*Multivalued mappings*

Let $(X, d)$ be a metric space. We denote

$P(X) = \{A \subset X \mid A \neq \emptyset\}\,, \quad P_{b,cl}(X) = \{A \subset P(X) \mid A \text{ is closed and bounded}\}$

and define the functional

$$D : P(X) \times P(X) \to \mathbb{R}_+, \quad D(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}\,.$$

We also consider the following generalized functionals:

$$\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}\,, \quad \rho(A, B) = \sup\{D(a, B) \mid a \in A\}\,,$$

$$\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}\,, \quad \delta(Y, Z) = \sup\{d(y, z) \mid y \in Y,\ z \in Z\}\,,$$

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}\,, \quad H_d(A, B) = \max\{\rho(A, B), \rho(B, A)\}\,.$$

It is well known that $H_d$ is a metric on $P_{b,cl}(X)$, commonly called *Hausdorff-Pompeiu metric*, and that, if $(X,d)$ is complete, then $(P_{b,cl}(X), H_d)$ is a complete metric space, too.

**Definition 1.7.** *Let $T : X \to P(X)$ be a multivalued operator. An element $x^* \in X$ is a fixed point of $T$ if and only if $x^* \in T(x^*)$.*

Denote, as in the single-valued case, by $F_T$ or $Fix(T)$ the set of all fixed points of $T$.

**Definition 1.8.** *Let $(X,d)$ be a metric space and $T : X \to P(X)$ be a multivalued operator. $T$ is said to be a multivalued weakly Picard operator if and only if for each $x \in X$ and any $y \in T(x)$, there exists a sequence $\{x_n\}_{n=0}^{\infty}$ such that:*

*(1) $x_0 = x$, $x_1 = y$;*
*(2) $x_{n+1} \in T(x_n)$, for all $n = 0, 1, 2, ...$;*
*(3) the sequence $\{x_n\}_{n=0}^{\infty}$ is convergent and its limit is a fixed point of $T$.*

*A sequence $\{x_n\}_{n=0}^{\infty}$ satisfying (1)-(2) is called sequence of successive approximations of a multivalued operator defined by the multivalued operator $T$ and starting values $(x,y)$.*

**Definition 1.9.** *A map $T : X \to P_{b,cl}(X)$ is called a multivalued contraction if and only if there exists a positive number $q < 1$ such that*

$$H_d(Tx, Ty) \le q d(x,y) \tag{1.1}$$

*holds for all $x, y \in X$.*

**Proposition 1.1.** *For any $A, B, C \in P_{b,cl}(X)$,*

*(1) $D(x,B) \le d(x,y)$, for any $y \in B$,*
*(2) $D(A,B) \le H_d(A,B)$,*
*(3) $H_d(A,C) \le H_d(A,B) + H_d(B,C)$.*

**Definition 1.10.** *A map $T : X \to P_{b,cl}(X)$ is said to be a generalized multivalued contraction if and only if there exists a positive number $q < 1$ such that*

$$H_d(Tx, Ty) \le q \max\left\{ d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty) + D(y,Tx)}{2} \right\} \tag{1.2}$$

*holds for all $x, y \in X$.*

The following result, usually referred as Nadler's fixed point theorem, gives a multi-valued version of the Contraction mapping principle, i.e., Theorem1.1.

**Theorem 1.2.** *Let* $(X, d)$ *be a complete metric space and* $T : X \to P_{b,cl}(X)$ *a multi-valued* $a$*-contraction, i.e., there exists a constant* $a \in (0, 1)$ *such that*

$$H_d(Tx, Ty) \leq ad(x, y),\ \forall x, y \in X.$$

*Then* $T$ *has at least one fixed point.*

*Difference inequalities*

In order to prove several convergence theorems, we shall use various elementary results concerning recurrent inequalities, as the following lemmas:

**Lemma 1.1.** *Let* $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ *be sequences of nonnegative numbers and a constant* $h$, $0 \leq h < 1$, *so that*

$$a_{n+1} \leq ha_n + b_n, \quad n \geq 0.$$

- *If* $\lim_{n\to\infty} b_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.
- *If* $\sum_{n=0}^{\infty} b_n < \infty$, *then* $\sum_{n=0}^{\infty} a_n < \infty$.

**Lemma 1.2.** *Let* $\{\epsilon_n\}_{n=0}^{\infty}$ *be a sequence of nonnegative real numbers. Then,*

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \sum_{i=0}^{n} k^{n-i}\epsilon_i = 0, \quad k \in [0, 1)\,.$$

*Commuting properties*

Let $X$ be a nonempty set and $S,\ T : X \to X$ be two operators. By definition, $S$ and $T$ are *commuting*, if $S \circ T = T \circ S$.

For this notion, in set-theoretic aspects and in order-theoretic aspects of the fixed point theory, we mention S. C. Chu and J. B. Diaz [**49**], J. B. Diaz [**56**], Z. Hedrlin [**64**] (see for example references in I. A. Rus, Teoria punctului fix în structuri algebrice, Univ. Babeş-Bolyai, 1971 [**141**]), H. Cohen [**54**], J. P. Huneke [**66**], W. M. Boyce [**40**], J. R. Jachymski (1971), A. A. Markov [**90**], S. Kakutani [**80**] (see other references in I. A. Rus, Fixed point structure theory, Cluj Univ. Press, 2006 [**137**])

As a generalization of this notion, Sessa [**146**] defined $S$ and $T$ to be *weakly commuting* if

$$d(STx, TSx) \leq d(Sx, Tx), \quad \forall\ x \in X.$$

There are several other concepts that weaken the notion of commuting mappings that were used for establishing common fixed point theorems. Here, we need the following concept, defined by Jungck [79].

**Definition 1.11.** *Let $(X, d)$ be a metric space and $S,\ T : X \to X$ be two mappings. We say that $S$ and $T$ are compatible, as a generalization of weakly commuting, if*

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

*whenever $\{x_n\}_{n=0}^{\infty}$ is a sequence in $X$ such that*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t, \quad t \in X.$$

Jungck [79] also showed that commuting implies weakly commuting which, in turn, implies compatibility property but the converse property is not true in general, as show the following illustrative example.

**Example 1.4.** *Let the functions $f(x) = x^3$ and $g(x) = 2x^3$, with $X = \mathbb{R}$. They are compatible, since*

$$|f(x) - g(x)| = \left|x^3\right| \to 0 \quad \Leftrightarrow \quad |fg(x) - gf(x)| = 6\left|x^9\right| \to 0,$$

*but the pair $(f, g)$ is not weakly commuting.*

**Definition 1.12.** *A point $x \in X$ is called a coincidence point of a pair of self-maps $S$, $T$, if there exists a point $u \in X$, usually called a point of coincidence in $X$, such that $u = Sx = Tx$.*

Moreover, Jungck [77] defined $S$ and $T$ to be *weaky compatible* if they commute at their coincidence points, i.e., if

$$Sz = Tz \quad \Rightarrow \quad STz = TSz, \quad z \in X.$$

Jungck [79] established the inclusions between these notions, respectively that the *commuting property* implies *weakly commuting property* which, in turn, implies *compatibility property* that implies *weakly compatibility property* but the reverse is not generally true.

Secondly, Aamri and Moutawakil [1] introduced a notion which is independent of the notion of weakly compatibility.

**Definition 1.13.** *$S$ and $T$ mappings satisfy (E.A) property if there exists a sequence $\{x_n\}_{n=0}^{\infty} \in X$ such that*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t, \quad \textit{for some} \quad t \in X.$$

The following example shows that a pair of mappings can satisfy the (E.A) property without being weakly compatible.

**Example 1.5.** *Let* $(\mathbb{R}_+, |\cdot|)$ *and define* $S$ *and* $T$ *by* $Sx = x^2$ *and* $Tx = x + 2$. *We have that* $Sx = Tx \Leftrightarrow x = 2$. *Let* $\{x_n\}_{n=0}^{\infty}$ *be a sequence in* $X$, *given by* $x_n = 2 + \frac{1}{n}$, $n \geq 1$. *Then,* $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 4$, *so,* $S$ *and* $T$ *satisfy property (E.A).*

*As* $ST(2) = S(4) = 16$, *and* $TS(2) = T(4) = 6$, $(S, T)$ *is not weakly compatible.*

In general, a pair satisfying (E.A) property need not follow the pattern of containment of range of one map into the range of other as it is generally utilized in proving common fixed point considerations but still it relaxes such requirements.

**Example 1.6.**

Consider $X = [-1, 1]$ with the usual metric. Define $S,\ T : X \to X$, as follows:

$$T(x) = \begin{cases} \frac{1}{2}, & x = -1, \\ \frac{x}{4}, & x \in (-1, 1), \\ \frac{3}{5}, & x = 1, \end{cases}$$

and

$$S(x) = \begin{cases} \frac{1}{2}, & x = -1, \\ \frac{x}{2}, & x \in (-1, 1), \\ \frac{-1}{2}, & x = 1. \end{cases}$$

Let the sequence $\{x_n\}_{n=0}\infty$ be given by $x_n = \frac{1}{n}$. Then,

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0,$$

so the pair $(S, T)$ satisfies (E.A) property.

The mappings $T$ and $S$ are also weakly compatible because $T(0) = S(0) = 0$ and $ST(0) = TS(0) = 0$.

On the other hand, $T(X) = \left\{\frac{1}{2}, \frac{3}{5}\right\} \cup \left(\frac{-1}{4}, \frac{1}{4}\right)$ and $S(X) = \left[\frac{-1}{2}, \frac{1}{2}\right]$. Hence, neither $T(X)$ is contained in $S(X)$ nor $S(X)$ is contained in $T(X)$.

## 2. Fixed point iteration procedures

Let $(X, d)$ be a metric space, $D \subset X$ a closed subset of $X$ (we often have $D = X$) and $T : D \to D$ a selfmap possessing at least one fixed point $p \in F_T$. For a given $x_0 \in X$ we consider the sequence of iterates $\{x_n\}_{n=0}^{\infty}$ determined by the successive iteration method

$$x_n = T(x_{n-1}) = T^n(x_0), \quad n = 1, 2, ... \tag{2.3}$$

As we already mentioned, the sequence defined by (2.3) is known as the *sequence of successive approximations* or, simply, *Picard iteration.*

Picard iteration appears to have been introduced by Liouville [**86**] and used by Cauchy. It was developed systematically for the first time by Picard [**118**] in his classical and well-known proof of the existence and uniqueness of the solution of initial value problems for ordinary differential equations, dating back in 1890.

When the contractive conditions imposed on the map T are slightly weaker, then the Picard iteration need not converge to a fixed point of the operator $T$ and some other iteration procedures must be considered.

All the next fixed point iteration schemes are introduced in a real normed space $(E, \|\cdot\|)$. Let $T : E \to E$ be a selfmap, $x_0 \in E$ and $\lambda \in ]0, 1[$. The sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \quad n = 0, 1, 2, ... \tag{2.4}$$

is called the *Krasnoselskij iteration procedure* or, simply, *Krasnoselskij iteration.*

It is easy to see that the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ given by (2.4) is exactly the Picard iteration corresponding to the averaged operator

$$T_\lambda = (1 - \lambda)I + \lambda T, \quad I = \text{ the identity operator}$$

and that for $\lambda = 1$ the Krasnoselskij iteration reduces to Picard iteration. Moreover, we have

$$Fix(T) = Fix(T_\lambda), \quad \forall \lambda \in (0, 1].$$

Krasnoselskij iteration, in the particular case $\lambda = \frac{1}{2}$, was first introduced by Krasnoselskij [**84**] in 1955 and in the general form by Schaefer [**145**] in 1957.

The normal *Mann iteration procedure* or *Mann iteration*, starting from $x_0 \in E$ is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1 - a_n)x_n + a_n T x_n, \quad n = 0, 1, 2, ... \tag{2.5}$$

where $\{a_n\}_{n=0}^{\infty} \subset ]0, 1[$ satisfies certain appropriate conditions.

If we consider

$$T_n = (1 - a_n)I + a_n T,$$

then we have $Fix(T) = Fix(T_n)$, for all $a_n \in (0, 1]$.

If the sequence $a_n = \lambda$ $(const)$, then the Mann iterative process obviously reduces to the Krasnoselskij iteration.

The original Mann iteration was defined in a matrix formulation by Mann [**89**] in 1953.

The *Ishikawa iteration scheme* or simply, *Ishikawa iteration* was introduced by Ishikawa [**71**] in order to establish the strong convergence to a fixed point for a Lipschitzian and pseudo-contractive selfmap of a convex compact subset of a Hilbert space.

It is defined by $x_0 \in X$ and

$$x_{n+1} = (1 - a_n)x_n + a_n T\left[(1 - b_n)x_n + b_n T x_n\right], \quad n = 0, 1, 2, ... \tag{2.6}$$

where $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty} \subset ]0, 1[$ satisfy certain appropriate conditions.

In the last three decades both Mann and Ishikawa schemes have been successfully used by various authors to approximate fixed points of different classes of operators in Banach spaces, see [**27**].

If we rewrite (2.6) in a system form

$$\begin{cases} y_n = (1 - b_n)x_n + b_n T x_n, \\ x_{n+1} = (1 - a_n)x_n + a_n T y_n, \quad n = 0, 1, 2, ..., \end{cases}$$

then we can regard the Ishikawa iteration as a sort of two-step Mann iteration, with two different parameter sequences.

Despite this apparent similarity and the fact that, for $b_n = 0$, Ishikawa iteration reduces to the Mann iteration, there is not a general dependence between convergence results for Mann iteration and Ishikawa iteration, see [**27**].

Some authors considered the so called *modified Mann iteration*, respectively *modified Ishikawa iteration*, by replacing the operator $T$ by its $n$-th iterate $T^n$.

For example, the modified Ishikawa iteration is defined by

$$\begin{cases} y_n = (1-b_n)x_n + b_n T^n x_n, \\ x_{n+1} = (1-a_n)x_n + a_n T^n y_n, \quad n = 0,1,2,\dots. \end{cases}$$

Recently, the so called *Ishikawa and Mann iteration procedures with errors*, for nonlinear mappings were introduced by Liu [87], [88] and by Xu [171], as follows:

(a) Let $K$ be a nonempty subset of a Banach space $E$ and $T : K \to K$ be an operator. The sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_0 \in K$ and

$$\begin{cases} y_n = (1-b_n)x_n + b_n T^n x_n + v_n, \\ x_{n+1} = (1-a_n)x_n + a_n T^n y_n + u_n, \quad n = 0,1,2,\dots, \end{cases} \tag{2.7}$$

where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are some sequences in $]0,1[$ satisfying appropriate conditions and $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are sequences in $K$ such that

$$\sum \|u_n\| < \infty, \quad \sum \|v_n\| < \infty, \tag{2.8}$$

is called *Ishikawa iteration process with errors.*

The *Mann iteration with errors* is similarly defined and could be obtained by simply taking $b_n = 0$ in (2.7).

We note from [27] that in spite of the fact that the fixed point iteration procedures are designed for numerical proposes and hence the consideration of errors is of both theoretical and practical importance, however it seems that the iteration process with errors is not quite satisfactory from a practical point of view.

Indeed, the condition (2.8) imply, in particular, that the errors tend to zero, which is not suitable for the randomness of the occurrence of errors in practical computations.

(b) Let $K$ be a nonempty convex subset of $E$ and $T : K \to K$ be a mapping. For any given $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined iteratively by

$$\begin{cases} x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n, \quad n = 0,1,2,\dots, \end{cases}$$

where $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$, $\{c'_n\}_{n=0}^{\infty}$ are sequences in the interval $(0,1)$ such that

$$a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$$

and $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ are bounded sequences in $K$, for all $n = 0,1,2,\dots$, is called the *Ishikawa iteration with errors.*

The *Mann iteration with errors* could be obtained by taking formally $b_n = b'_n = 0$, for all integers $n \geq 0$.

We also mention in the end other two fixed point iterations methods for which some stability results have been obtained by Harder and Hicks [**61**], [**62**].

The *Kirk's iteration procedure* was introduced by Kirk [**83**] and it is defined by $x_0 \in E$ and

$$x_{n+1} = \alpha_0 x_n + \alpha_1 T x_n + \alpha_2 T^2 x_n + ... + \alpha_k T^k x_n,$$

where $k$ is a fixed integer, $k \geq 1$, $\alpha_i \geq 0$, for $i = 0, 1, ..., k$, $\alpha_1 > 0$ and

$$\alpha_1 + \alpha_2 + ... + \alpha_k = 1.$$

This scheme reduces to Picard iteration, for $k = 0$ and to Krasnoselskij iteration, for $k = 1$.

The Kirk, Krasnoselskij, Mann and Ishikawa iteration procedures are mainly used to generate iterative methods for approximating fixed points of various classes of mappings in normed linear spaces, for which the Picard iteration does not converge.

Let $C$ be a closed, bounded and convex set. The sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_0 \in C$ and

$$x_n = T_n^{n^2} x_{n-1}, \quad n = 1, 2, ...,$$

where $T_n x = \frac{n}{n+1} T x$, $n \geq 1$, will be called the *Figueiredo iteration procedure.*

This iteration scheme is attributed to Figueiredo in Istratescu [**72**].

We note from [**27**] that the Figueiredo iteration converges strongly to a fixed point of nonexpansive operators $T : C \to C$.

CHAPTER 2

# Stability of fixed point, common fixed point and coincidence point iterative procedures for mappings satisfying an explicit contractive condition

This chapter presents the concept of stability of fixed point iteration procedures and surveys the most significant contributions in this area.

The concept of stability is fundamental in various mathematical domains, such as Differential Equations, Difference Equations, Dynamical Systems, Numerical Analysis etc. Our interest is for stability theory in Discrete Dynamical Systems.

In this context, one of the concepts of stability that we use in the paper is the one considered by Harder [**60**], Harder and Hicks [**61**], [**62**], who has been systematically studied this problem.

The stability of Picard iterative procedure for a fixed point equation was first studied by Ostrowski [**115**] on metric spaces. This subject was formally developed by several authors.

One of the extensions was made by Berinde [**27**] who introduced a weaker and more natural notion of stability, called *weak stability*, by adopting approximate sequences instead of arbitrary sequences in the definition of stability. Following this concept, we continued to study the problem of weak stability of common fixed point iterative procedures for some classes of contractive type mappings.

The author's original contributions in this chapter are: Definition 5.19, Theorem 4.8, Theorem 4.9, Examples 6.8-6.10, Example 6.11, Example 6.12, Definition 7.22 and Theorem 7.12.

Most of them were published in [**158**] (Timiş, I., *On the weak stability of fixed point iterative methods*, presented at ICAM7, Baia Mare, 1-4 Sept. 2010), [**159**] (Timiş, I., *On the weak stability of Picard iteration for some contractive type mappings*, An. Univ. Craiova Ser. Mat. Inform. 37 (2) (2010), 106-114), [**160**] (Timiş, I., *On the weak stability of Picard iteration for some contractive type mappings and coincidence theorems*, International Journal of Computer Applications

37 (4) (2012), 9-13) and [**169**] (Timiş, I. and Berinde, V., *Weak stability of iterative procedures for some coincidence theorems*, Creative Math. Inform. 19 (2010), 85-95).

### 1. Stability of fixed point iteration procedures

Intuitively, a fixed point iteration procedure is numerically stable if, "small" modifications in the initial data or in the data that are involved in the computation process will produce a "small" influence on the computed value of the fixed point.

Let $(X, d)$ be a metric space and we define a fixed point iteration procedure by a general relation of the form

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, ...,$$

and considering that $f(T, x_n)$ does contain all parameters that define the fixed point iteration procedure, where $T : X \to X$ is an operator and $x_0 \in X$, with $F_T \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ a sequence generated by a fixed point iteration procedure that ensure its convergence to a fixed point $p$ of $T$.

In practical applications, when calculating $\{x_n\}_{n=0}^{\infty}$, we usually follow the steps:

(1) We choose the initial approximation $x_0 \in X$;
(2) We compute $x_1 = f(T, x_0)$ but, due to various errors that occur during the computations (rounding errors, numerical approximations of functions, derivatives or integrals etc.), we do not get the exact value of $x_1$, but a different one, say $y_1$, which is however close enough to $x_1$, i.e., $y_1 \approx x_1$.
(3) Consequently, when computing $x_2 = f(T, x_1)$, we will actually compute $x_2$ as $x_2 = f(T, y_1)$ and so, instead of the theoretical value $x_2$, we will obtain in fact another value, say $y_2$, again close enough but generally different of $x_2$, i.e., $y_2 \approx x_2$, ..., and so on.

In this way, instead of the theoretical sequence $\{x_n\}_{n=0}^{\infty}$, defined by the given iterative method, we will practically obtain an approximate sequence $\{y_n\}_{n=0}^{\infty}$. We shall consider the given fixed point iteration method to be numerically **stable** if and only if, for $y_n$ close enough (in some sense) to $x_n$ at each stage, the approximate sequence $\{y_n\}_{n=0}^{\infty}$ still converges to a fixed point of $T$.

Following basically this idea, the next concept of stability was introduced.

**Definition 1.14.** [60] *Let $(X,d)$ be a metric space and $T : X \to X$ a mapping, $x_0 \in X$ and let assume that the sequence generated by the iteration procedure*

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, 2, ..., \tag{1.9}$$

*converges to a fixed point $p$ of $T$.*

*Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in $X$ and set*

$$\epsilon_n = d(y_{n+1}, f(T, y_n)), \quad n = 0, 1, 2, ...$$

*We shall say that the fixed point iteration procedure (1.9) is $T$-stable or stable with respect to $T$ if and only if*

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} y_n = p.$$

**Remark 1.5.** *We note from [27] that the Picard iteration is $T$-stable with respect to any $\alpha$-contraction $T$ and also with respect to any Zamfirescu mapping $T$, both of these results being established in the framework of a metric space setting.*

**Remark 1.6.** *It has also been shown in [27] that in a normed linear space setting, certain Mann iterations are $T$-stable with respect to any Zamfirescu mapping.*

*In the same setting, a similar result was proved for Kirk's iteration procedure, in the class of c-contractions $(0 \le c < 1)$.*

**Remark 1.7.** *One of the most general contractive definition for which corresponding stability results have been obtained in the case of Kirk, Mann and Ishikawa iteration procedures in arbitrary Banach spaces appears to be the following class of mappings: for $(X,d)$ a metric space, $T : X \to X$ is supposed to satisfy the condition*

$$d(Tx, Ty) \le ad(x,y) + Ld(x,Tx), \tag{1.10}$$

*for some $a \in [0,1)$, $L \ge 0$ and for all $x, y \in D \subset X$.*

*This condition appears in [110] and other related results may be found in [107], [132], [133].*

*We note from [27] that any a-contractive and any Zamfirescu operator satisfy (1.10).*

*However, if a mapping $T$ satisfies only (1.10), it need not have a fixed point in general. But, in the case of Zamfirescu mappings, Kannan mappings or weak contractions, if $T$ has a fixed point and satisfies (1.10), then the fixed point is unique.*

We shall present in the following some general stability results for mappings satisfying (1.10).

**Theorem 1.3.** [110] *Let $(X, d)$ be a metric space and $T : X \to X$ a mapping satisfying (1.10). Suppose $T$ has a fixed point $x^*$. Let $x_0 \in X$ and $x_{n+1} = Tx_n$, $n \geq 0$.*

*Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to $x^*$ and is stable with respect to $T$.*

**Theorem 1.4.** [110] *Let $E$ be a normed linear space and $T : E \to E$ a mapping satisfying (1.10). Suppose $T$ has a fixed point $x^*$. Let $x_0$ be arbitrary in $E$ and define*

$$z_n = (1 - \beta_n)\, x_n + \beta_n T x_n, \quad n \geq 0$$

*and*

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T z_n, \quad n \geq 0,$$

*where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$ such that $0 < \alpha \leq \alpha_n$, for some $\alpha$.*

*Let $\{y_n\}_{n=0}^{\infty}$ be a given sequence in $E$ and define*

$$s_n = (1 - \beta_n)\, y_n + \beta_n T y_n, \quad n \geq 0$$

*and*

$$\epsilon_n = \left\| y_{n+1} - (1 - \alpha_n)\, y_n - \alpha_n T s_n \right\|, \quad n \geq 0.$$

*Then $\{x_n\}_{n=0}\infty$ converges strongly to $x^*$ and is stable with respect to $T$.*

Similar results can be proved in a normed linear setting for Kirk's iteration procedure and for self-operator $T$ satisfying (1.10).

On the other hand, there are several examples of fixed point iteration procedures which are not stable with respect to certain operators.

**Remark 1.8.** *Harder and Hicks [62] showed that neither Picard iteration, nor Mann or Kirk's iterations are $T$-stable with respect to a nonexpansive self-operator of a closed convex bounded set in a Hilbert space, but the next theorem shows that Figueiredo's iteration is $T$-stable with respect to nonexpansive mappings.*

**Theorem 1.5.** [62] *Let $K$ be a closed, bounded and convex subset of a Hilbert space $H$ containing $0$. If $T : K \to K$ is a nonexpansive mapping, then for any $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$, defined by*

$$x_n = T_n^{n^2} x_{n-1}, \quad n = 1, 2, \ldots$$

*and $T_n x = \frac{n}{n+1} T x$, is $T$-stable.*

## 2. Stability of common fixed point iterative procedures

The concept of stability of common fixed point iterative procedures for a pair of mappings $(S, T)$ with a coincidence fixed point was introduced by Singh, Bhatnagar and Mishra [**151**].

Let $X$ be an arbitrary nonempty set and $(X, d)$ a metric space.

Let $S, T : X \to X$ be two mappings, such that $T(X) \subseteq S(X)$. For any $x_0 \in X$, consider the common fixed point iteration procedure

$$Sx_{n+1} = Tx_n, \quad n = 0, 1, ...,$$

which is the iterative procedure introduced by Jungck [**78**].

The common fixed point iteration procedure becomes the Picard iterative procedure when $S = I$, the identity map on $X$.

Jungck [**78**] showed that the mappings $S$ and $T$ satisfying

$$d(Tx, Ty) \le kd(Sx, Sy), \quad 0 \le k < 1, \quad \forall x, y \in X, \tag{2.11}$$

have a common fixed point in $X$, provided that $S$ and $T$ are commuting, $T(X) \subseteq S(X)$ and $S$ is continuous.

The following significantly improved version of this result is generally called the Jungck contraction principle, obtained by Singh and Prasad [**152**].

**Theorem 2.6.** [**152**] *Let $(X, d)$ be a metric space and let $S, T : X \to X$ satisfying (2.11). If $T(X) \subseteq S(X)$ and $S(X)$ or $T(X)$ is a complete subspace of $X$, then $S$ and $T$ have a coincidence point.*

*For any $x_0 \in X$, there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in $X$ such that $Sx_{n+1} = Tx_n$, $n = 0, 1, 2, ...,$ and assume that $\{Sx_n\}_{n=0}^{\infty}$ converges to $Sz$ for some $z$ in $X$ and $Sz = Tz = u$, respectively the point of coincidence of $S$ and $T$.*

*If $S$ and $T$ commute just at $z$, then $S$ and $T$ have an unique common fixed point.*

As concerns the construction of the sequences $\{Sx_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ under the procedure $Sx_{n+1} = Tx_n, \quad n = 0, 1, 2, ...,$ we may calculate $a_1 = Tx_0$ and then may proceed to solve the equation $Sx_1 = a_1$.

If the map $S$ is not an injection, then we have multiple choices for $x_1$, as $x_1 \in S^{-1}a_1$. So, in practice, instead of getting an exact sequence $\{Sx_n\}_{n=0}^{\infty}$, we get an approximative sequence $\{Sy_n\}_{n=0}^{\infty}$ and this is the main problem that stability plays a very important role in actual numerical computations.

**Definition 2.15.** [152] *Let $(X,d)$ be a metric space and let $S,\ T : X \to X$. Let $z$ to be a coincidence point of $T$ and $S$, that is, $Sz = Tz = u$.*

*For any $x_0 \in X$, the sequence $\{Sx_n\}_{n=0}^{\infty}$ generated by the general iterative procedure*

$$Sx_{n+1} = Tx_n, \quad n = 1, 2, ..., \tag{2.12}$$

*and suppose that it converges to $u \in X$. Let $\{Sy_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and set*

$$\epsilon_n = d(Sy_{n+1}, Ty_n), \quad n = 0, 1, 2, \dots .$$

*Then the iterative procedure 2.12 is $(S,T)$-stable or stable with respect to $(S,T)$ if and only if*

$$\lim_{n\to\infty} \epsilon_n = 0 \implies \lim_{n\to\infty} Sy_n = u.$$

Some authors name (2.12) to be the Jungck common fixed point iteration procedure.

Definition 2.15 reduces to that of the stability of the fixed point iterative procedure due to Harder and Hicks [61], [62] when $S = I$, the identity map on $X$.

For several examples discussing the practical aspect and theoretical importance of the stability when $S$ is the identity map on $X$ in the above definition, see Berinde [27].

## 3. Several studies about stability

As we mentioned in section 1, the first stability result for fixed point iteration procedures has been obtained by Ostrowski[115].

Harder [60] introduced the concept of stability for general fixed point iteration procedures and made a systematical study by obtaining stability results that extend Ostrowski's theorem to mappings satisfying more general contractive conditions for various fixed point iteration procedures.

Harder and Hicks [62] showed that the function iteration, for mappings $T$ satisfying various contractive definitions is $T$-stable, as well as for several iteration schemes other that function iteration. Rhoades [132] extended some of the results of Harder and Hicks [62] to an independent contractive definition and also proved stability theorems for additional iteration procedures.

Moreover, Rhoades [133] continued the study of stability results by using a more general contractive definition than the ones studied by Harder and Hicks [62]: for $(E, \|\cdot\|)$ a normed linear space, $T$ a selfmap of $E$, there exists a constant $C$, $0 \leq C < 1$ such that for each $x, y \in E$,

$$\|Tx - Ty\| \leq CM(x,y), \tag{3.13}$$

where

$$M(x,y) := \max\left\{\|x-y\|, \frac{\|x-Tx\| + \|y-Ty\|}{2}, \right.$$
$$\left. \|x-Ty\|, \|y-Tx\|\right\},$$

and then proved several stability results which are generalizations and extensions of most of the results of Harder and Hicks [62] and Rhoades [132]. Osilike [111] continued the study of stability results of iteration procedures for mappings satisfying (3.13).

Osilike and Udomene [114] gave short proofs of several stability results for fixed point iteration procedures established by Harder and Hicks [62], Rhoades [132], [133], Osilike [111], [110]. This method of proof yielded both the convergence of the sequence of iterates to the fixed point of the mappings as well as the stability of the iteration procedure. These stability results have also been applied by Imoru and Olatinwo [69], Imoru, Olatinwo and Owojori [70], [104] and some others.

Olatinwo [99] also proved stability results for two newly introduced hybrid fixed point iterative algorithms of Kirk-Ishikawa and Kirk-Mann type in normed linear space, using certain contractive condition, in extension and improvement of the results of Harder and Hicks [62], Rhoades [132], [133], Berinde [26], [27], [28], [29] and Osilike [110].

Moreover, Olatinwo [100] made generalizations and obtained first stability results using the concepts of pointwise convergence of sequences of operators and the fixed point iteration procedure was investigated for the case of two metrics.

The Ishikawa and Mann iteration processes have been studied extensively by various authors and have been successfully employed to approximate fixed points of several nonlinear operator equations in Banach spaces. Rhoades [131] compared the performance of these two iteration schemes and showed that even though they are similar, they may exhibit different behaviors for different classes of nonlinear mappings.

In its original form, the Ishikawa procedure does not include the Mann process as a special case because of the condition $0 \leq \alpha_n \leq \beta_n \leq 1$. In an effort to have an Ishikawa type iteration sheme which does include the Mann iteration process as a special case, some authors (see for example Rhoades [**131**] and Osilike [**111**]) have modified the inequality condition to read $0 \le \alpha_n, \beta_n \le 1$.

In his study of stability of iteration procedures for mappings satisfying (3.13), Rhoades [**133**] proved that Picard iteration, the Mann iteration and the iteration method of Kirk are $T$-stable. Osilike [**111**] generalized and extended these results and proved that the Ishikawa process is $T$-stable for a mapping satisfying (3.13).

For a contractive definition which is more general than the contractive definition (3.13), a mapping $T$ is said to be quasi-contractive if there exists a $k \in [0,1)$ such that

$$\|Tx - Ty\| \le k \max\{\|x-y\|, \|x-Tx\|, \|y-Ty\|,$$

$$\|x-Ty\|, \|y-Tx\|\}, \tag{3.14}$$

for all $x, y \in E$. Is is clear that condition (3.13) implies (3.14). Furthermore, it is shown in Rhoades [**130**] that the contractive definition (3.14) is one of the most general contractive-type definitions for which Picard iteration yields an unique fixed point.

Osilike [**107**] proved that certain Mann iteration procedure is $T$-stable for quasi-contractive maps in Banach spaces which are either $q$-uniformly smooth or $p$-uniformly convex. These Banach spaces include all Hilbert spaces, $L_p$ or $l_p$ spaces, $1 < p < \infty$, and Sobolev spaces, $W_m^p$, $1 < p < \infty$. Moreover, Osilike [**108**] extended these results to certain Ishikawa iteration method and included all the results of Osilike [**107**] as special cases.

Osilike [**113**] established stability results for the important class of strongly pseudo-contractive operators. Furthermore, he construct certain $T$-stable Mann and Ishikawa iteration methods which converge strongly to the fixed point of $T$. A related result dealt with the construction of stable iteration methods for the iterative approximation of solutions of nonlinear operator equations of the accretive and strongly accretive types. These stability and convergence results are improvements of several results that have appeared for fixed points of Lipschitz strong pseudo-contractions (see, for example, Chidume [**46**] and Chidume [**47**]). Furthermore, Osilike [**112**] extended all these results from real $q$-uniformly smooth Banach spaces to arbitrary real Banach spaces.

Zhou [**172**] studied the stability of the Mann and Ishikawa iteration procedures for strong pseudo-contractions without Lipschitz assumptions in real uniformly Banach spaces. Then, Fang [**58**] improved and extended the corresponding results of Osilike [**113**] and Zhou [**172**] by developing some new Ishikawa iteration procedures with errors for approximating the fixed points of strong pseudo-contractions and discussed the stability for the strong pseudocontractions without Lipschitz assumptions in uniformly smooth Banach spaces.

Then, Zhou [**174**] examined the weak stability of the Ishikawa iteration procedures for Lipschitzian and $\phi$-hemicontractive mappings in real Banach spaces, under a strict condition of the function $\phi$ defined in $\phi$-hemicontractive operators, $\lim_{n\to\infty} \frac{\phi(t)}{t} > 0$. This condition is not desirable because a lot of the strictly increasing functions $\phi$ cannot satisfy the strict requirement.

Furthermore, Huang [**65**] proved the weak $T$-stability of the Mann and Ishikawa iterative sequences with errors without the strict restriction $\lim_{n\to\infty} \frac{\phi(t)}{t} > 0$ on the Lipschitzian $\phi$-hemicontractive operators in arbitrary Banach spaces.

Jungck [**78**] generalized the Banach's contraction principle, by replacing the identity map with a continuous map, thus obtaining a common fixed point theorem. Following the Jungck's contraction principle, many authors proved general common fixed points theorems and coincidence theorems (see Imdad and Ali [**67**], Aamri and Moutawakil [**1**]).

Stability results of common fixed point iterative procedures and coincidence points were obtained by some authors. Czerwik [**55**] extended Ostrowski's classical theorem for the stability of iterative procedures to the setting of $b$-metric spaces. Then, Singh, Bhatnagar and Mishra [**151**] discussed the stability of Jungck type iterative procedures for the coincidence equation $Sx = Tx$, where $Y$ is an arbitrary nonempty set, $S, T$ are maps on $Y$ with values in a space $X$ and $T(Y) \subseteq S(Y)$. They established some stability results for Jungck and Jungck-Mann iteration procedures by employing two contractive definitions which generalized those of Osilike [**110**] but independent of that of Imoru and Olatinwo [**69**]. Furthermore, Singh and Prasad [**152**] studied the problem of stability for this coincidence equation on $b$-metric spaces.

Moreover, Olatinwo [**97**], [**101**] obtained some stability results for nonself mappings in normed linear spaces which are generalizations and extensions of Berinde [**29**], Imoru and Olatinwo [**69**], Imoru, Olatinwo and Owojori[**70**]. Olatinwo and Postolache [**106**] also studied the stability in convex metric spaces for nonself mappings satisfying certain general contractive definitions in the case of Jungck-Mann and Jungck-Ishikawa iteration procedures.

## 4. Stability results for common fixed point iteration procedures using certain classes of contractive nonself mappings

Let $(X, d)$ be a metric space, $Y \subset X$ and $S, T : Y \to X$ two nonself mappings, satisfying the following contraction condition: $\exists q \in (0, 1)$ such that

$$d(Tx, Ty) \leq q d(Sx, Sy), \quad \forall\, x, y \in Y. \tag{4.15}$$

Goebel [59] proved that $S$ and $T$ have a coincidence point in $X$ (see Buică [41]) and Jungck [77] showed that the maps $S$ and $T$ satisfying (4.15) have an unique common fixed point in a complete space $(X, d)$, provided that

(1) $T(X) \subseteq S(X)$;
(2) $S$ is continuous;
(3) $S$ and $T$ commute.

The next theorem is an improved version of the Jungck's contraction principle [77], which has been obtained by Singh and Prasad [152].

**Theorem 4.7.** [152] *Let $(X, d)$ be a metric space, $Y$ a subset of $X$ and let $S, T : Y \to X$ be two mappings satisfying (4.15).*

*If $T(Y) \subseteq S(Y)$ and $S(Y)$ or $T(Y)$ is a complete subspace of $X$, then $S$ and $T$ have a coincidence point (that is, there exists $z \in Y$, such that $Sz = Tz$).*

*Moreover, for any $x_0 \in Y$, there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in $Y$, such that*

*(1) $Sx_{n+1} = Tx_n$, $n = 0, 1, 2, ...$,*
*(2) $\{Sx_n\}_{n=0}^{\infty}$ converges to $Sz$ for some coincidence point $z$ in $Y$.*

*Further, if $Y = X$ and $S$ and $T$ commute (just) at $z$, then $S$ and $T$ have an unique common fixed point, that is, $Sz = Tz = z$.*

Starting from the stability results of Singh and Prasad [152], we study the problem of stability of common fixed point iterative procedures for some classes of contractive type mappings.

As we have seen previously, the definition of $(S, T)$-stable iterative procedures used in [152] is based on the choice of an *arbitrary* sequence $\{Sy_n\}_{n=0}^{\infty}$. But, as shown in the paper [27], it is not natural to consider an arbitrary sequence in Definition 2.15, because in this way, we do not treat the problem of stability in its general context.

Our main result in this respect is given by the next theorem, which completes Theorem 4.7 by the result regarding the $(S,T)$-stability of the Jungck type iteration procedure.

**Theorem 4.8.** *(**Timiş**, [**169**]) Let $(X,d)$ be a metric space, $Y$ a subset of $X$ and let $S,T : Y \to X$ be two mappings satisfying*

$$d(Tx,Ty) \leq qd(Sx,Sy),\ \forall x,y \in Y,\ q \in [0,1). \tag{4.16}$$

*If $T(Y) \subseteq S(Y)$ and $S(Y)$ is a complete subspace of $X$, then $S$ and $T$ have an unique coincidence point (that is, there exists $z \in Y$, such that $Sz = Tz = u$.).*

*Moreover, for any $x_0 \in Y$, there exists a sequence $\{Sx_n\}_{n=0}^{\infty} \in Y$ such that*

*(i) $Sx_{n+1} = Tx_n$, $n = 0,1,2,...,$*

*(ii) $\{Sx_n\}_{n=0}^{\infty}$ converges to $u$.*

*Let $\{Sy_n\}_{n=0}^{\infty} \subset Y$ be an approximate sequence of $\{Sx_n\}_{n=0}^{\infty}$ and define*

$$\epsilon_n = d(Sy_{n+1}, Ty_n),\ n = 0,1,2,....$$

*Then,*

*(1) $d(u,Sy_{n+1}) \leq d(u,Sx_{n+1}) + q^{n+1}d(Sx_0,Sy_0) + \sum_{r=0}^{n} q^{n-r}\epsilon_r$;*

*(2) $\lim_{n\to\infty} Sy_n = u$, if and only if $\lim_{n\to\infty} \epsilon_n = 0$, that is, the iterative procedure is $(S,T)$-stable.*

Proof. Let $x_0$ to be an arbitrary point in $Y$. Since $T(Y) \subseteq S(Y)$, we can choose $x_1 \in Y$, such that $Tx_0 = Sx_1$, in order to generate the sequence $\{Sx_n\}_{n=0}^{\infty}$, defined by (i).

If $x := x_n$ and $y := x_{n-1}$ are two successive terms of the sequence $\{Sx_n\}_{n=0}^{\infty}$, then, by (4.16), we have

$$d(Sx_{n+1}, Sx_n) = d(Tx_n, Tx_{n-1}) \leq qd(Sx_n, Sx_{n-1}). \tag{4.17}$$

Now, by induction, we obtain

$$d(Sx_{n+k}, Sx_{n+k-1}) \leq q^k d(Sx_n, Sx_{n-1}),\ n = 0,1,...,\ k = 1,2,...,$$

and then,

$$\begin{aligned} d(Sx_{n+p}, Sx_n) &\leq d(Sx_{n+p}, Sx_{n+p-1}) + \ldots + d(Sx_{n+1}, Sx_n) \leq \\ &\leq q^p d(Sx_n, Sx_{n-1}) + \ldots + qd(Sx_n, Sx_{n-1}) = \\ &= q\left(1 + \ldots + q^{p-1}\right) d(Sx_n, Sx_{n-1}) = q \cdot \frac{1-q^p}{1-q} \cdot d(Sx_n, Sx_{n-1}) < \\ &< \frac{q}{1-q} d(Sx_n, Sx_{n-1}) \leq \ldots \leq \frac{q^n}{1-q} d(Sx_1, Sx_0),\quad n = 0,1,..., \end{aligned}$$

which shows that $\{Sx_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

Since $S(Y)$ is a complete subspace of $X$, there exists $u \in S(Y)$ and $z \in Y$, such that

$$\lim_{n\to\infty} Sx_{n+1} = u = Sz.$$

Now, we shall prove that $Sz = Tz$. Indeed, from (4.17), we have

$$d(Sx_n, Tz) = d(Tx_{n-1}, Tz) \leq qd(Sx_{n-1}, Sz) \leq q^{n-1}d(Sx_1, Sz). \quad (*)$$

Letting $n \to \infty$ in $(*)$, we obtain

$$\lim_{n\to\infty} d(Sx_n, Tz) = 0,$$

which means that

$$\lim_{n\to\infty} Sx_{n+1} = Tz,$$

and hence, we get

$$Sz = Tz,$$

that is, $z$ is a coincidence point of $S$ and $T$.

Now let us show that $T$ and $S$ have a unique coincidence point. Assume there exists $z' \in Y$ such that $Tz' = Sz'$. Then, by (4.16), we get

$$d(Sz', Sz) = d(Tz', Tz) \leq qd(Sz', Sz),$$

which shows that $Sz' = Sz = u$, that is, $T$ and $S$ have a unique point of coincidence, $z$.

Now, by the triangle inequality, we have

$$d(u, Sy_{n+1}) \leq d(u, Sx_{n+1}) + qd(Sx_n, Sy_n) + \epsilon_n \leq$$

$$\leq d(u, Sx_{n+1}) + q\left[d(Tx_{n-1}, Ty_{n-1}) + d(Ty_{n-1}, Sy_n)\right] + \epsilon_n.$$

After iterating $n-1$ times this process, one yields (1).

To prove (2), first suppose that $\lim_{n\to\infty} Sy_n = u$. Then,

$$\epsilon_n = d(Sy_{n+1}, Ty_n) \leq d(Sy_{n+1}, Sx_{n+1}) + d(Tx_n, Ty_n) \leq$$

$$\leq d(Sy_{n+1}, Sx_{n+1}) + qd(Sx_n, Sy_n) \leq$$

$$\leq d(Sy_{n+1}, u) + d(u, Sx_{n+1}) + qd(Sx_n, u) + qd(u, Sy_n) \to 0, \ as \ n \to \infty,$$

because $\{Sx_n\}_{n=0}^{\infty}$ converges to $u$ and $\lim_{n\to\infty} Sy_n = u$.

Now, suppose that $\lim_{n\to\infty} \epsilon_n = 0$ and prove that $\lim_{n\to\infty} Sy_n = u$.

Then,

$$d(Sy_{n+1}, u) \leq d(Sy_{n+1}, Ty_n) + d(Tx_n, Ty_n) + d(Tx_n, u) \leq$$

$$\leq \epsilon_n + qd(Sx_n, Sy_n) + d(Sx_{n+1}, u).$$

Because $\lim_{n\to\infty} Sx_n = u$ and applying Lemma 1.2, we get the conclusion, $\lim_{n\to\infty} d(u, Sy_{n+1}) = 0$. □

**Remark 4.9.** *One can obtain the last part of the proof directly by inequality (i), without using Lemma 1.2.*

**Remark 4.10.** ***Particular cases of Theorem 4.8.***

*(1) If $Y = X$, then by Theorem 4.8, we obtain an improved result of stability for the Jungck's contraction principle, see Singh and Prasad [152].*

*(2) If $f$ $Y = X$ and $S = I$ (the identity map on $X$), then by Theorem 4.8, then we obtain an improved result of stability for Banach's contraction mapping principle, see Ostrowski [115] and Harder and Hicks [62].*

**Theorem 4.9.** *(**Timiş**, [169]) Let $(X, d)$ be a metric space, $Y$ a subset of $X$ and let $S, T : Y \to X$ be two mappings satisfying*

$$d(Tx, Ty) \leq qd(Sx, Sy) + Ld(Sx, Tx),\ \forall x, y \in Y,\ q \in (0, 1),\ L \geq 0. \tag{4.18}$$

*If $T(Y) \subseteq S(Y)$ and $S(Y)$ is a complete subspace of $X$, then $S$ and $T$ have an unique coincidence point (that is, there exists $z \in Y$, such that $Tz = Sz = u$).*

*Moreover, for any $x_0 \in Y$, there exists a sequence $\{Sx_n\}_{n=0}^{\infty} \in Y$ such that*

*(i) $Sx_{n+1} = Tx_n$, $n = 0, 1, 2, ...,$*

*(ii) $\{Sx_n\}_{n=0}^{\infty}$ converges to $u$.*

*Let $\{Sy_n\}_{n=0}^{\infty} \subset Y$ be an approximate sequence of $\{Sx_n\}_{n=0}^{\infty}$ and define*

$$\epsilon_n = d(Sy_{n+1}, Ty_n),\ n = 0, 1, 2, ....$$

*Then,*

*(1) $d(u, Sy_{n+1}) \leq d(u, Sx_{n+1}) + q^{n+1}d(Sx_0, Sy_0) + L\sum_{r=0}^{n} q^{n-r}d(Sx_r, Tx_r) + \sum_{r=0}^{n} q^{n-r}\epsilon_r$;*

*(2) $\lim_{n\to\infty} Sy_n = u$ if and only if $\lim_{n\to\infty} \epsilon_n = 0$.*

Proof. Let $x_0$ to be an arbitrary point in $Y$. Since $T(Y) \subseteq S(Y)$, we can choose $x_1 \in Y$, such that $Tx_0 = Sx_1$, in order to generate the sequence $\{Sx_n\}_{n=0}^{\infty}$, defined by (i).

If $x := x_n$ and $y := x_{n-1}$ are two successive terms of the sequence $\{Sx_n\}_{n=0}^{\infty}$, then, by (4.18), we have

$$d(Sx_{n+1}, Sx_n) = d(Tx_n, Tx_{n-1}) \leq qd(Sx_n, Sx_{n-1}) + Ld(Sx_n, Tx_n) =$$

$$= qd(Sx_n, Sx_{n-1}) + Ld(Sx_n, Sx_{n+1}),$$

so, we obtain

$$d(Sx_{n+1}, Sx_n) \leq \frac{q}{1-L} d(Sx_n, Sx_{n-1}). \tag{4.19}$$

Now, by induction, we obtain

$$d(Sx_{n+k}, Sx_{n+k-1}) \leq \left(\frac{q}{1-L}\right)^k d(Sx_n, Sx_{n-1}),\ n = 0, 1, ...,\ k = 1, 2, ...,$$

and then,

$$\begin{aligned}
d(Sx_{n+p}, Sx_n) &\leq d(Sx_{n+p}, Sx_{n+p-1}) + \ldots + d(Sx_{n+1}, Sx_n) \leq \\
&\leq \left(\frac{q}{1-L}\right)^p d(Sx_n, Sx_{n-1}) + \ldots + \frac{q}{1-L} d(Sx_n, Sx_{n-1}) = \\
&= \frac{q}{1-L}\left[1 + \ldots + \left(\frac{q}{1-L}\right)^{p-1}\right] d(Sx_n, Sx_{n-1}) = \\
&= \frac{q}{1-L} \cdot \frac{1 - \left(\frac{q}{1-L}\right)^p}{1 - \frac{q}{1-L}} \cdot d(Sx_n, Sx_{n-1}) = \\
&= \frac{q}{1-L-q} \cdot \left[1 - \left(\frac{q}{1-L}\right)^p\right] \cdot d(Sx_n, Sx_{n-1}) < \\
&< \frac{q}{1-L-q} d(Sx_n, Sx_{n-1}) \leq \ldots \leq \frac{q}{1-L-q}\left(\frac{q}{1-L}\right)^{n-1} d(Sx_1, Sx_0),
\end{aligned}$$

for $n = 0, 1, ...$, which shows that $\{Sx_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

Since $S(Y)$ is a complete subspace of $X$, there exists $u \in S(Y)$ and $z \in Y$, such that

$$\lim_{n\to\infty} Sx_{n+1} = u = Sz.$$

Now, we shall prove that $Sz = Tz$. Indeed, from (4.19), we have

$$\begin{aligned}
d(Sx_n, Tz) = d(Tx_{n-1}, Tz) &\leq qd(Sx_{n-1}, Sz) + Ld(Sx_{n-1}, Tx_{n-1}) = \\
&= qd(Sx_{n-1}, Sz) + Ld(Sx_{n-1}, Sx_n) \leq \\
&\leq qd(Sx_{n-1}, u) + L \cdot \left(\frac{q}{1-L}\right)^{n-1} d(Sx_1, Sx_0). \quad (**)
\end{aligned}$$

Letting $n \to \infty$ in $(**)$, we obtain

$$\lim_{n\to\infty} d(Sx_n, Tz) = 0,$$

which means that

$$\lim_{n\to\infty} Sx_{n+1} = Tz,$$

and hence, we get

$$Sz = Tz,$$

that is, $z$ is a coincidence point of $S$ and $T$.

Now let us show that $T$ and $S$ have a unique coincidence point. Assume there exists $z' \in Y$ such that $Tz' = Sz'$. Then, by (4.18), we get

$$d(Sz', Sz) = d(Tz', Tz) \leq qd(Sz', Sz) + Ld(Sz', Tz') = qd(Sz', Sz),$$

which shows that $Sz' = Sz = u$, that is, $T$ and $S$ have an unique coincidence point, $z$.

For any nonnegative integer $n$, we have

$$d(Sx_{n+1}, Sy_{n+1}) = d(Tx, Sy_{n+1}) \leq d(Tx_n, Ty_n) + d(Ty_n, Sy_{n+1}) \leq$$

$$\leq qd(Sx_n, Sy_n) + Ld(Sx_n, Tx_n) + \epsilon_n \leq$$

$$\leq q^2 d(Sx_{n-1}, Sy_{n-1}) + qLd(Sx_{n-1}, Tx_{n-1}) + Ld(Sx_n, Tx_n) + q\epsilon_{n-1} + \epsilon_n.$$

After iterating $n-1$ times, we obtain

$$d(Sx_{n+1}, Sy_{n+1}) \leq q^{n+1} d(Sx_0, Sy_0) + L \sum_{r=0}^{n} q^{n-r} d(Sx_r, Ty_r) + \sum_{r=0}^{n} q^{n-r} \epsilon_r.$$

Therefore,

$$d(u, Sy_{n+1}) \leq d(u, Sx_{n+1}) + d(Sx_{n+1}, Sy_{n+1}) \leq$$

$$\leq d(u, Sx_{n+1}) + q^{n+1} d(Sx_0, Sy_0) + L \sum_{r=0}^{n} q^{n-r} d(Sx_r, Ty_r) + \sum_{r=0}^{n} q^{n-r} \epsilon_r.$$

This provides (1).

In order to prove (2), assume that $\lim_{n\to\infty} Sy_n = u$. Then,

$$\epsilon_n = d(Sy_{n+1}, Ty_n) \leq d(Sy_{n+1}, Sx_{n+1}) + d(Tx_n, Ty_n) \leq$$

$$\leq d(Sy_{n+1}, Sx_{n+1}) + qd(Sx_n, Sy_n) + Ld(Sx_n, Tx_n) \leq$$

$$\leq d(Sy_{n+1}, u) + d(u, Sx_{n+1}) + qd(Sx_n, u) + qd(u, Sy_n) + Ld(Sx_n, Sx_{n+1}).$$

For $n \to \infty$, we obtain that $\epsilon_n \to 0$, since $\{Sx_n\}_{n=0}^{\infty}$ converges to $u$ and $\lim_{n\to\infty} Sy_n = u$.

Now, suppose that $\lim_{n\to\infty} \epsilon_n = 0$.

Then,

$$d(Sy_{n+1}, u) \leq d(Sy_{n+1}, Ty_n) + d(Tx_n, Ty_n) + d(Tx_n, u) \leq$$

$$\leq \epsilon_n + qd(Sx_n, Sy_n) + Ld(Sx_n, Tx_n) + d(Sx_{n+1}, u).$$

We have $d(Sx_n, Tx_n) = d(Sx_n, Sx_{n-1})$ and since $\{Sx_n\}_{n=0}^{\infty}$ converges to $u$, we get $\lim_{n\to\infty} d(Sx_n, Tx_n) = 0$ and hence, applying Lemma 1.2, we get the conclusion, $\lim_{n\to\infty} d(u, Sy_{n+1}) = 0$. $\square$

**Remark 4.11.** *One can obtain the last part of the proof directly by inequality (i), without using Lemma 1.2.*

**Remark 4.12.** ***Particular cases of Theorem 4.9.***

(1) *If $Y = X$, then by Theorem 4.9, we obtain an improved result of stability for the Jungck's contraction principle, see Singh and Prasad* [**152**].

(2) *If $Y = X$ and $S = I$ (the identity map on $X$), then by Theorem 4.9, we obtain an improved result of stability for Banach's contraction mapping principle, see Ostrowski* [**115**].

(3) *If $Y = X$ and $S = I$ (the identity map on $X$), then by Theorem 4.9, we obtain an improved result of stability for the Kannan's fixed point theorem* [**81**]*, see Harder and Hicks* [**62**].

(4) *If $Y = X$ and $S = I$ (the identity map on $X$), then by Theorem 4.9, we obtain an improved result of stability for the Zamfirescu's fixed point theorem, that is, Theorem 2 from Harder and Hicks* [**62**].

(5) *If $Y = X$ and $S = I$ (the identity map on $X$), then by Theorem 4.9, we obtain an improved result of stability for the Chatterjea's fixed point theorem* [**45**].

## 5. Weak stability concept of fixed point iteration procedures and common fixed point iteration procedures

In this section, we review some existing results on the weak stability of fixed point iteration procedures and we transpose this concept to a pair of mappings with a coincidence point.

The concept of (almost) stability is slightly not very precise because of the sequence $\{y_n\}_{n=0}^{\infty}$ which is *arbitrary* taken. From a numerical point of view, $\{y_n\}_{n=0}^{\infty}$ must be an *approximate* sequence of $\{x_n\}_{n=0}^{\infty}$.

By adopting a concept of such kind of approximate sequences, Berinde [**27**] introduced a weaker and more natural concept of stability, called *weak stability.* So, any stable iteration will be also weakly stable but the reverse is not generally true.

**Definition 5.16.** [**27**] *Let $(X, d)$ be a metric space and $\{x_n\}_{x=0}^{\infty} \subset X$ be a given sequence. We shall say that $\{y_n\}_{n=0}^{\infty} \in X$ is an approximate sequence of $\{x_n\}_{n=0}^{\infty}$*

*if, for any $k \in \mathbb{N}$, there exists $\eta = \eta(k)$ such that*

$$d(x_n, y_n) \leq \eta, \quad for\ all\ n \geq k.$$

**Remark 5.13.** *We can have approximate sequences of both convergent and divergent sequences.*

The following result will be useful in the sequel.

**Lemma 5.3.** [27] *The sequence $\{y_n\}_{n=0}^{\infty}$ is an approximate sequence of $\{x_n\}_{n=0}^{\infty}$ if and only if there exists a decreasing sequence of positive numbers $\{\eta_n\}_{n=0}^{\infty}$ converging to some $\eta \geq 0$ such that*

$$d(x_n, y_n) \leq \eta_n, \quad for\ any\ n \geq k\ (fixed)\ .$$

**Definition 5.17.** [27] *Let $(X, d)$ be a metric space and $T : X \to X$ be a map. Let $\{x_n\}_{n=0}^{\infty}$ be an iteration procedure defined by $x_0 \in X$ and*

$$x_{n+1} = f(T, x_n), \quad n \geq 0.$$

*Suppose that $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point $p$ of $T$. If for any approximate sequence $\{y_n\}_{n=0}^{\infty} \subset X$ of $\{x_n\}_{n=0}^{\infty}$*

$$\lim_{n\to\infty} d(y_{n+1}, f(T, y_n)) = 0 \quad implies \quad \lim_{n\to\infty} y_n = p,$$

*then we shall say that the iteration procedure is weakly $T$-stable or weakly stable with respect to $T$.*

**Remark 5.14.** *It is obvious that any stable iteration procedure is also weakly stable, but the reverse is generally not true.*

**Definition 5.18.** [65] *Let $\{\alpha_n\}_{n=0}^{\infty}$ be a nonnegative real sequence in $[0, 1]$. Suppose $E$ is a real Banach space and $T : E \to E$ a mapping, with $F_T \neq \emptyset$.*

*Let $x_0 \in E$ and let $\{x_n\}_{n=0}^{\infty}$ be an iteration procedure given by*

$$x_{n+1} = f\left(T, \alpha_n, x_n\right), \quad n = 0, 1, 2, ...,$$

*that converges strongly to a fixed point $x^* \in F_T$.*

*Let $\{y_n\}_{n=0}^{\infty}$ be a sequence in $E$ and $\{\epsilon_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers given by*

$$\epsilon_n = \left\| y_{n+1} - f\left(T, \alpha_n, y_n\right) \right\| .$$

*If $\sum_{n=1}^{\infty} \epsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = x^*$, then the iteration procedure is said to be almost $T$-stable or almost stable with respect to $T$.*

*If $\epsilon_n = o(\alpha_n)$ implies that $\lim_{n\to\infty} y_n = x^*$, then the iteration procedure is said to be pseudo $T$-stable with respect to $T$ or pseudo stable with respect to $T$.*

**Remark 5.15.** *It is obvious that if an iteration* $\{x_n\}_{n=1}^{\infty}$ *is* $T$*-stable, then it is weakly* $T$*-stable and if the iteration* $\{x_n\}_{n=1}^{\infty}$ *is weakly* $T$*-stable, then it is both almost* $T$*-stable and pseudo* $T$*-stable.*

*Conversely, an iteration* $\{x_n\}_{n=1}^{\infty}$ *which is either almost* $T$*-stable and pseudo* $T$*-stable may fail to be weakly* $T$*-stable. Accordingly, it is of important theoretical interest to study the weak stability.*

All examples given by various authors that have studied the stability of the fixed point iteration procedures - examples intended to illustrate non stable fixed point iteration procedures - do not consider approximate sequences of $\{x_n\}_{n=0}^{\infty}$.

Berinde [**27**] presented in detail some of the aforementioned examples, in order to show how important and natural is to restrict the stability concept to approximate sequences $\{y_n\}_{n=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$.

**Example 5.7.** [**27**]

Let $T : \mathbb{R} \to \mathbb{R}$ be given by $Tx = \frac{1}{2}x$, where $\mathbb{R}$ is endowed with the usual metric. $T$ is an $\frac{1}{2}$-contraction, $F_T = \{0\}$.

By Theorem 1.4, the Ishikawa iteration $\{x_n\}_{n=1}^{\infty}$ is $T$-stable, hence it is almost $T$-stable and weakly $T$-stable, too.

However, Osilike [**109**] claimed that the Ishikawa iteration is not $T$-stable. To show this, it was used the sequence $\{y_n\}_{n=1}^{\infty}$ given by

$$y_n = \frac{n}{1+n}, \ n \geq 0.$$

But this is obviously nonsense, because $x_n \to 0$, as $n \to \infty$, the unique fixed point of $T$, while $y_n \to 1$, as $n \to \infty$, so, by construction, $\{y_n\}_{n=1}^{\infty}$ would have to be an approximate sequence of $\{x_n\}$.

Hence, using arbitrary sequences, the Ishikawa iteration is not $T$-stable.

In the following, we transpose the concept of $(S, T)$-stability used by Singh and Prasad [**152**] to $(S, T)$-weak stability in a metric space.

**Definition 5.19.** *(**Timiş**, [**169**]) Let* $(X, d)$ *be a metric space and two mappings* $S,\ T : X \to X$ *be such as* $T(X) \subseteq S(X)$ *and let* $z$ *is a coincidence point of* $S$ *and* $T$*, that is, a point for which we have* $Sz = Tz = u \in X$*.*

*For any $x_0 \in X$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$ generated by the general iterative procedure*

$$Sx_{n+1} = f(T, x_n), \quad n = 1, 2, ..., \tag{5.20}$$

*and assume that it converges to $u$.*

*If for any approximate sequence $\{Sy_n\}_{n=0}^{\infty} \subset X$ of $\{Sx_n\}_{n=0}^{\infty}$, we have that*

$$\lim_{n\to\infty} d(Sy_{n+1}, f(T, y_n)) = 0 \quad \textit{implies} \quad \lim_{n\to\infty} Sy_n = u,$$

*then we shall say that (5.20) is weakly $(S,T)$-stable or weakly stable with respect to $(S,T)$.*

## 6. Examples of weak stable but not stable iterations

Harder and Hicks [**62**] presented some examples of mappings which satisfy various contractive conditions for which the corresponding iteration procedures are not stable.

In the following, we present some of these examples in order to study their weak stability.

We also present some examples of mappings with a coincidence point which satisfy certain contractive conditions in order to study their stability with respect to $(S,T)$.

**Example 6.8.** ***(Timiş***, [**159**]*)*

Let $T : [0,1] \to [0,1]$ be given by

$$Tx = \begin{cases} \frac{1}{2}, & x \in \left[0, \dfrac{1}{2}\right] \\ 0, & x \in \left(\dfrac{1}{2}, 1\right] \end{cases},$$

where $[0,1]$ is endowed with the usual metric. $T$ is continuous at each point of $[0,1]$ except at $\frac{1}{2}$ and $T$ has an unique fixed point at $\frac{1}{2}$, see Harder and Hicks [**62**].

As shown in [**62**], $T$ satisfies the condition

$$d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty)\}, \quad \forall\, x, y \in X, \quad x \neq y.$$

Indeed, first let $x \in \left[0, \frac{1}{2}\right]$ and $y \in \left(\frac{1}{2}, 1\right]$.

Then, $|Tx - Ty| = |\frac{1}{2} - 0| = \frac{1}{2} < \max\{|x - Tx|, |y - Ty|\} =$

$$= \max\left\{|\frac{1}{2} - x|, |y - 0|\right\} = \max\left\{\left(\frac{1}{2} - x\right), y\right\} = y.$$

Now, let $x \in \left[0, \frac{1}{2}\right]$ and $y \in \left[0, \frac{1}{2}\right]$, with $x \neq y$.

Then, $|Tx - Ty| = 0 < \max\{|x - Tx|, |y - Ty|\} =$

$$= \max\left\{|\frac{1}{2} - x|, |\frac{1}{2} - y|\right\} = \max\left\{\left(\frac{1}{2} - x\right), \left(\frac{1}{2} - y\right)\right\}.$$

If $x \in \left(\frac{1}{2}, 1\right]$ and $y \in \left(\frac{1}{2}, 1\right]$, with $x \neq y$,

$$|Tx - Ty| = 0 < \max\{|x - Tx|, |y - Ty|\} = \max\{x, y\}.$$

In order to study the $T$-stability, let $x_0$ be any point in $[0, 1]$ and $x_{n+1} = Tx_n$, for $n = 0, 1, 2, \cdots$ be the Picard iteration procedure.

Then,

$$x_1 = Tx_0 = \begin{cases} \frac{1}{2}, & x_0 \in \left[0, \dfrac{1}{2}\right] \\ 0, & x_0 \in \left(\dfrac{1}{2}, 1\right] \end{cases},$$

and $x_2 = Tx_1 = \frac{1}{2}$ for either case.

Furthermore, $x_n = \frac{1}{2}$, $\forall n \geq 2$ and hence, $\lim_{n\to\infty} x_n = \frac{1}{2} = T\left(\frac{1}{2}\right)$.

Now, let $\{y_n\}_{n=0}^{\infty} = \frac{1}{2}, \frac{1}{4}, \frac{1}{2} + \frac{1}{4^2}, \frac{1}{4^3}, \frac{1}{2} + \frac{1}{4^4}, \frac{1}{4^5}, \cdots$. Observe that $\{y_n\}_{n=0}^{\infty}$ is a divergent sequence.

If $n$ is a positive even integer, then

$$\epsilon_n = \left|\frac{1}{4^{n+1}} - T\left(\frac{1}{2} + \frac{1}{4^n}\right)\right| = \left|\frac{1}{4^{n+1}} - 0\right| = \frac{1}{4^{n+1}}.$$

If $n$ is a positive odd integer, then

$$\epsilon_n = \left|\left(\frac{1}{2} + \frac{1}{4^{n+1}}\right) - T\left(\frac{1}{4^n}\right)\right| = \frac{1}{2} + \frac{1}{4^{n+1}} - \frac{1}{2} = \frac{1}{4^{n+1}}.$$

Thus,

$$\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \frac{1}{4^{n+1}} = 0,$$

but $\{y_n\}_{n=0}^{\infty}$ does not converge to $\frac{1}{2}$. So, the Picard iteration is not $T$-stable.

In order to study the $T$-weak stability, we take an approximate sequence $\{y_n\}_{n=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$. Then, there exists a decreasing sequence of nonnegative numbers $\{\eta_n\}$ converging to some $\eta \geq 0$ for $n \to \infty$ such that

$$|x_n - y_n| \leq \eta_n, \quad n \geq k, \quad k \ fixed.$$

Then, $-\eta_n \leq x_n - y_n \leq \eta_n$ and it results that $0 \leq y_n \leq x_n + \eta_n, \quad n \geq k$.

Since $x_n = \frac{1}{2}$, for $n \geq 2$, we obtain $0 \leq y_n \leq \frac{1}{2} + \eta_n, \quad n \geq k_1 = \max\{2, k\}$.

For every choice of nonnegative $\eta_n$ we have that $0 \le y_n \le 1, \quad \forall n \ge k_1$.

For example, take $\eta_n = \frac{1}{n}$, then $y_n = \frac{1}{2} + (-1)^n \frac{1}{n}$ is an approximate sequence of $\{x_n\}_{n=0}^{\infty}$.

Since $Ty_n = 0$, if $n$ is even, and $Ty_n = \frac{1}{2}$, if $n$ is odd, it follows that $\{y_n\}_{n=0}^{\infty}$ does not converge and hence, the Picard iteration is not $T$-weakly stable.

**Example 6.9.** ***(Timiş,*** *[**159**])*

Let $T : [0,1] \to [0,1]$ be given by

$$Tx = \begin{cases} 0, & x \in \left[0, \frac{1}{2}\right] \\ \frac{1}{2}, & x \in \left(\frac{1}{2}, 1\right] \end{cases},$$

where $[0,1]$ is endowed with the usual metric. $T$ is continuous at every point of $[0,1]$ except at $\frac{1}{2}$ and 0 is the only fixed point of $T$, see [**62**].

For each $x, y \in [0,1]$, $x \neq y$, $T$ satisfies the condition

$$d(Tx, Ty) < \max\{d(x, Ty), d(y, Tx)\}.$$

Indeed, first let $x \in \left[0, \frac{1}{2}\right]$, $y \in \left[0, \frac{1}{2}\right]$ and $x \neq y$. Then, $|Tx - Ty| = 0 < \max\{x, y\} = \max\{|x - Ty|, |y - Tx|\}$.

If $x \in \left(\frac{1}{2}, 1\right]$, $y \in \left(\frac{1}{2}, 1\right]$ and $x \neq y$, then $|Tx - Ty| = 0 <$

$$< \max\left\{\left(x - \frac{1}{2}\right), \left(y - \frac{1}{2}\right)\right\} = \max\{|x - Ty|, |y - Tx|\}.$$

If $x \in \left[0, \frac{1}{2}\right]$ and $y \in \left(\frac{1}{2}, 1\right]$, then $|Tx - Ty| = \left|0 - \frac{1}{2}\right| = \frac{1}{2} <$

$$< y = \max\left\{\left(\frac{1}{2} - x\right), y\right\} = \max\{|x - Ty|, |y - Tx|\}.$$

We will show that the Picard iteration is not $T$-stable but it is $T$-weakly stable.

In order to prove the first claim, let $\{y_n\}_{n=0}^{\infty}$ be given by

$$y_n = \frac{n+2}{2n}, \quad n \ge 1.$$

Then,

$$\epsilon_n = |y_{n+1} - f(T, y_n)| = |y_{n+1} - Ty_n| = \left|\frac{n+3}{2(n+1)} - \frac{1}{2}\right|,$$

because $y_n \ge \frac{1}{2}$, for $n \ge 1$.

Therefore, $\lim_{n\to\infty} \epsilon_n = 0$ but $\lim_{n\to\infty} y_n = \frac{1}{2}$, so the Picard iteration is not $T$-stable.

In order to show the $T$-weak stability, we take an approximate sequence $\{y_n\}_{n=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$. Then, there exists a decreasing sequence of nonnegative numbers $\{\eta_n\}_{n=0}^{\infty}$ converging to some $\eta \geq 0$ for $n \to \infty$ such that

$$|x_n - y_n| \leq \eta_n, \quad n \geq k.$$

Then, $-\eta_n \leq x_n - y_n \leq \eta_n$ and it results that $0 \leq y_n \leq x_n + \eta_n, \quad n \geq k$.

Since $x_n = 0$, for $n \geq 2$, we obtain $0 \leq y_n \leq \eta_n, \quad n \geq k_1 = \max\{2, k\}$.

We can choose $\{\eta_n\}$ such that $\eta_n \leq \frac{1}{2}$, $n \geq k_1$ and therefore $0 \leq y_n \leq \frac{1}{2}, \quad \forall n \geq k_1$. So, $Ty_n = 0$ and it results that $\epsilon_n = |y_{n+1} - Ty_n| = |y_{n+1}| = y_{n+1}$.

Now, it is obvious that $\lim_{n\to\infty} \epsilon_n = 0 \quad \Rightarrow \quad \lim_{n\to\infty} y_n = 0$, so the iteration $\{y_n\}_{n=0}^{\infty}$ is $T$-weakly stable.

**Example 6.10.** *(**Timiş**, [**159**])*

Let $T : \mathbb{R} \to \left\{0, \frac{1}{4}, \frac{1}{2}\right\}$ be defined by

$$Tx = \begin{cases} \frac{1}{2}, & x < 0 \\ \frac{1}{4}, & x \in \left[0, \dfrac{1}{2}\right] \\ 0, & x > \frac{1}{2} \end{cases},$$

where $\mathbb{R}$ is endowed with the usual metric. $T$ is continuous at every point in $\mathbb{R}$ except at 0 and $\frac{1}{2}$. The only fixed point of $T$ is $\frac{1}{4}$, see [**62**].

For each $x, y \in \mathbb{R}$, $x \neq y$, $T$ satisfies the condition

$$d(Tx, Ty) < \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Indeed, first choose $x < 0$ and $y \in \left[0, \frac{1}{2}\right]$.

Then, $|Tx - Ty| = |\frac{1}{2} - \frac{1}{4}| = \frac{1}{4}$ and $\frac{|x-Tx|+|y-Ty|}{2} \geq \frac{1}{2}\left|x - \frac{1}{2}\right| = \frac{1}{4} - \frac{x}{2} > \frac{1}{4}$.

If $x < 0$ and $y > \frac{1}{2}$, then $|Tx - Ty| = |\frac{1}{2} - 0| = \frac{1}{2}$ and

$$\frac{|x - Tx| + |y - Ty|}{2} = \frac{|x - \frac{1}{2}| + |y - 0|}{2} \geq \frac{1}{4} + \frac{y - x}{2} > \frac{1}{4} + \frac{1}{4} > \frac{1}{2}.$$

If $x \in \left[0, \frac{1}{2}\right]$ and $y > \frac{1}{2}$, then $|Tx - Ty| = |\frac{1}{4} - 0| = \frac{1}{4}$ and

$$\frac{|x - Tx| + |y - Ty|}{2} = \frac{|x - \frac{1}{4}| + y}{2} \geq \frac{y}{2} > \frac{1}{4}.$$

If $x < 0$, $y < 0$ and $x \neq y$, then $|Tx - Ty| = 0 < |x - y|$.

If $x > \frac{1}{2}$, $y > \frac{1}{2}$ and $x \neq y$, then $|Tx - Ty| = 0 < |x - y|$.

If $x \in \left[0, \frac{1}{2}\right]$, $y \in \left[0, \frac{1}{2}\right]$ and $x \neq y$, then $|Tx - Ty| = 0 < |x - y|$.

Thus, $|Tx - Ty| < \max\left\{|x-y|, \frac{|x-Tx|+|y-Ty|}{2}, \frac{|x-Ty|+|y-Tx|}{2}\right\}$, for each $x, y \in \mathbb{R}$ such that $x \neq y$.

In order to study the $T$-stability of Picard iteration procedure associated to $T$, let $x_0$ be any real number and $x_{n+1} = Tx_n$, for $n = 0, 1, 2, \cdots$, be the Picard iteration procedure starting at $x_0$.

Then,

$$x_1 = Tx_0 = \begin{cases} \frac{1}{2}, & x_0 < 0 \\ \frac{1}{4}, & x_0 \in \left[0, \frac{1}{2}\right] \\ 0, & x_0 > \frac{1}{2} \end{cases} .$$

In either case, $x_2 = Tx_1 = \frac{1}{4}$ and hence, $x_n = \frac{1}{4}$, $\forall n \geq 2$. So, $\lim_{n\to\infty} x_n = \frac{1}{4} = T\left(\frac{1}{4}\right)$.

To show that the Picard iteration is not $T$-stable, let $\{y_n\}_{n=0}^{\infty}$ be the sequence of real numbers such that $y_0 = x_0$, $y_n = \frac{1}{2} + \frac{1}{n}$, for each positive odd integer and $y_n = -\frac{1}{n}$, for each positive even integer.

If $n$ is a positive even integer, then

$$\epsilon_n = \left|\frac{1}{2} + \frac{1}{n+1} - T\left(-\frac{1}{n}\right)\right| = \left|\frac{1}{2} + \frac{1}{n+1} - \frac{1}{2}\right| = \frac{1}{n+1}.$$

If $n$ is a positive odd integer, then

$$\epsilon_n = \left|\left(-\frac{1}{n+1}\right) - T\left(\frac{1}{2} + \frac{1}{n}\right)\right| = \left|-\frac{1}{n+1} - 0\right| = \frac{1}{n+1}.$$

Thus,

$$\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \frac{1}{n+1} = 0,$$

but $\{y_n\}_{n=0}^{\infty}$ does not converge to $\frac{1}{4}$. So, the Picard iteration is not $T$-stable.

Now, in order to study the $T$-weak stability, we take an approximate sequence $\{y_n\}_{n=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$.

Then, there exists a decreasing sequence of nonnegative numbers $\{\eta_n\}$ converging to some $\eta \geq 0$ for $n \to \infty$ such that

$$|x_n - y_n| \leq \eta_n, \quad n \geq k.$$

Then, $-\eta_n \leq x_n - y_n \leq \eta_n$ and it results that $0 \leq y_n \leq x_n + \eta_n, \quad n \geq k$.

Since $x_n = \frac{1}{4}$, for $n \geq 2$, we obtain $0 \leq y_n \leq \frac{1}{4} + \eta_n, \quad n \geq k_1 = \max\{2, k\}$.

We can choose $\{\eta_n\}$ such that $\eta_n \leq \frac{1}{4}$, $n \geq k_1$ and therefore $0 \leq y_n \leq \frac{1}{2}$, $\forall n \geq k_1$. So, $Ty_n = \frac{1}{4}$ and by $\lim_{n\to\infty} |y_{n+1} - Ty_n| = 0$ it results that $\lim_{n\to\infty} y_n = \frac{1}{4} = T\left(\frac{1}{4}\right)$.

This shows that the Picard iteration is weakly $T$-stable.

**Example 6.11.** ***(Timiş,*** *[**160**])*

Let $S,\ T : [0,1] \to [0,1]$ be given by

$$Tx = \begin{cases} 0, & x \in \left[0, \frac{1}{2}\right] \\ \frac{1}{2}, & x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

and

$$Sx = \begin{cases} \frac{1}{2} - x, & x \in \left[0, \frac{1}{2}\right] \\ x - \frac{1}{4}, & x \in \left(\frac{1}{2}, 1\right] \end{cases},$$

where $[0,1]$ is endowed with the usual metric. $S$ and $T$ are continuous at every point of $[0,1]$ except at $\frac{1}{2}$ which is their coincidence point, i.e., $T\left(\frac{1}{2}\right) = S\left(\frac{1}{2}\right) = 0 = u$ and $T([0,1]) = \left\{0, \frac{1}{2}\right\} \subseteq S([0,1]) = \left[0, \frac{1}{2}\right] \cup \left(\frac{1}{4}, \frac{3}{4}\right] = \left[0, \frac{3}{4}\right]$.

For each $x, y \in [0,1]$, $x \neq y$, $T$ and $S$ satisfy the condition

$$d(Tx, Ty) < \max\{d(Sx, Ty), d(Sy, Tx)\}.$$

Indeed, first let $x \in \left[0, \frac{1}{2}\right]$, $y \in \left[0, \frac{1}{2}\right]$ and $x \neq y$. Then,

$$|Tx - Ty| = 0 < \max\left\{\left|\frac{1}{2} - x - 0\right|, \left|\frac{1}{2} - y - 0\right|\right\} =$$

$$= \max\left\{\left|\frac{1}{2} - x\right|, \left|\frac{1}{2} - y\right|\right\} \neq 0,$$

since $y \neq x$.

If $x \in \left(\frac{1}{2}, 1\right]$, $y \in \left(\frac{1}{2}, 1\right]$ and $x \neq y$, then

$$|Tx - Ty| = 0 < \max\left\{\left|x - \frac{1}{4} - \frac{1}{2}\right|, \left|y - \frac{1}{4} - \frac{1}{2}\right|\right\} =$$

$$= \max\left\{\left|x - \frac{3}{4}\right|, \left|y - \frac{3}{4}\right|\right\} \neq 0,$$

since $y \neq x$.

If $x \in \left[0, \frac{1}{2}\right]$ and $y \in \left(\frac{1}{2}, 1\right]$, then

$$|Tx - Ty| = \left|0 - \frac{1}{2}\right| = \frac{1}{2} < \max\left\{\left|\frac{1}{2} - x - \frac{1}{2}\right|, \left|y - \frac{1}{4} - 0\right|\right\} =$$

$\max\left\{|x|, \left|y - \frac{1}{4}\right|\right\} \neq 0$, since we cannot have simultaneously $x = 0$ and $y = \frac{1}{4}$.

We will show that the Picard iteration is not $(S,T)$-stable nor $(S,T)$-weakly stable.

In order to prove the first claim, let $(Sy_n)$, with $Sy_n = \frac{n+2}{2n}, \quad n \geq 1$.

Then
$$\epsilon_n = |Sy_{n+1} - Ty_n| = |\frac{n+3}{2(n+1)} - \frac{1}{4} - \frac{1}{2}|,$$
because $Sy_n > \frac{1}{2}$, for $n \geq 1$.

According to Definition 2.15, assuming that $\lim_{n\to\infty} \epsilon_n = 0$, we should obtain that $\lim_{n\to\infty} Sy_n = \frac{3}{4}$ but in fact, $\lim_{n\to\infty} Sy_n = \frac{1}{2}$, so the Picard iteration is not $(S,T)$-stable.

Studying the $(S,T)$-weak stability, we take an approximate sequence $\{Sy_n\}_{n=0}^{\infty}$ of $Sx_n$. Then, there exists a decreasing sequence of nonnegative numbers $\{\eta_n\}$ converging to some $\eta \geq 0$ for $n \to \infty$ such that
$$|Sx_n - Sy_n| \leq \eta_n, \quad n \geq k.$$

Then, $-\eta_n \leq Sx_n - Sy_n \leq \eta_n$ and it results that $0 \leq Sy_n \leq Sx_n + \eta_n, \quad n \geq k$.

If $x_0 \in \left[0, \frac{1}{2}\right]$, then $Sx_1 = Tx_0 = 0$, therefore $Sx_n = 0, \quad \forall n \geq 1$. On the other hand, if $x_0 \in \left(\frac{1}{2}, 1\right]$, then $Sx_1 = Tx_0 = \frac{1}{2}$ and $Sx_2 = Tx_1 = 0$, so $Sx_n = 0, \quad \forall n \geq 2$.

If $x_n \in \left[0, \frac{1}{2}\right]$, then $Sx_n = \frac{1}{2} - x_n$. So, $0 \leq x_n \leq \frac{1}{2} \quad \Leftrightarrow \quad 0 \geq -x_n \geq -\frac{1}{2} \quad \Leftrightarrow \quad \frac{1}{2} \geq \frac{1}{2} - x_n \geq 0 \quad \Leftrightarrow \quad 0 \leq \frac{1}{2} - x_n = Sx_n \leq \frac{1}{2}$. Hence, in this situation, $Sx_n$ can have the value of 0. If $x_n \in \left(\frac{1}{2}, 1\right]$, then $Sx_n = x_n - \frac{1}{4}$. So, $\frac{1}{2} < x_n \leq 1 \quad \Leftrightarrow \quad \frac{1}{4} < x_n - \frac{1}{4} = Sx_n \leq \frac{3}{4}$. In this case, $Sx_n$ can not be 0. Therefore, $x_n \in \left[0, \frac{1}{2}\right]$ and then, $Tx_n = 0$.

Since $Sx_n = 0$, for $n \geq 2$, we obtain that $0 \leq Sy_n \leq \eta_n, \quad n \geq k_1 = \max\{2, k\}$. We can choose $\{\eta_n\}$ such that $\eta_n \leq \frac{1}{2}$, $n \geq k_1$ and therefore $0 \leq Sy_n \leq \frac{1}{2}, \quad \forall n \geq k_1$.

If $y_n \in \left[0, \frac{1}{2}\right]$, then $Sy_n = \frac{1}{2} - y_n$, so $0 \leq y_n \leq \frac{1}{2} \quad \Leftrightarrow \quad 0 \geq -y_n \geq -\frac{1}{2} \quad \Leftrightarrow \quad -\frac{1}{2} \leq -y_n \leq 0 \quad \Leftrightarrow \quad 0 \leq \frac{1}{2} - y_n = Sy_n \leq \frac{1}{2}$, situation that can be possible. In this case, for $y_n \in \left[0, \frac{1}{2}\right]$, we have that $Ty_n = 0$.

If $y_n \in \left(\frac{1}{4}, \frac{3}{4}\right] \cap \left(\frac{1}{2}, 1\right] = \left(\frac{1}{2}, \frac{3}{4}\right]$, then $Sy_n = y_n - \frac{1}{4}$, so $\frac{1}{2} < y_n \leq \frac{3}{4} \quad \Leftrightarrow \quad \frac{1}{4} < y_n - \frac{1}{4} = Sy_n \leq \frac{1}{2}$ and this can be possible, too. Hence, for $y_n \in \left(\frac{1}{2}, \frac{3}{4}\right]$, we have that $Ty_n = \frac{1}{2}$.

According to Definition 5.19, if $d\left(Sy_{n+1}, Ty_n\right) \to 0$, as $n \to \infty$, implies that $d\left(Sy_n, u\right) \to 0$, for $n \to \infty$, the $(S,T)$-weak stability should be obtained.

But, if $y_n \in \left(\frac{1}{2}, \frac{3}{4}\right]$, then from $d\left(Sy_{n+1}, Ty_n\right) = d\left(Sy_{n+1}, \frac{1}{2}\right) \to 0$, as $n \to \infty$, we obtain that $Sy_{n+1} \to \frac{1}{2}$, so $Sy_n \to \frac{1}{2}$. Therefore, the Picard iteration is not $(S,T)$-weakly stable.

**Example 6.12.** ***(Timiş***, [160]*)*

Let $S,\ T:[0,1]\to[0,1]$ be given by

$$Tx=\begin{cases}\frac{x+1}{2}, & x\in\left[0,\frac{1}{2}\right]\\ \frac{1}{2}, & x\in\left(\frac{1}{2},1\right]\end{cases}$$

and

$$Sx=\begin{cases}\frac{1}{2}-x, & x\in\left[0,\frac{1}{2}\right]\\ x-\frac{1}{4}, & x\in\left(\frac{1}{2},1\right]\end{cases},$$

where $[0,1]$ is endowed with the usual metric. $S$ and $T$ have a two coincidence points, i.e., $T(0)=S(0)=T\left(\frac{3}{4}\right)=S\left(\frac{3}{4}\right)=\frac{1}{2}=u$ and $T([0,1])=\left[\frac{1}{2},\frac{\frac{1}{2}+1}{2}\right]\cup\left\{\frac{1}{2}\right\}=\left[\frac{1}{2},\frac{3}{4}\right]\subseteq S([0,1])=\left[0,\frac{1}{2}\right]\cup\left(\frac{1}{4},\frac{3}{4}\right]=\left[0,\frac{3}{4}\right]$.

For each $x,y\in[0,1]$, $x\neq y$, $T$ and $S$ satisfy the condition

$$d(Tx,Ty)<\max\{d(Sx,Ty),d(Sy,Tx)\}.$$

Indeed, first let $x,\ y\in\left[0,\frac{1}{2}\right]$, with $x\neq y$. Then,

$$|Tx-Ty|=\left|\frac{x}{2}+\frac{1}{2}-\frac{y}{2}-\frac{1}{2}\right|=\frac{1}{2}|x-y|=\left[0,\frac{1}{4}\right]<$$

$$<\max\left\{\left|\frac{1}{2}-x-\frac{y}{2}-\frac{1}{2}\right|,\left|\frac{1}{2}-y-\frac{x}{2}-\frac{1}{2}\right|\right\}=$$

$=\max\left\{\left|x+\frac{y}{2}\right|,\left|y+\frac{x}{2},\right|\right\}=\max\left\{\left[0,\frac{1}{2}\right]+\left[0,\frac{1}{4}\right],\left[0,\frac{1}{2}\right]+\left[0,\frac{1}{4}\right]\right\}=$
$=\max\left\{\left[0,\frac{3}{4}\right],\left[0,\frac{3}{4}\right]\right\}=\left[0,\frac{3}{4}\right]$.

If $x,\ y\in\left(\frac{1}{2},1\right]$ and $x\neq y$, then $|Tx-Ty|=0<$

$$<\max\left\{\left|x-\frac{1}{4}-\frac{1}{2}\right|,\left|y-\frac{1}{4}-\frac{1}{2}\right|\right\}=$$

$$=\max\left\{\left|x-\frac{3}{4}\right|,\left|y-\frac{3}{4}\right|\right\}=\left[0,\frac{1}{4}\right].$$

If $x\in\left[0,\frac{1}{2}\right]$ and $y\in\left(\frac{1}{2},1\right]$, then $|Tx-Ty|=\left|\frac{x}{2}+\frac{1}{2}-\frac{1}{2}\right|=\frac{1}{2}|x|=\left[0,\frac{1}{4}\right]<$ $\max\left\{\left|\frac{1}{2}-x-\frac{1}{2}\right|,\left|y-\frac{1}{4}-\frac{x}{2}-\frac{1}{2}\right|\right\}=\max\left\{|x|,\left|y-\frac{x}{2}-\frac{3}{4}\right|\right\}=$
$=\max\left\{\left[0,\frac{1}{2}\right],\left(\frac{1}{2},1\right]-\left[0,\frac{1}{4}\right]-\frac{3}{4}=\left[0,\frac{1}{4}\right]\right\}=\left[0,\frac{1}{2}\right]$.

We will show that the Picard iteration is not $(S,T)$-stable nor $(S,T)$-weakly stable.

In order to prove the first claim, let $(Sy_n)$, with $Sy_n=\frac{n+2}{2n},\ \ n\geq 1$.

Then
$$\epsilon_n = |Sy_{n+1} - Ty_n| = |\frac{n+3}{2(n+1)} - \frac{1}{4} - \frac{1}{2}|,$$
because $Sy_n > \frac{1}{2}$, for $n \geq 1$.

According to Definition 2.15, assuming that $\lim_{n\to\infty} \epsilon_n = 0$, we should obtain that $\lim_{n\to\infty} Sy_n = \frac{3}{4}$ but in fact, $\lim_{n\to\infty} Sy_n = \frac{1}{2}$, so the Picard iteration is not $(S,T)$-stable.

Studying the $(S,T)$-weak stability, according to Definition 5.19, for any $x_0 \in [0,1]$, the sequence $\{Sx_n\}_{n=0}^{\infty}$ generated by the iterative procedure $Sx_{n+1} = Tx_n$, $n > 0$, converges to $u = \frac{1}{2}$.

Indeed, if $x_0 \in \left[0, \frac{1}{2}\right]$, then $Sx_1 = Tx_0 = \frac{x_0+1}{2} \in \frac{\left[0,\frac{1}{2}\right]+1}{2} = \frac{\left[1,\frac{3}{2}\right]}{2} = \left[\frac{1}{2}, \frac{3}{4}\right]$. Now, if $x_1 \in \left[0, \frac{1}{2}\right]$, then $Sx_1 = \frac{1}{2} - x_1 \in \frac{1}{2} - \left[0, \frac{1}{2}\right] = \left[0, \frac{1}{2}\right]$. Only for $x_1 = 0$, we have that $Sx_1 = \frac{1}{2} \in \left[\frac{1}{2}, \frac{3}{4}\right]$. Hence, $Sx_2 = Tx_1 = \frac{1}{2}$, so $Sx_n = Tx_n = \frac{1}{2}$, $\forall n \geq 2$. On the other hand, if $x_1 \in \left(\frac{1}{2}, 1\right]$, then $Sx_1 = x_1 - \frac{1}{4} \in \left(\frac{1}{2}, 1\right] - \frac{1}{4} = \left(\frac{1}{4}, \frac{3}{4}\right]$. Only for $x_1 \in \left(\frac{3}{4}, 1\right]$, we have that $Sx_1 \in \left(\frac{1}{2}, \frac{3}{4}\right] \in \left[\frac{1}{2}, \frac{3}{4}\right]$. Hence, $Sx_2 = Tx_1 = \frac{1}{2}$, so $Sx_n = Tx_n = \frac{1}{2}$, $\forall n \geq 2$.

If $x_0 \in \left(\frac{1}{2}, 1\right]$, then $Sx_1 = Tx_0 = \frac{1}{2}$, so $Sx_n = Tx_n = \frac{1}{2}$, $\forall n \geq 1$.

We take an approximate sequence $\{Sy_n\}_{n=0}^{\infty}$ of $Sx_n$. Then, there exists a decreasing sequence of nonnegative numbers $\{\eta_n\}$ converging to some $\eta \geq 0$ for $n \to \infty$ such that
$$|Sx_n - Sy_n| \leq \eta_n, \quad n \geq k.$$

Then, $-\eta_n \leq Sx_n - Sy_n \leq \eta_n$ and it results that $0 \leq Sy_n \leq Sx_n + \eta_n$, $\ n \geq k$. Since $Sx_n = \frac{1}{2}$, for $n \geq 2$, we obtain that $0 \leq Sy_n \leq \frac{1}{2} + \eta_n$, $\ n \geq k_1 = \max\{2, k\}$. We can choose $\{\eta_n\}$ such that $\eta_n \leq \frac{1}{4}$, $n \geq k_1$ and therefore $0 \leq Sy_n \leq \frac{3}{4}$, $\ \forall n \geq k_1$.

According to Definition 5.19, if $d\left(Sy_{n+1}, Ty_n\right) \to 0$, as $n \to \infty$, implies that $d\left(Sy_n, u\right) \to 0$, for $n \to \infty$, the $(S,T)$-weak stability should be obtained.

If $y_n \in \left[0, \frac{1}{2}\right]$, then $Sy_n = \frac{1}{2} - y_n = \frac{1}{2} - \left[0, \frac{1}{2}\right] = \left[0, \frac{1}{2}\right] \in \left[0, \frac{3}{4}\right]$ and $Ty_n = \frac{y_n+1}{2} \in \frac{1}{2}\left[1, \frac{3}{2}\right] = \left[\frac{1}{2}, \frac{3}{4}\right]$. Therefore, $d\left(Sy_{n+1}, Ty_n\right) = \left|\left[0, \frac{1}{2}\right] - \left[\frac{1}{2}, \frac{3}{4}\right]\right| = \left[\frac{1}{4}, \frac{1}{2}\right]$ and then $\lim_{n\to\infty} d\left(Sy_{n+1}, Ty_n\right)$ can not be 0. So, in this situation, the Picard iteration is not $(S,T)$-weak stable.

## 7. Stability and weak stability of fixed point iterative procedures for multivalued mappings

By extending the contraction mapping principle form single-valued mappings to multivalued mappings, Nadler [94] proved that a multivalued contraction on a complete metric space has a fixed point. Ciric [52] extended this result for generalized multivalued contractions on metric spaces.

The concept of weak contraction from the case of single-valued mappings was extended to multi-valued mappings and then corresponding convergence theorems for the Picard iteration associated to a multi-valued weak contraction are obtained. M. Berinde and V. Berinde [15] extended, improved and unified a multitude of classical results in the fixed point theory of single and multi-valued contractive mappings.

On the other hand, Singh and Chadha [153] extended Ostrowski's stability theorem (Theorem 7.10, in this paper) to multivalued contractions using Nadler's theorem and introduced the following definition of stability of iterative procedures for multivalued maps.

**Definition 7.20.** [153] *Let $X$ be a metric space and $T : X \to P_{b,cl}(X)$. Let $x_0 \in X$ and $x_{n+1} \in Tx_n$ denote the Picard iterative procedure for $T$.*

*Let $\{x_n\}_{n=0}^{\infty}$ be convergent to a fixed point $u$ of $T$ and $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence.*

*Set $\epsilon_n = H(y_{n+1}, Ty_n)$, $n = 0, 1, 2, \ldots$.*

*The iterative procedure $Tx_n$ is said to be $T$-stable provided that*

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \textit{implies} \quad \lim_{n\to\infty} y_n = u.$$

The first result on the stability of Picard iterative procedure for multivalued mappings is due to Singh and Chadha [153] and it is stated as follows.

**Theorem 7.10.** [153] *Let $X$ be a complete metric space and $T : X \to P_{b,cl}(X)$.*

*Suppose there exists a positive number $q < 1$ such that $T$ satisfies the condition*

$$H_d(Tx, Ty) \leq qd(x, y), \quad \forall x, y \in X.$$

*Let $x_0$ be an arbitrary point in $X$ and assume that $\{x_n\}_{n=0}^{\infty}$ is a sequence which converges to a fixed point $u$ of $T$.*

*Let $\{y_n\}_{n=0}^{\infty}$ be a sequence in $X$ and set $\epsilon_n = H_d(y_{n+1}, Ty_n)$, $n = 0, 1, 2, \ldots$.*

*If $Tu$ is singleton then $\lim_{n\to\infty} y_n = u$ if and only if $\lim_{n\to\infty} \epsilon_n = 0$.*

Afterwards, Czerwik, Dlutek and Singh [55] studied the stability of Picard iterative procedures for multivalued maps in $b$-metric spaces. Furthermore, Singh, Bhatnagar and Mishra [150] obtained a fixed point theorem for generalized multivalued contractions in $b$-metric spaces and further studied the stability of Picard iterative procedures for such maps.

**Definition 7.21.** [143] *Let $X$ be a nonempty set. If $T : X \to P(X)$ is a multivalued operator, then an element $x \in X$ is called a fixed point for $T$, if and only if $x \in T(x)$.*

*On the other hand, a strict fixed point for $T$ is an element $x \in X$ with the property $\{x\} = T(x)$. The set of all strict fixed points of $T$ is denoted by $SFix(T)$.*

**Theorem 7.11.** [140] *Let $(X, d)$ be a complete metric space and $T : X \to P_b(X)$ be a multivalued operator. We suppose that*

*i) $x \in Tx$, $\forall x \in X$;*

*ii) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and a Picard sequence $x_{n+1} \in Tx_n$, $n \in \mathbb{N}$, such that*

$$\delta(Tx_{n+1}) \leq \varphi(\delta(Tx_n)),\ n \in \mathbb{N}.$$

*Then, $x_n \to x^*$, as $n \to \infty$ and $x^* \in SFix(T)$.*

In the following, we give a stability result for multivalued mappings satisfying an almost contraction condition.

**Theorem 7.12.** *(**Timiş**, [168]) Let $(X, d)$ be a complete metric space and $T : X \to P_{b,cl}(X)$ a mapping with $SFix(T) \neq \phi$, satisfying*

$$H_d(Tx, Ty) \leq q \cdot d(x, y) + L \cdot D(x, Tx),$$

*for all $x, y \in X$, $q \in [0, 1)$ and $L \geq 0$.*

*Let $\{x_n\}_{n=0}^{\infty}$ an iterative procedure defined by $x_0 \in X$ and $x_{n+1} \in Tx_n$, for all $n \geq 0$ and assume that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to $x^*$, the unique strict fixed point of $T$.*

*Then, the Picard iteration is $T$-stable.*

Proof. Consider $\{y_n\}_{n=0}^{\infty}$ to be an arbitrary sequence. Then, according to Definition 7.20, if $\lim_{n\to\infty} H_d(y_{n+1}, Ty_n) = 0$ implies that $\lim_{n\to\infty} y_n = x^*$, then the Picard iteration is $T$-stable.

In order to prove this, we suppose that $\lim_{n\to\infty} H_d(y_{n+1}, Ty_n) = 0$. Therefore, $\forall \epsilon > 0$, $\exists n_0 = n(\epsilon)$ such that $H_d\left(y_{n+1}, Ty_n\right) < \epsilon$, $\forall n \geq n_0$.

So,

$$d(y_{n+1}, x^*) = H_d(\{y_{n+1}\}, \{x^*\}) \leq$$
$$\leq H_d(\{y_{n+1}\}, Ty_n) + H_d(Ty_n, Tx_n) + H_d(Tx_n, \{x^*\}) \leq$$
$$\leq H_d(y_{n+1}, Ty_n) + q \cdot d(x_n, y_n) + L \cdot D(x_n, Tx_n) + H_d(Tx_n, x^*) \leq$$
$$\leq H_d(y_{n+1}, Ty_n) + q \cdot d(y_n, x^*) + q \cdot d(x^*, x_n) + L \cdot D(x_n, Tx_n) + H_d(Tx_n, x^*).$$

From the hypothesis, by $x_n \to x^*$, as $n \to \infty$, we have that $H_d(x^*, Tx_n) \to 0$ and $H_d(y_{n+1}, Ty_n) \to 0$, as $n \to \infty$.

Also,

$$D(x_n, Tx_n) \leq d(x_n, x_{n+1}) \to 0, \quad n \to \infty.$$

Then applying Lemma 1.1, for

$$\epsilon_n := H_d(y_{n+1}, Ty_n) + q \cdot d(x^*, x_n) + L \cdot d(x_n, x_{n+1}) + H_d(Tx_n, x^*),$$

where $\lim_{n\to\infty} \epsilon_n = 0$ and by taking to the limit, we obtain that $H_d(y_{n+1}, x^*) \to 0$, as $n \to \infty$, and this shows that the Picard iteration is stable with respect to $T$. $\square$

**Remark 7.16.** *Theorem 7.12 extends Theorem 7.10 of Singh and Chadha* [**153**]*. If we take $L = 0$ in Theorem 7.12, we get the stability result from Theorem 7.10.*

As argued in Section 2 of this Chapter, from a numerical point of view, the concept of weak stability is more natural than the one of usual stability considered in [**55**], [**150**], [**153**] etc., because of the *arbitrary* sequence taken. So, any stable iteration will be also weakly stable but the reverse is not generally true.

In the sequel, we give the transposition to multivalued mapping of Definition 5.17 of the weak stability with respect to $T$.

**Definition 7.22.** *(**Timiş**, [**168**]) Let $(X, d)$ be a metric space and $T : X \to P_{b,cl}(X)$ be a multivalued mapping. Let $\{x_n\}_{n=0}^{\infty}$ be an iteration procedure defined by $x_0 \in X$ and*

$$x_{n+1} = f(T, x_n), \quad n \geq 0.$$

*Suppose that $\{x_n\}_{n=0}^{\infty}$ converges to a strict fixed point $p$ of $T$. If for any approximate sequence $\{y_n\}_{n=0}^{\infty} \subset X$ of $\{x_n\}_{n=0}^{\infty}$,*

$$\lim_{n\to\infty} H_d(y_{n+1}, f(T, y_n)) = 0 \quad \textit{implies} \quad \lim_{n\to\infty} y_n = p,$$

*then we shall say that the iteration procedure $\{x_n\}_{n=0}^{\infty}$ is weakly $T$-stable or weakly stable with respect to $T$.*

CHAPTER 3

# Stability of fixed point, common fixed point and coincidence point iterative procedures for contractive mappings defined by implicit relations

Several classical fixed point theorems and common fixed point theorems have been recently unified by considering general contractive conditions expressed by an implicit relation. This development has been initiated by Popa [119], [120], [121] and following this approach, a consistent part of the literature on fixed point, common fixed point and coincidence theorems, both for single valued and multi-valued mappings, in various ambient spaces have been accomplished.

Bouhadjera and Djoudi [39] proved a common fixed point theorem for four weakly compatible mappings satisfying an implicit relation without need of continuity. This theorem generalizes some results on compatible continuous mappings of Popa [121].

Aliouche [7] proved common fixed point theorems for weakly compatible mappings in metric spaces satisfying an implicit relation using (E.A) property and a common (E.A) property, which generalizes the results of Aamri and Moutawakil [1].

Aliouche [8] also proved common fixed point theorems for weakly compatible mappings satisfying implicit relations without the condition that the map to be decreasing in any variable. These theorems improve results of Ali and Imdad [5], Jeong and Rhoades [75] and Popa [122].

On the other hand, Pathak and Verma [117] proved some coincidence and common fixed point results by using an implicit relation for four weakly compatible mappings which satisfy (E.A) property in symmetric spaces. These are generalizations of related results of symmetric spaces and they also improve the results of Imdad, Ali and Khan [68].

For these new fixed point theorems did not exist corresponding stability results and Berinde [19], [30] filled this gap and established corresponding stability results for fixed point iterative procedures associated to contractive mappings defined by an implicit relation.

We continue to study the stability of Picard iterative procedure and also of Jungck iterative procedure for common fixed points and coincidence points, for contractive mappings satisfying various implicit relations, with different number of parameters.

Since a metrical common fixed point theorem generally involves conditions of commutativity, a lot of researches in this domain are aimed at weakening these conditions. The evolution of weak commutativity of Sessa [**146**] and compatibility of Jungck [**79**] developed weak conditions in order to improve common fixed points theorems. We also give a general stability result for the common fixed point iteration procedure of Jungck-type in the class of weakly compatible mappings defined by means of an implicit contraction condition.

The results obtained in this chapter are generalizations of fixed point theorems and stability theorems for Picard iteration existing in literature: see Berinde [**20**], [**24**], [**27**], [**29**], [**31**], Chatterjea [**45**], Harder and Hicks [**61**], [**62**], Hardy and Rogers [**63**], Imoru and Olatinwo [**69**], Jungck [**78**], Kannan [**81**], Olatinwo [**100**], Osilike [**111**], [**110**], Ostrowski [**115**], Popa [**120**], Reich [**127**], Reich and Rus [**154**], Rhoades [**130**], [**132**], [**133**], Rus [**138**], [**139**], Zamfirescu [**173**] and most of their references.

The author's original contributions in this chapter are: Example 1.15, Theorem 1.14, Corollary 1.1, Corollary 1.2, Theorem 2.15, Examples 3.23-3.25, Examples 3.27-3.29, Theorem 3.16, Corollary 3.3 and Corollary 3.4.

Most of them were published in [**161**] (Timiş, I., *Stability of Jungck-type iterative procedure for some contractive type mappings via implicit relations*, Miskolc Math. Notes 13 (2) (2012), 555-567), [**163**] (Timiş, I., *Stability of Jungck-type iterative procedure for common fixed points and contractive mappings via implicit relations*, presented at ICAM8, Baia Mare, 27-30 Oct. 2011) and [**164**] (Timiş, I., *Stability of the Picard iterative procedure for mappings which satisfy implicit relations*, Comm. Appl. Nonlinear Anal. 19 (2012), no. 4, 37-44).

## 1. Stability of fixed point iterative procedure for contractive mappings satisfying implicit relations

Berinde [30] gave a general stability result of the Picard iteration for mappings satisfying an implicit relation with six parameters, using the set of all continuous real functions $F:\mathbb{R}_+^6\to R_+$ introduced by Popa [120], [121], with the following conditions:

$(F_{1a})$ $F$ is non-increasing in the fifth variable and $F(u,v,v,u,u+v,0)\le 0$ for $u,v\ge 0\Longrightarrow \exists h\in[0,1)$ such that $u\ge hv$;

$(F_{1b})$ $F$ is non-increasing in the fourth variable and $F(u,v,0,u+v,u,v)\le 0$ for $u,v\ge 0\Longrightarrow \exists h\in[0,1)$ such that $u\ge hv$;

$(F_{1c})$ $F$ is non-increasing in the third variable and $F(u,v,u+v,0,v,u)\le 0$ for $u,v\ge 0\Longrightarrow \exists h\in[0,1)$ such that $u\ge hv$;

$(F_2)$ $F(u,u,0,0,u,u)>0$, for all $u>0$.

**Theorem 1.13.** [30] *Let $(X,d)$ be a complete metric space, $T:X\to X$ a self mapping for which there exists $F\in\mathbb{F}$ such that for all $x,y\in X$*

$$F\left(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\right)\le 0.$$

*If $F$ satisfies $(F_{1a})$ and $(F_2)$, then $T$ has an unique fixed point.*

*If, additionally, $F$ satisfies $(F_{1b})$, then Picard iteration is: a) $T$-stable; b) summable almost $T$-stable.*

In the following, we study the stability of the Picard iterative procedure for mappings satisfying an implicit relation but we reduce the number of parameters to five.

Popa [119] introduced $\mathbb{F}$ to be the set of all continuous real functions $F:\mathbb{R}_+^5\to\mathbb{R}$ with the following conditions:

(1) $F$ is continuous in each coordinate variable,

(2) there exists $h\in[0,1)$ such that, for all $u,v,w\ge 0$ satisfying
- (2a) $F(u,v,u,v,w)\le 0$ or
- (2b) $F(u,v,v,u,w)\le 0$,

we have that $u\le h\max\{v,w\}$.

In the following, there are some examples of functions that satisfy some of the above conditions:

**Example 1.13.** [119] *Define $F(t_1,t_2,t_3,t_4,t_5):\mathbb{R}_+^5\to\mathbb{R}$ as*

(1) $F(t_1, ..., t_5) = t_1 - c \max\{t_2, t_3, t_4, t_5\}, \quad c \in [0, 1)$;
(2) $F(t_1, ..., t_5) = t_1^2 - c \max\{t_2 t_3, t_2 t_4, t_3 t_4, t_5^2\}, \quad c \in [0, 1)$;
(3) $F(t_1, ..., t_5) = t_1^2 - (a t_1 t_2 + b t_1 t_3 + c t_1 t_4 + d t_5^2)$, *where* $a, b, c, d > 0$ *and* $0 < a + b + c + d < 1$.

**Example 1.14.** [27] *Define* $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ *as*

(1) $F(t_1, ..., t_5) = t_1 - a t_2, \quad a \in [0, 1)$;
(2) $F(t_1, ..., t_5) = a \max\{t_1, t_2, t_3, t_4, t_5\}, \quad a \in [0, 1)$;
(3) $F(t_1, ..., t_5) = a \max\left\{t_1, t_2, t_3, t_4, \frac{t_4+t_5}{2}\right\}, \quad a \in [0, 1)$;
(4) $F(t_1, ..., t_5) = a(t_2 + t_3), \quad a \in \left[0, \frac{1}{2}\right)$;
(5) $F(t_1, ..., t_5) = a t_1 + b(t_2 + t_3), \quad a, b \in \mathbb{R}_+, \quad a + 2b < 1$;
(6) $F(t_1, ..., t_5) = a \max\{t_2, t_3\}, \quad a \in (0, 1)$;
(7) $F(t_1, ..., t_5) = \left(\sum_{i=1}^5 a_i t_i^p\right)^{\frac{1}{p}}, \quad a_i \in \mathbb{R}_+, \quad \sum_{i=1}^5 a_i < 1, \quad p \geq 1$;
(8) $F(t_1, ..., t_5) = \max\{a t_1, b(t_2 + t_4), c(t_3 + t_5)\}$, *where* $a \in [0, 1), \quad b, c \in \left[0, \frac{1}{2}\right)$.

**Example 1.15.** *(**Timiş**, [164]) Define* $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ *as*

$$F(t_1, ..., t_5) = t_1 - c t_2 - t_5, \quad c \in [0, \frac{1}{2}).$$

We establish the following general stability theorem for the Picard iteration procedure:

**Theorem 1.14.** *(**Timiş**, [164]) Let $(X, d)$ be a complete metric space, $T : X \to X$ a map with $Fix(X) \neq \emptyset$ for which there exists $F \in \mathbb{F}$ such that for all $x, y \in X$,*

$$F\left(d(Tx, Ty), d(x, y), d(x, Ty), d(y, Tx), \frac{d(x, Tx) + d(y, Ty)}{2}\right) \leq 0. \tag{1.21}$$

*If $F$ satisfyes (2a) then*

(1) *the fixed point $p$ is unique in $X$;*
(2) *the Picard iteration is $T$-stable.*

Proof. (1) Suppose that there exists $p_1, p_2 \in F_X$, such that $p_1 \neq p_2$. Then, by taking $x := p_1$ and $y := p_2$ in (1.21) and by denoting $\delta := d(p_1, p_2) > 0$ we get $F(\delta, \delta, \delta, \delta, 0) \leq 0$.

By (2a), there exists $h \in [0, 1)$ such that $\delta \leq h \max\{\delta, 0\} \Leftrightarrow \delta \leq h\delta$ and this is a contradiction, as long as $h \in [0, 1)$. So, we have an unique fixed point $p$.

(2) Let $\{x_n\}_{n=0}^{\infty}$ be the associated Picard iteration of $T$ with the general form $x_{n+1} = T x_n$, converging to the fixed point $p$ of $T$, which exists and it is unique by (1).

Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in $X$ and define $\{\epsilon_n\}_{n=0}^{\infty}$ by

$$\epsilon_n = d(y_{n+1}, Ty_n), \quad n = 0, 1, 2, ...$$

In order to prove that the Picard iteration is $T$-stable, we need to prove that

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \Rightarrow \quad \lim_{n\to\infty} y_n = p.$$

Assume that $\lim_{n\to\infty} \epsilon_n = 0$. We have

$$d(y_{n+1}, p) \leq d(y_{n+1}, Ty_n) + d(Ty_n, p) = \epsilon_n + d(Ty_n, p).$$

By taking $x := p$ and $y := y_n$ in (1.21) and denoting

$$u := d(Ty_n, p),\ v := d(y_n, p),\ w := \frac{d(y_n, Ty_n)}{2},$$

we obtain that

$$F(u, v, u, v, w) \leq 0.$$

Now, since $F$ satisfies (2a), there exists $h \in [0, 1)$ such that $u \leq h \max\{v, w\}$, respectively $d(Ty_n, p) \leq h \max\left\{d(y_n, p), \frac{d(y_n, Ty_n)}{2}\right\}$. We discuss two cases.

In the first case, when $\max = d(y_n, p)$, it yields that $d(Ty_n, p) \leq hd(y_n, p)$, and then

$$d(y_{n+1}, p) \leq \epsilon_n + hd(y_n, p)$$

and applying Lemma 1.1 we get the conclusion.

For the second case, if $\max = \frac{d(y_n, Ty_n)}{2}$, we have that

$$d(Ty_n, p) \leq \frac{h}{2} d(Ty_n, y_n) \leq \frac{h}{2} d(Ty_n, p) + \frac{h}{2} d(p, y_n).$$

Then,

$$(1 - \frac{h}{2}) d(Ty_n, p) \leq \frac{h}{2} d(p, y_n),$$

so,

$$d(Ty_n, p) \leq \frac{\frac{h}{2}}{1 - \frac{h}{2}} d(y_n, p).$$

We denote $q := \frac{\frac{h}{2}}{1-\frac{h}{2}} \in [0, 1)$, because $h \in [0, 1)$ and then we get

$$d(Ty_n, p) \leq qd(y_n, p),$$

so,

$$d(y_{n+1}, p) \leq qd(y_n, p) + \epsilon_n.$$

Consequently, the conclusion follow by applying Lemma 1.1. $\square$

**Corollary 1.1.** *(**Timiş**, [**164**]) Let $(X, d)$ be a complete metric space, $T : X \to X$ a map with $Fix(X) \neq \emptyset$ for which there exists $F \in \mathbb{F}$ such that for all $x, y \in X$,*

$$F\left(d(Tx, Ty), d(x, y), d(x, Ty), d(y, Tx), \frac{d(x, Tx) + d(y, Ty)}{2}\right) \leq 0.$$

*If $F$ satisfies (2a) then*

*(1) the fixed point $p$ is unique in $X$;*

*(2) the Picard iteration corresponding to the fixed point theorem obtained by Reich [**128**] and Rus [**139**] (see Taskovic [**154**]) is $T$-stable.*

PROOF. We use Theorem 1.14 for $F$ given by Example 1.14 (5). □

**Corollary 1.2.** *(**Timiş**, [**164**]) Let $(X, d)$ be a complete metric space, $T : X \to X$ a map with $Fix(X) \neq \emptyset$ for which there exists $F \in \mathbb{F}$ such that for all $x, y \in X$,*

$$F\left(d(Tx, Ty), d(x, y), d(x, Ty), d(y, Tx), \frac{d(x, Tx) + d(y, Ty)}{2}\right) \leq 0.$$

*If $F$ satisfies (2a) then*

*(1) the fixed point $p$ is unique in $X$;*

*(2) the Picard iteration corresponding to the fixed point theorem obtained by Bianchini [**34**] and Dugundij (1976) (see Rus [**139**]) is $T$-stable.*

PROOF. We use Theorem 1.14 for $F$ given by Example 1.14 (6). □

**Remark 1.17.** ***Some other important particular cases:***

*(1) If $F$ is given by Example 1.14 (1), then we obtain a stability result for Banach's contraction mapping principle, see Ostrowski [**115**].*

*(2) If $F$ is given by Example 1.14 (2), then we obtain a stability result for the Ciric's fixed point theorem [**50**], see Harder and Hicks [**62**].*

*(3) If $F$ is given by Example 1.14 (4), then we obtain a stability result for the Kannan's fixed point theorem [**81**], see Harder and Hicks [**62**].*

*(4) If $F$ is given by Example 1.14 (8), then we obtain a stability result for Zamfirescu's fixed point theorem, that is, Theorem 2 from Harder and Hicks [**62**].*

*(5) If $F$ is given by Example 1.15, then we obtain a stability result for Reich's fixed point theorem, that is, for Theorem 3 from Reich [**129**].*

**Remark 1.18.** *The contractive conditions obtained from (1.21) with $F$ as in Examples 1-2 imply the contraction condition used by Rhoades in [**130**], [**132**], [**133**]* *and furthermore, they involve stability results for other well-known fixed point theorems.*

## 2. Stability of fixed point iterative procedure for common fixed points and coincidence points and contractive mappings satisfying implicit relations with six parameters

Popa [**120**], [**121**] also introduced $\mathbb{F}$ to be the set of all continuous real functions $F : \mathbb{R}_+^6 \to R_+$ with the following conditions:

(1) (a) $F$ is non-increasing in the fifth variable and $F(u, v, v, u, u+v, 0) \leq 0$ for $u, v \geq 0 \Longrightarrow \exists h \in [0, 1)$ such that $u \geq hv$;
(b) $F$ is non-increasing in the fourth variable and $F(u, v, 0, u+v, u, v) \leq 0$ for $u, v \geq 0 \Longrightarrow \exists h \in [0, 1)$ such that $u \geq hv$;
(c) $F$ is non-increasing in the third variable and $F(u, v, u+v, 0, v, u) \leq 0$ for $u, v \geq 0 \Longrightarrow \exists h \in [0, 1)$ such that $u \geq hv$;

(2) $F(u, u, 0, 0, u, u) > 0$, for all $u > 0$.

The following examples of such functions appearing in Popa [**121**] correspond to well-known fixed point theorems and satisfy the above conditions.

**Example 2.16.** [**121**] *Define* $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to R_+$ *as*

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}, \quad k \in (0, 1).$$

**Example 2.17.** [**121**] *Define* $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to R_+$ *as*

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - b(t_3 + t_4), \quad b \in \left[0, \frac{1}{2}\right).$$

**Example 2.18.** [**121**] *Define* $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to R_+$ *as*

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - c(t_5 + t_6), \quad c \in \left[0, \frac{1}{2}\right).$$

**Example 2.19.** [**121**] *Define* $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to R_+$ *as*

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - c_1 \max\{t_2^2, t_3^2, t_4^2\} - c_2 \max\{t_3 t_5, t_4 t_6\} - c_3 t_5 t_6,$$

*where* $c_1 > 0$, $c_2, c_3 \geq 0$, $c_1 + 2c_2 < 1$ *and* $c_1 + c_3 < 1$.

**Example 2.20.** [121] *Define* $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to R_+$ *as*

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - t_1 (at_2 + bt_3 + ct_4) - dt_5 t_6,$$

*where* $a > 0$, $b, c, d \geq 0$, $a + b + c < 1$ *and* $a + d < 1$.

**Example 2.21.** [121] *Define* $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to R_+$ *as*

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^3 - at_1^2 t_2 - bt_1 t_3 t_4 - ct_5^2 t_6 - dt_5 t_6^2,$$

*where* $a > 0$, $b, c, d \geq 0$, $a + c + d < 1$ *and* $a + b < 1$.

Imdad and Ali [67] proved a general common fixed point theorem for a pair of mappings using implicit functions due to Popa [120], [121].

In the following, using the results obtained in [67], we give a stability result for the common fixed point iterative procedure.

**Theorem 2.15.** *(**Timiş**, [162]) Let* $(X, d)$ *be a complete metric space and* $S, T : X \to X$ *two mappings such that*

- $T$ *and* $S$ *satisfy (E.A) property;*
- $\forall x, y \in X$, *there exists* $F \in \mathbb{F}$,

$$F\left(d(Tx, Ty), d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\right) \leq 0, \tag{2.22}$$

- $S(X)$ *is a complete subspace of* $X$.

*Then*

*(i) the pair* $(T, S)$ *has a point of coincidence;*

*(ii) the pair* $(T, S)$ *has a unique common fixed point, as long as the pair* $(T, S)$ *is also weakly compatible;*

*(iii) if, additionally,* $F$ *satisfies (1b), then the associated iterative procedure is* $(S, T)$*-stable.*

Proof. Since $T$ and $S$ satisfy (E.A) property, then there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in $X$ such that

$$\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Sx_n = t, \quad t \in X.$$

As long as $S(X)$ is a complete subspace of $X$, every convergent sequence of points of $S(X)$ has a limit in $S(X)$. Therefore,

$$\lim_{n \to \infty} Sx_n = t = Sa = \lim_{n \to \infty} Tx_n = t, \quad a \in X$$

which in turn yields that $t = Sa \in S(X)$. Now assert that $Sa = Ta$. If it is not, then $d(Ta, Sa) > 0$. Using (2.22), we have that

$$F\left(d(Ta, Tx_n), d(Sa, Sx_n), d(Sa, Ta), d(Sx_n, Tx_n), d(Sa, Tx_n), d(Sx_n, Ta)\right) \leq 0$$

which on letting $n \to \infty$ reduces to

$$F\left(d(Ta, t), d(Sa, t), d(Sa, Ta), d(t, t), d(Sa, t), d(t, Ta)\right) \leq 0$$

or

$$F\left(d(Ta, Sa), 0, d(Sa, Ta), 0, 0, d(Sa, Ta)\right) \leq 0$$

and according to (1b), $d(Ta, Sa) \leq 0$.

Hence, $Ta = Sa$, which shows that $a$ is a coincidence point of $T$ and $S$.

Since $S$ and $T$ are weakly compatible, we have

$$St = STa = TSa = Tt.$$

Now assert that $Tt = t$. If not, then $d(Tt, t) > 0$. Again, using (2.22), we get

$$F\left(d(Tt, Ta), d(Sa, St), d(St, Tt), d(Sa, Ta), d(St, Ta), d(Sa, Tt)\right) \leq 0$$

or

$$F\left(d(Tt, t), d(Tt, t), 0, 0, d(Tt, t), d(t, Tt)\right) \leq 0,$$

which contradicts (2).

Hence, $Tt = t$ which shows that $t$ is a common fixed point of $T$ and $S$.

Now, we shall prove the uniqueness of $t$. Assume the contrary, respectively, there exists $t' \in Fix(T)$, such that $t \neq t'$. Then, by taking $x := t$ and $y := t'$ in (2.22) and by denoting $\delta := d(t, t') > 0$, we get

$$F(\delta, \delta, 0, 0, \delta, \delta) \leq 0,$$

which contradicts (2), and this proves that the pair $(S, T)$ has a unique common fixed point.

In order to prove the $(S, T)$-stability, we take the sequence $\{Sx_n\}_{n=0}^{\infty}$ generated by the general iterative procedure $Sx_{n+1} = Tx_n, \quad n = 1, 2, ...,$ for any $x_0 \in X$, which converges to $a \in X$, the coincidence point of the iterative procedure.

Let $\{Sy_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and set

$$\epsilon_n = d(Sy_{n+1}, Ty_n), \quad n = 0, 1, 2, ... \, .$$

Then, in order to show that the iterative procedure is $(S, T)$-stable or stable with respect to $(S, T)$, we have to prove the implication:

$$\lim_{n \to \infty} \epsilon_n = 0 \quad \Longrightarrow \quad \lim_{n \to \infty} Sy_n = a.$$

Assume that $\lim_{n\to\infty} \epsilon_n = 0$. Then

$$d(Sy_{n+1}, a) \leq d(Sy_{n+1}, Ty_n) + d(Ty_n, a) = \epsilon_n + d(Ty_n, a). \tag{2.23}$$

If we take $x := a$ and $y := y_n$ in (2.22), then we obtain $F(u, v, 0, w, u, v) \leq 0$, where $u := d(Ty_n, a)$, $v := d(Sy_n, a)$, $w := d(Sy_n, Ty_n)$. By the triangle inequality, $d(Sy_n, Ty_n) \leq d(Ty_n, a) + d(Sy_n, a)$, that is, $w \leq u + v$.

Now, according to (1b), since $F$ is non-increasing in the fourth variable, we have that

$$F(u, v, 0, u + v, u, v) \leq F(u, v, 0, w, u, v) \leq 0.$$

Then, by the same assumption (1b), there exists $h \in [0, 1)$ such that $u \leq hv$, that is, $d(Ty_n, a) \leq hd(Sy_n, a)$, which, by (3.25) yields $d(Sy_{n+1}, a) \leq hd(Sy_n, a) + \epsilon_n$ and applying Lemma 1.1, we get the conclusion. □

**Remark 2.19.** ***Particular cases:***

(1) *In the case of $F$ given in Example 2.16, from Theorem 2.15 we obtain a stability result for the Ciric's fixed point theorem* [**51**].

(2) *In the case of $F$ given in Example 2.17, from Theorem 2.15 we obtain a stability result for the Kannan's fixed point theorem* [**81**].

(3) *In the case of $F$ given in Example 2.18, from Theorem 2.15 we obtain a stability result for the Chatterjea's fixed point theorem* [**45**].

**Remark 2.20.** *Theorem 2.15 gives a stability result for the common fixed point iteration procedure corresponding to Theorem 3.1 in* [**67**].

## 3. Stability of fixed point iterative procedure for common fixed points and coincidence points for contractive mappings satisfying implicit relations with five parameters

From the class of implicit functions due to Popa [**119**], [**120**], [**121**], now let $\mathbb{F}$ be the set of all continuous real functions $F : \mathbb{R}_+^5 \to \mathbb{R}$, satisfying the following conditions:

(1) $F$ is continuous in each coordinate variable,

(2) there exists $h \in [0, 1)$ such that, $\forall u, v, w \geq 0$ satisfying

- (2a) $F(u, v, u, v, w) \leq 0$ or
- (2b) $F(u, v, v, u, w) \leq 0$

then we have that $u \leq h \max\{v, w\}$.

(3) $F(u, u, u, u, 0) > 0$, for all $u > 0$.

In the sequel, we present some examples of functions depending on five parameters, satisfying some of the conditions above.

**Example 3.22.** [119] *The function* $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ *given by*

$$F(t_1, ..., t_5) = t_1 - at_2,$$

*where* $a \in [0, 1)$, *satisfies (1),(2a),(2b) and (3), with* $h = a$.

**Example 3.23.** *(**Timiş**, [161]) The function* $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ *given by one of the following:*

*(1)* $F(t_1, ..., t_5) = t_1 - at_2$,

*(2)* $F(t_1, ..., t_5) = t_1 - bt_5$,

*(3)* $F(t_1, ..., t_5) = t_1 - c(t_3 + t_4)$,

*where* $a, b \in [0, 1)$, $c \in \left[0, \frac{1}{2}\right)$, *satisfies (1),(2a),(2b) and (3), with* $h = a$, $b$, *respectively* $\frac{b}{1-b} < 1$.

**Example 3.24.** *(**Timiş**, [161]) The function* $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ *given by*

$$F(t_1, ..., t_5) = t_1 - kt_5,$$

*where* $k \in (0, 1)$, *satisfies (1),(2a),(2b) and (3), with* $h = k$.

**Example 3.25.** *(**Timiş**, [161]) The function* $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ *given by*

$$F(t_1, ..., t_5) = t_1 - at_2 - bt_5,$$

*where* $a, b \in (0, 1)$, *with* $a + 2b < 1$, *satisfies (1), (2a), (2b) and (3), with* $h = a$, *if* $\max\{v, w\} = v$ *and* $h = b$, *if* $\max\{v, w\} = w$.

**Example 3.26.** [119] *The function* $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ *given by*

$$F(t_1, ..., t_5) = t_1 - a(t_3 + t_4),$$

*where* $a \in \left(0, \frac{1}{2}\right)$, *satisfies (1), (2a), (2b) and (3), with* $h = \frac{a}{1-a} \in (0, 1)$.

**Example 3.27.** *(**Timiş**, [161]) The function* $F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}$ *given by*

$$F(t_1, ..., t_5) = t_1 - h \max\{t_3, t_4\},$$

*where* $h \in [0, 1)$, *satisfies (1), (2a), (2b) and (3).*

**Example 3.28.** *(**Timiş**, [161]) The function $F(t_1,t_2,t_3,t_4,t_5):\mathbb{R}_+^5\to\mathbb{R}$ given by*

$$F(t_1,...,t_5)=t_1-at_2-bt_3-ct_4,$$

*where $a,b,c\in[0,1)$, with $a+b+c<1$, satisfies (1), (2a) with $h=\frac{a+c}{1-b}\in[0,1)$, (2b) with $h=\frac{a+b}{1-c}\in[0,1)$, and (3).*

**Example 3.29.** *(**Timiş**, [161]) The function $F(t_1,t_2,t_3,t_4,t_5):\mathbb{R}_+^5\to\mathbb{R}$ given by*

$$F(t_1,...,t_5)=t_1-at_2-bt_3-ct_4-dt_5,$$

*where $a,b,c,d\in[0,1)$, with $a+b+c+2d<1$, satisfies (1), (2a) with $h=\frac{a+c}{1-b}\in[0,1)$, (2b) with $h=\frac{a+b}{1-c}\in[0,1)$, and (3), where $h=\frac{a+c}{1-b}\in[0,1)$, if $\max\{v,w\}=v$ and $h=\frac{a+b}{1-c}\in[0,1)$, if $\max\{v,w\}=w$.*

**Example 3.30.** [117] *The function $F(t_1,t_2,t_3,t_4,t_5):\mathbb{R}_+^5\to\mathbb{R}$ given by*

$$F(t_1,...,t_5)=t_1-a\max\left\{t_2,\frac{t_3+t_4}{2},t_5\right\},$$

*where $a\in[0,1)$, satisfies (1), (2a), (2b) and (3), respectively when $\max=t_2$ or $\max=t_5$, then $h=a$, when $\max=\frac{t_3+t_4}{2}$, then $h=\frac{\frac{a}{2}}{1-\frac{a}{2}}$.*

**Example 3.31.** [119] *The function $F(t_1,t_2,t_3,t_4,t_5):\mathbb{R}_+^5\to\mathbb{R}$ given by*

$$F(t_1,...,t_5)=t_1-c\max\{t_2,t_3,t_4,t_5\},$$

*where $h=c\in[0,1)$, satisfies (1), (3), when $\max=t_2$, $\max=t_4$ or $\max=t_5$ is satisfied (2a) and when $\max=t_3$ is satisfied (2b).*

**Example 3.32.** [119] *The function $F(t_1,t_2,t_3,t_4,t_5):\mathbb{R}_+^5\to\mathbb{R}$ given by*

$$F(t_1,...,t_5)=t_1^2-c\max\left\{t_2t_3,t_2t_4,t_3t_4,t_5^2\right\},$$

*where $c\in[0,1)$, satisfies (1), (2a) and (3), with $h=c$.*

Using the common fixed point theorem of Imdad and Ali [67], we give the following general stability result for the common fixed point iteration procedure of Jungck-type using weakly compatible mappings satisfying (E.A) property and defined by an implicit contraction condition.

**Theorem 3.16.** *(**Timiş**, [161]) Let $(X,d)$ be a complete metric space and $S,T:X\to X$ be two mappings, such that $T$ and $S$ satisfy (E.A) property and $S(X)$ is a complete subspace of $X$.*

*Assume there exists $F \in \mathbb{F}$ such that*

(3.24)
$$F\left(d(Tx,Ty), d(Sx,Sy), d(Sx,Ty), d(Sy,Tx), \frac{d(Sx,Tx)+d(Sy,Ty)}{2}\right) \leq 0,$$

*for all $x, y \in X$. Then*

(1) *if $F$ satisfies (2b), then the pair $(T,S)$ has a point of coincidence;*
(2) *if $F$ satisfies (3), the pair $(T,S)$ has a unique common fixed point as long as the pair $(T,S)$ is also weakly compatible;*
(3) *if, additionally, $F$ satisfies (2a), then the associated iterative procedure is $(S,T)$-stable.*

Proof. Since $T$ and $S$ satisfy (E.A) property, there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in $X$ such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t, \quad t \in X.$$

Since $S(X)$ is a complete subspace of $X$, every convergent sequence of points of $S(X)$ has a limit in $S(X)$. Therefore,

$$\lim_{n\to\infty} Sx_n = t = Sz = \lim_{n\to\infty} Tx_n = t, \quad z \in X$$

which in turn yields that $t = Sz \in S(X)$.

Assert that $Sz = Tz$. If not, then $d(Tz, Sz) > 0$ and using (3.24), we have

$$F\left(d(Tz,Tx_n), d(Sz,Sx_n), d(Sz,Tx_n), d(Sx_n,Tz), \frac{d(Sz,Tz)+d(Sx_n,Tx_n)}{2}\right) \leq 0$$

which by letting $n \to \infty$ reduces to

$$F\left(d(Tz,t), d(Sz,t), d(Sz,t), d(t,Tz), \frac{d(Sz,Tz)+d(t,t)}{2}\right) \leq 0$$

or to

$$F\left(d(Tz,Sz), 0, 0, d(Sz,Tz), \frac{d(Sz,Tz)+0}{2}\right) \leq 0. \quad (*)$$

By $(*)$ and according to (2b), there exists $h \in [0,1)$ such that

$$d(Tz,Sz) \leq h \max\left\{0, \frac{d(Sz,Tz)}{2}\right\} = h\frac{d(Sz,Tz)}{2} < d(Sz,Tz),$$

a contradiction.

Hence $Tz = Sz$, so $z$ is a coincidence point of $T$ and $S$.

Since $S$ and $T$ are weakly compatible, then

$$St = STz = TSz = Tt.$$

Now, assert that $Tt = t$. If not, then $d(Tt,t) > 0$. Again, using (3.24),

$$F\left(d(Tt,Tz), d(Sz,St), d(St,Tz), d(Sz,Tt), \frac{d(St,Tt)+d(Sz,Tz)}{2}\right) \leq 0$$

or

$$F\left(d(Tt,t), d(Tt,t), d(Tt,t), d(Tt,t), 0\right) \leq 0$$

which contradicts property (3). Hence, $Tt = t$ which shows that $t$ is a common fixed point of $T$ and $S$.

Now, we shall prove the uniqueness of $t$. Assume the contrary, respectively, there exists $t' \in Fix(T)$, such that $t \neq t'$. Then, by taking $x := t$ and $y := t'$ in (3.24) and by denoting $\delta := d(t,t') > 0$, we get

$$F(\delta, \delta, \delta, \delta, 0) \leq 0,$$

which contradicts (2), and this proves that the pair $(S,T)$ has a unique common fixed point.

In order to prove the $(S,T)$-stability of Jungck type iteration procedure, we take the sequence $\{Sx_n\}_{n=0}^{\infty}$ generated by $Sx_{n+1} = Tx_n, \quad n = 1,2,...,$ for any $x_0 \in X$, which converges to $t$, the common fixed point of the iterative procedure, as long as the pair $(S,T)$ has an unique common fixed point.

Let $\{Sy_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and set

$$\epsilon_n = d(Sy_{n+1}, Ty_n), \quad n = 0,1,2,\ldots\ .$$

By definition, the iterative procedure is $(S,T)$-stable if and only if

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} Sy_n = t.$$

Assume that $\lim_{n\to\infty} \epsilon_n = 0$. Then

$$d(Sy_{n+1}, t) \leq d(Sy_{n+1}, Ty_n) + d(Ty_n, t) = \epsilon_n + d(Ty_n, t). \tag{3.25}$$

If we take $x := t$ and $y := y_n$ in (3.24), then we obtain $F(u,v,u,v,w) \leq 0$, where $u := d(Ty_n,t)$, $v := d(Sy_n,t)$, $w := \frac{1}{2}d(Sy_n,Ty_n)$. Now, since $F$ satisfies (2a), there exists $h \in [0,1)$ such that $u \leq h \max\{v,w\}$, respectively $d(Ty_n,t) \leq h \max\left\{d(Sy_n,t), \frac{d(Sy_n,Ty_n)}{2}\right\}$. We discuss two cases.

Case 1. We take $\max\left\{d(Sy_n,t), \frac{d(Sy_n,Ty_n)}{2}\right\} = d(Sy_n,t)$ and it yields that $d(Ty_n,t) \leq hd(Sy_n,t)$, and then

$$d(Sy_{n+1},t) \leq hd(Sy_n,t) + \epsilon_n$$

and applying Lemma 1.1 we get the conclusion, i.e., $\lim_{n\to\infty} Sy_{n+1} = t$.

Case 2. If $\max\left\{d(Sy_n,t), \frac{d(Sy_n,Ty_n)}{2}\right\} = \frac{d(Sy_n,Ty_n)}{2}$, we have

$$d(Ty_n,t) \leq \frac{h}{2}d(Ty_n,Sy_n) \leq \frac{h}{2}d(Ty_n,t) + \frac{h}{2}d(t,Sy_n).$$

Then,

$$(1-\frac{h}{2})d(Ty_n,t) \leq \frac{h}{2}d(t,Sy_n), \quad \text{so,}$$

$$d(Ty_n,t) \leq \frac{\frac{h}{2}}{1-\frac{h}{2}}d(Sy_n,p).$$

We denote $q := \frac{\frac{h}{2}}{1-\frac{h}{2}} \in [0,1)$, because $h \in [0,1)$ and then we get

$$d(Ty_n,t) \leq qd(Sy_n,t).$$

So,

$$d(Sy_{n+1},t) \leq qd(Sy_n,t) + \epsilon_n,$$

and again, the conclusion follows by applying Lemma 1.1. □

**Remark 3.21.** *Theorem 3.16 completes Theorem 3.1 in Imdad and Ali [67] with the information about the stability of the Jungck-type iterative procedure with respect to the mappings $S$ and $T$, provided that the function $F$ satisfies an additional condition.*

**Corollary 3.3.** *(**Timiş**, [161]) Let $(X,d)$ be a complete metric space and $S,T : X \to X$ be two mappings, such that $T$ and $S$ satisfy (E.A) property and $S(X)$ is a complete subspace of $X$.*

*Suppose there exists $F \in \mathbb{F}$ such that $S$ and $T$ satisfy (3.24), for all $x,y \in X$. Then, the Jungck-type iterative procedure is $(S,T)$-stable.*

Proof. We apply Theorem 3.16, with $F$ given by Example 3.22 which satisfies all conditions (1)-(3) and then we obtain a stability result for Jungck's contraction principle given in [78]. □

**Corollary 3.4.** *(**Timiş**, [161]) Let $(X,d)$ be a complete metric space and $S,T : X \to X$ be two mappings, such that $T$ and $S$ satisfy (E.A) property and $S(X)$ is a complete subspace of $X$.*

*Suppose there exists $F \in \mathbb{F}$ such that $S$ and $T$ satisfy (3.24), for all $x,y \in X$.*

*Then, in the case of the contraction conditions of Zamfirescu's type, the associated common fixed point iterative procedure is $(S,T)$-stable.*

PROOF. We apply Theorem 3.16, for $S = I$, the identity map on $X$, and with $F$ given by Example 3.23 and then we obtain a stability result for the Zamfirescu's fixed point theorem, see [**173**], corresponding to a pair of mappings with a common fixed point. □

**Remark 3.22.** ***Other particular cases.***

(1) *If $F$ is given by Example 3.24 and $S = I$, the identity map on $X$, then we obtain a stability result for the Kannan's fixed point theorem, see* [**81**], *corresponding to a pair of mappings with a common fixed point;*
(2) *If $F$ is given by Example 3.25 and $S = I$, the identity map on $X$, then we obtain a stability result for a fixed point theorem obtained by Reich (1971) and Rus (1971), see* [**154**], *corresponding to a pair of mappings with a common fixed point;*
(3) *If $F$ is given by Example 3.26 and $S = I$, the identity map on $X$, then we obtain a stability result for the Chatterjea's fixed point theorem, see* [**45**], *corresponding to a pair of mappings with a common fixed point;*
(4) *If $F$ is given by Example 3.29 and $S = I$, the identity map on $X$, then we obtain a stability result for the Hardy and Rogers's fixed point theorem, see* [**63**], *corresponding to a pair of mappings with a common fixed point;*
(5) *If $F$ is given by Example 3.30 and $S = I$, the identity map on $X$, then we obtain a stability result for the Pathak and Verma's fixed point theorem, see* [**117**], *corresponding to a pair of mappings with a common fixed point in symmetric spaces;*
(6) *If $F$ is given by Examples 3.31, 3.32 and $S = I$, the identity map on $X$, then we obtain stability results for the Popa's fixed point theorem, see* [**119**], *corresponding to two pairs of mappings on two metric spaces.*

**Remark 3.23.** *The contractive conditions obtained from (3.24) with $F$ as in above examples also imply contractive conditions used by Rhoades in* [**130**], [**132**], [**133**], [**134**].

### Conclusions:

Because of the inclusions between the commutativity definitions, the weakly compatible pair of mappings is the most general type from the mentioned notions and it includes the others. The above theorem use this kind of weakly compatible mappings and it follows that it holds also for compatible, commuting and weakly commuting pair of mappings.

In order to extend and generalize all the mentioned common fixed point theorems, it can be established corresponding stability results for fixed point iteration procedures associated to contractive mappings defined by a suitable implicit relation.

CHAPTER 4

# A new point of view on the stability of fixed point iterative procedures

By taking account of the notions of stability in difference equations, dynamical systems, differential equations, operator theory and numerical analysis, Rus [**136**] unified these notions by new ones.

We consider these new notions in this chapter and study the stability of Picard iteration for mappings which satisfy certain contractive conditions. We also give some illustrative examples.

The author's original contributions in this chapter are: Theorem 1.17, Proposition 1.2, Corollary 1.5, Corollary 1.6, Corollary 1.7, Example 1.33, Corollary 1.8, Theorem 2.18, Corollary 2.9, Example 2.34, Theorem 2.19, Corollary 2.10, Examples 3.35 - 3.42, Definition 4.26, Definition 4.27, Proposition 4.3, Theorem 4.20, Theorem 4.21, Theorem 5.22.

Some of them are included in [**156**] (Timiş, I., *New stability results of Picard iteration for common fixed points and contractive type mappings*, presented at SYNASC 2012, Timişoara, 26-29 Sept. 2012).

## 1. New stability concept for Picard iterative procedures

Eirola, Nevanlinna and Pilyugin [**57**] introduced the notion of *limit shadowing property* and Rus [**136**] adopted it, in order to introduce a new concept of stability for fixed point iteration procedures which appears to be more general than the notion of stability introduced by Harder [**60**].

**Definition 1.23.** *(Rus,[**136**]) On the metric space $(X, d)$, the operator $T : X \to X$ has stable Picard iterates at $x_0 \in X$, if for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$, such that*

$$x \in X,\ d(x, x_0) < \delta(\epsilon) \quad \Rightarrow \quad d\left(T^n x, T^n x_0\right) < \epsilon, \quad \forall n \in \mathbb{N}.$$

*The operator $T$ $Y \subset X$, if it has stable Picard iterates at all $x_0 \in Y$.*

**Definition 1.24.** [57] *The operator $T$ has the limit shadowing property with respect to Picard iteration, if*

$$y_n \in X, \quad n \in \mathbb{N}, \quad d(y_{n+1}, Ty_n) \to 0 \quad as \quad n \to \infty$$

*imply that there exists $x_0 \in X$, such that*

$$d(y_n, T^n x_0) \to 0 \quad as \quad n \to \infty.$$

**Definition 1.25.** [136] *Picard iteration is stable with respect to an operator $T$ if it is convergent with respect to $T$ and the operator $T$ has the limit shadowing property with respect to this iterative procedure.*

**Theorem 1.17.** *(**Timiş**, [157]) Let $(X, d)$ be a metric space and $T : X \to X$ be an $a$-contraction, i.e., $T$ satisfies*

$$d(Tx, Ty) \leq ad(x, y), \quad \forall x, y \in X,$$

*with $a \in [0, 1)$ fixed.*

*Then, $T$ has stable Picard iterates on $X$.*

Proof. When $a = 0$, the operator $T$ is constant and we have nothing to prove. When $a \in (0, 1)$, for $x_0 \in X$ and $\epsilon > 0$ arbitrarily chosen, we are looking for an $\delta(\epsilon) > 0$, such that

$$d(x, x_0) < \delta(\epsilon) \;\Rightarrow\; d\left(T^n x, T^n x_0\right) < \epsilon, \quad \forall n \in \mathbb{N}. \tag{1.26}$$

Since $T$ is an $a$-contraction, $d\left(T^n x, T^n x_0\right) \leq a^n d(x, x_0) \leq ad(x, x_0)$, $\forall n \geq 1$. It suffices to impose the condition $a\delta < \epsilon$. So, (1.26) holds, as soon as $\delta(\epsilon)$ is an arbitrary number in the interval $\left(0, \frac{a}{2}\right)$.

According to Definition 1.23, $T$ has stable Picard iterates at $x_0 \in X$, and because $x_0$ was arbitrary taken, then $T$ has stable Picard iterates on $X$. □

In the following, we study the relationship between the two stability definitions, the one of Harder [60] and the other one due to Rus [136].

**Proposition 1.2.** *(**Timiş**, [157])*

*Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping. Let $x_0 \in X$ and let use assume that the Picard iteration procedure $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, converges to a fixed point $p$ of $T$.*

*Suppose that Picard iteration is stable in the sense of Harder. Then, it is also stable in the sense of Rus. (Definition 1.23)*

Proof. Let $(X, d)$ be a metric space and $T : X \to X$ a mapping, $x_0 \in X$ and let us assume that the iteration procedure $x_{n+1} = Tx_n, \quad n = 0, 1, 2, ...$, converges to a fixed point $p$ of $T$.

Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in $X$, such that $\epsilon_n = d(y_{n+1}, Ty_n) \to 0$, as $n \to \infty$. As the Picard iteration is $T$-stable in the sense of Harder, $\lim_{n\to\infty} y_n = p$.

Then, $x_0 = p$ satisfies the condition from Definition 1.25, because,

$$d(y_n, T^n x_0) = d(y_n, x_n) \leq d(y_n, p) + d(p, x_n) \to 0,$$

so the Picard iteration is also $T$-stable in the sense of Rus. $\square$

**Corollary 1.5.** *(**Timiş**, [157]) Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping satisfying the contraction condition of Zamfirescu, i.e., there exists real numbers $\alpha, \beta, \gamma$, satisfying $0 \leq \alpha < 1,\ 0 \leq \beta, \gamma < \frac{1}{2}$, such that, for each $x, y \in X$, at least one of the following is true:*

*(1)* $d(Tx, Ty) \leq \alpha d(x, y)$;
*(2)* $d(Tx, Ty) \leq \beta \left[d(x, Tx) + d(y, Ty)\right]$;
*(3)* $d(Tx, Ty) \leq \gamma \left[d(x, Ty) + d(y, Tx)\right]$.

*Let $x_0 \in X$ and let use assume that the Picard iteration procedure $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, converges to a fixed point $p$ of $T$.*

*Suppose that Picard iteration is stable in the sense of Harder. Then, it is also stable in the sense of Rus (Definition 1.23).*

**Remark 1.24.** *Corollary 1.5 gives a stability result corresponding to the fixed point theorem of Zamfirescu* [**173**].

**Corollary 1.6.** *(**Timiş**, [157]) Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping satisfying Kannan's contraction condition, i.e., there exists $a \in [0, 1)$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq a \left[d(x, Tx) + d(y, Ty)\right].$$

*Let $x_0 \in X$ and let use assume that the Picard iteration procedure $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, converges to a fixed point $p$ of $T$.*

*Suppose that Picard iteration is stable in the sense of Harder. Then, it is also stable in the sense of Rus (Definition 1.23).*

**Remark 1.25.** *Corollary 1.6 gives a stability result corresponding to the fixed point theorem of Kannan* [**81**].

**Corollary 1.7.** *(**Timiş**, [157]) Let $(X,d)$ be a metric space and $T : X \to X$ be a mapping satisfying Chatterjea's contraction condition, i.e., there exists $a \in \left[0, \frac{1}{2}\right)$ such that for all $x, y \in X$,*

$$d\left(Tx, Ty\right) \leq a\left[d(x, Ty) + d(y, Tx)\right].$$

*Let $x_0 \in X$ and let use assume that the Picard iteration procedure $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, converges to a fixed point $p$ of $T$.*

*Suppose that Picard iteration is stable in the sense of Harder. Then, it is also stable in the sense of Rus (Definition 1.23).*

**Remark 1.26.** *Corollary 1.7 gives a stability result corresponding to the fixed point theorem of Chatterjea* [45].

**Remark 1.27.** *The converse of Proposition 1.2 is not generally true, as shown by the following example.*

**Example 1.33.** *(**Timiş**, [157])*

Let $T : [0,1] \to [0,1]$ be identity mapping on $[0,1]$, that is, $Tx = x$, for each $x \in [0,1]$, where $[0,1]$ is endowed with the usual metric. Every point in $[0,1]$ is a fixed point of $T$ and $T$ is nonexpansive, but not a contraction.

Harder [62] showed in this case that Picard iteration is not $T$-stable. Let now study the stability in sense of Rus. For any $y_n \in X$, with $n \in \mathbb{N}$, we have to prove that $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$ implies that there exists $x_0 \in X$, such that $\lim_{n\to\infty} d(y_n, T^n x_0) = 0$.

Indeed, for any $y_n \in [0,1]$, we get $Ty_n = y_n$, and suppose that

$$\lim_{n\to\infty} d(y_{n+1}, Ty_n) = \lim_{n\to\infty} d(y_{n+1}, y_n) = 0.$$

Now, there exists $x_0 \in X$, where $x_0 = l := \lim_{n\to\infty} y_n$ such that

$$\lim_{n\to\infty} d(y_n, T^n x_0) = \lim_{n\to\infty} d(y_n, x_0) = 0.$$

Hence, Picard iteration is stable in the sense of Rus.

**Corollary 1.8.** *(**Timiş**, [157])*

*Let $(X,d)$ be a metric space and $T : X \to X$ a mapping, $x_0 \in X$ and let us assume that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point $p$ of $T$.*

*If Picard iteration procedure is stable in the sense of Harder, then the fixed point is unique.*

PROOF. Suppose that $Fix(T) = \{p, q\}$, with $p \neq q$.

For the sequence $\{y_n\}_{n=0}^{\infty}$, $y_n = q$, with $Ty_n = q$, we have that $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$, but $\lim_{n\to\infty} y_n = q \neq p$.

So, Picard iteration procedure is stable in the sense of Harder if and only if $Fix(T) = \{p\}$. $\square$

**Remark 1.28.** *Corollary 1.8 has been suggested by Professor I. A. Rus (private communication).*

## 2. Stability results of Picard iteration for mappings satisfying certain contractive conditions

According to above stability definitions of Rus [136], in the following we study the stability of Picard iterative procedure as well as the stability of Picard iterates at $x_0 \in X$, with respect to $T$.

A generalized contraction condition introduced by Berinde [20], named *almost contraction* condition has some surprising properties: it ensures the convergence of Picard iteration to a fixed point and under adequate conditions, an unique fixed point, but it does not require the sum of the coefficients on the right side of the contractive condition to be less than 1.

In a metric space $(X, d)$, a self mapping $T : X \to X$ is called an *almost contraction* if there exists two constants $\delta \in [0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx),$$

for any $x, y \in X$. Here, $\delta + L$ is not restricted to be less than 1.

Almost contractions have a very similar behavior to that of Banach contractions, which explains their name, except for the fact that the fixed point is generally not unique.

In order to ensure this uniqueness, Berinde [20] considered another condition, similar to the above one, namely

$$d(Tx, Ty) \leq \delta_u d(x, y) + L_u d(x, Tx), \tag{2.27}$$

for any $x, y \in X$, where $\delta_u \in [0, 1)$ and $L_u \geq 0$ are constants.

Note that (2.27) has been used by Osilike [108], [110], Osilike and Udomene [114] in order to establish several stability results.

Berinde [21] also proved the existence of coincidence points and common fixed points for a large class of almost contractions in cone metric spaces.

Moreover, Berinde [18] proved the existence of coincidence points and common fixed points of noncommuting almost contractions in metric spaces and a method for approximating the coincidence points or the common fixed points is also constructed, for which both a priori and a posteriori error estimates are obtained.

Using this condition, we obtain the following stability result:

**Theorem 2.18.** *(**Timiş**, [157]) Let $(X,d)$ be a metric space and $T : X \to X$ be a self mapping satisfying the contraction condition (2.27), i.e., for some $\delta_u \in [0,1)$ and $L_u \geq 0$. For all $x, y \in X$, we have*

$$d\,(Tx, Ty) \leq \delta_u d(x,y) + L_u d(x, Tx).$$

*Then, the associated Picard iteration is $T$-stable in the sense of Definition 1.25.*

PROOF. Osilike [110] established the stability in the sense of Harder for Picard iteration and using a mapping satisfying (2.27).

Further, by Proposition 1.2, stability in the sense of Harder involve stability in the sense of Rus, so, we get the conclusion. □

**Remark 2.29.** *For a metric space $(X,d)$ and a self mapping $T$ satisfying the almost contraction condition (2.27), the associated Picard iteration is $T$-stable in the sense of Rus, provided it is $T$-stable in the sense of Harder.*

**Corollary 2.9.** *(**Timiş**, [157]) Let $(X,d)$ be a metric space and a mapping $T : X \to X$, satisfying Banach's contraction condition, i.e., there exists $a \in [0,1)$ and for all $x, y \in X$, we have that $T$ satisfy the contraction condition*

$$d\,(Tx, Ty) \leq a d(x,y).$$

*Then, the associated Picard iteration is $T$-stable in the sense of Definition 1.25.*

PROOF. Applying Theorem 2.18 for $L = 0$, we obtain a stability result for Banach's contraction principle [14]. □

**Example 2.34.** *(**Timiş**, [157])*

Let $X = \left\{0, \frac{1}{2}, \frac{1}{2^2}, \ldots\right\}$ with the usual metric. Define $T : X \to X$ by $T(0) = \frac{1}{2}$, $T\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+1}}$, $n = 1, 2, 3, \ldots$

Babu, Sandhya and Kameswari [13] proved that $T$ satisfies the almost contraction condition (2.27), with $\delta = \frac{1}{2}$, $L = 1$, and $\delta + L = \frac{3}{4} > 1$.

Because $T$ has no fixed points, Picard iteration is not stable in the sense of Harder. Now, we study the stability in the sense of Rus.

For an arbitrary sequence $\{y_n\}_{n=0}^{\infty} \in X$, with $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$, where $\lim_{n\to\infty} y_n := l$, there obviously exists $x_0 \in X$, with $\lim_{n\to\infty} x_n = l$, such that $\lim_{n\to\infty} d(y_n, x_n) = 0$.

Because Picard iteration is also convergent with respect to $T$, then it is stable in the sense of Rus.

Babu, Sandhya and Kameswari [13] found a different contractive condition that ensures the uniqueness of fixed points of almost contractions: if there exists $\delta \in (0,1)$ and $L \geq 0$, such that for all $x, y \in X$,

$$d(Tx,Ty) \leq \delta d(x,y) + L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}. \tag{2.28}$$

Using this condition, we obtain the following stability result:

**Theorem 2.19.** *(**Timiş**, [157]) Let $(X,d)$ be a metric space and a self mapping $T : X \to X$, satisfying the almost contraction condition (2.28), i.e., there exists $\delta \in (0,1)$ and $L \geq 0$, such that*

$$d(Tx,Ty) \leq \delta d(x,y) + L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\},$$

*for all $x, y \in X$.*

*Then, the associated Picard iteration is $T$-stable in the sense of Harder.*

Proof. Let the Picard iteration with the initial value $x_0 \in X$, $\{x_n\}_{n=1}^{\infty}$, which converges to a fixed point $p$ of $T$, see [13].

Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in $X$, satisfying condition

$$\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0.$$

The fixed point iteration is $T$-stable in the sense of Harder, if this implies

$$\lim_{n\to\infty} d(y_n, p) = 0.$$

We have

$$d(y_{n+1}, p) \leq d(y_{n+1}, Ty_n) + d(Ty_n, Tx_n) + d(Tx_n, p) \leq d(y_{n+1}, Ty_n) +$$

$$+\delta d(x_n, y_n) + L\min\{d(x_n,Tx_n), d(y_n,Ty_n), d(x_n,Ty_n), d(y_n,Tx_n)\} + d(Tx_n, p).$$

We discuss four cases.

Case 1.

$$\min\{d(x_n,Tx_n), d(y_n,Ty_n), d(x_n,Ty_n), d(y_n,Tx_n)\} := d(x_n,Tx_n).$$

Then, $d(y_{n+1}, p) \le \epsilon_n + \delta d(x_n, y_n)$, where $\epsilon_n := d(y_{n+1}, Ty_n) + Ld(x_n, Tx_n) + d(Tx_n, p) \to 0$, as $n \to \infty$, and applying Lemma 1.1 for $\delta \in (0, 1)$, we get the conclusion.

Case 2.

$$\min\{d(x_n, Tx_n), d(y_n, Ty_n), d(x_n, Ty_n), d(y_n, Tx_n)\} := d(y_n, Ty_n).$$

As $d(y_n, Ty_n) \le d(x_n, Tx_n)$, then, $d(y_{n+1}, x_{n+1}) \le d(y_{n+1}, Ty_n) + \delta d(x_n, y_n) + Ld(y_n, Ty_n) + d(Tx_n, p) \le d(y_{n+1}, Ty_n) + \delta d(x_n, y_n) + Ld(x_n, Tx_n) + d(Tx_n, p) \le \epsilon'_n + \delta d(x_n, y_n)$, where $\epsilon'_n := d(y_{n+1}, Ty_n) + Ld(x_n, Tx_n) + d(Tx_n, p) \to 0$, as $n \to \infty$, and applying again Lemma 1.1 for $\delta \in (0, 1)$, we get the conclusion.

Case 3.

$$\min\{d(x_n, Tx_n), d(y_n, Ty_n), d(x_n, Ty_n), d(y_n, Tx_n)\} := d(x_n, Ty_n).$$

As $d(x_n, Ty_n) \le d(x_n, Tx_n)$, we follow the same steps as in above case in order to get the conclusion.

Case 4.

$$\min\{d(x_n, Tx_n), d(y_n, Ty_n), d(x_n, Ty_n), d(y_n, Tx_n)\} := d(y_n, Tx_n).$$

As $d(y_n, Tx_n) \le d(x_n, Tx_n)$, we follow the same steps as in above case in order to get the conclusion.

In a similar way, we treat the last two cases.

Therefore, the fixed point iteration procedure is stable with respect to $T$, in the sense of Harder. □

**Corollary 2.10.** *(**Timiş**, [157]) Let $(X, d)$ be a metric space and a self mapping $T : X \to X$, satisfying the almost contraction condition (2.28), i.e., there exists $\delta \in (0, 1)$ and $L \ge 0$, such that*

$$d(Tx, Ty) \le \delta d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

*for all $x, y \in X$.*

*Then, the associated Picard iteration is $T$-stable in the sense of Rus, provided it is $T$-stable in the sense of Harder.*

**Conclusions:**

1. A fixed point iteration procedure which is stable in the sense of Harder is also stable in the sense of Rus. But the reverse is not generally true, because Harder stability implies the uniqueness of fixed point, while the new one of Rus does not.

2. The stability of a fixed point iteration procedure in the sense of Rus may imply stability in the sense of Harder, if and only if the iterative procedure converges to the fixed point.

3. On the other hand, there are many examples of mappings that satisfy certain contractive conditions and for which the associated Picard iteration is not stable in the sense of Harder but it is actually stable in the sense of Rus.

In the following examples, we will present some nonexpansive mappings and almost contractions for which the associated Picard iteration is stable in the sense of Rus but it is not stable in the sense of Harder.

**Open problem:** Study the stability in the sense of Rus for general nonexpansive mappings as well as for general almost contractions (that do not satisfy a certain uniqueness condition).

## 3. Examples

In the following, we give some examples of mappings satisfying certain contractive conditions for which the associated Picard iteration is not stable in the sense of Harder but it is actually stable in the sense of Rus.

**Example 3.35.** ***(Timiş,*** [**157**]***)***

Let $T : [0, 2] \to [0, 2]$ be given by

$$Tx = \begin{cases} \frac{x}{2}, & x \in [0, 1) \\ \\ 2, & x \in [1, 2], \end{cases}$$

where $[0, 2]$ is endowed with the usual metric. $T$ has two fixed points, $Fix(T) = \{0, 2\}$.

Păcurar [**116**] showed that $T$ is an almost contraction, i.e., there exists the constants $\delta = \frac{1}{2} \in [0, 1)$ and $L = 3 \geq 0$, such that, for any $x, y \in [0, 2]$, we have that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx).$$

Note that $\delta + L = \frac{7}{2} > 1$.

In the following, we show that Picard iteration is not $T$-stable in sense of Harder but it is $T$-stable in sense of Rus.

Let $x_0 \in X$ and assume that Picard iteration procedure $\{x_n\}_{n=1}^{\infty}$, given by $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, converges to a fixed point $p$ of $T$.

According to Definition 1.14 of Harder, the fixed point iteration procedure is $T$-stable if and only if for every sequence $\{y_n\}_{n=0}^{\infty}$ in $X$,

$$\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} y_n = p.$$

Let $x_0 \in [0, 1)$, so $x_n = \frac{1}{2^n}x_0$, with $\lim_{n\to\infty} x_n = 0 = p$. Then, $Tx_n = \frac{1}{2^{n+1}}x_0$.

Let us consider the sequence $\{y_n\}_{n=0}^{\infty}$ in $X$, defined by $y_0 = 1$ and $y_n = \frac{2n-1}{n}$, for $n \geq 1$.

Then, $Ty_n = 2$ and

$$\lim_{n\to\infty} d(y_{n+1}, Ty_n) = \lim_{n\to\infty} d\left(\frac{2n+1}{n+1}, 2\right) = 0.$$

On the other hand, $\lim_{n\to\infty} d\left(\frac{2n-1}{n}, 0\right) = 2 \neq 0$, so the Picard iteration is not $T$-stable in sense of Harder.

Now, according to Definition 1.25 of Rus, Picard iteration is Rus-stable if $y_n \in X$, $n \in \mathbb{N}$, $d(y_{n+1}, Ty_n) \to 0 \ \ as \ \ n \to \infty$ implies that there exists $x_0 \in X$, such that $d(y_n, T^n x_0) \to 0 \ \ as \ \ n \to \infty$. We discuss two cases.

Case 1. If $y_n \in [0, 1)$, then $y_n = \frac{1}{2^n} y_0$, with $Ty_n = \frac{1}{2^{n+1}} y_0$.

So, $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = \lim_{n\to\infty} d\left(\frac{1}{2^{n+1}} y_0, \frac{1}{2^{n+1}} y_0\right) = 0$ and therefore, there exists $x_0 \in X$ such that

$$\lim_{n\to\infty} d(y_n, x_n) = \lim_{n\to\infty} d\left(\frac{1}{2^{n+1}} y_0, \frac{1}{2^{n+1}} x_0\right) = \lim_{n\to\infty} \frac{1}{2^{n+1}} d\left(y_0, x_0\right) = 0.$$

Case 2. If $y_n \in [1, 2]$, then $y_n = 2 = Ty_n$.

So, $d(y_{n+1}, Ty_n) = d(y_{n+1}, 2)$ and from $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$ we obtain that $\{y_n\}_{n=0}^{\infty}$ converges to 2. Now, just take $x_0 \in [1, 2]$ arbitrary, to get $x_n = 2$, $n \geq 0$, and hence, $\lim_{n\to\infty} d(y_n, x_n) = 0.$ as required.

Therefore, the Picard iteration is $T$-stable in sense of Rus.

**Example 3.36.** ***(Timiş,* [157]*)***

Let $T : [0, 1] \to [0, 1]$ be given by

$$Tx = \begin{cases} \frac{2}{3}x, & x \in \left[0, \frac{1}{2}\right) \\ \\ \frac{2}{3}x + \frac{1}{3}, & x \in \left[\frac{1}{2}, 1\right] \end{cases},$$

where $[0, 1]$ is endowed with the usual metric.

$T$ has two fixed points, $Fix(T) = \{0, 1\}$.

Păcurar [116] showed that $T$ is an almost contraction, i.e., there exists the constants $\delta = \frac{2}{3} \in [0, 1)$ and $L = 6 \geq 0$, such that, for any $x, y \in [0, 1]$, we have that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx).$$

Note that $\delta + L = 6 + \frac{2}{3} > 1$.

In the following, we show that Picard iteration is not $T$-stable in sense of Harder but it is $T$-stable in sense of Rus.

Let $x_0 \in X$ and assume that Picard iteration procedure $\{x_n\}_{n=1}^{\infty}$, given by $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, converges to a fixed point $p$ of $T$.

Let $x_0 \in \left[0, \frac{1}{2}\right)$, so $x_n = \left(\frac{2}{3}\right)^n x_0$, with $\lim_{n\to\infty} x_n = 0 = p$.

Let us consider the sequence $\{y_n\}_{n=1}^{\infty}$ in $X$, defined by $y_n = \frac{n-1}{n} \in [\frac{1}{2}, 1]$, for $n \geq 1$, and $\lim_{n\to\infty} y_n = 1$.

Then, $Ty_n = \frac{2}{3}y_n + \frac{1}{3}$ and

$$\lim_{n\to\infty} d\left(y_{n+1}, Ty_n\right) = \lim_{n\to\infty} d\left(\frac{n}{n+1}, \frac{2}{3}y_n + \frac{1}{3}\right) = \lim_{n\to\infty} d\left(\frac{n}{n+1}, \frac{2}{3}\cdot\frac{n-1}{n} + \frac{1}{3}\right) = 0.$$

On the other hand, $\lim_{n\to\infty} d\left(\frac{n-1}{n}, 0\right) = 1 \neq 0$, so the Picard iteration is not $T$-stable in sense of Harder.

Now, according to Definition 1.25 of Rus, if $y_n \in X$, $n \in \mathbb{N}$, $d(y_{n+1}, Ty_n) \to 0 \ \ as \ \ n \to \infty$ implies that there exists $x_0 \in X$, such that $d(y_n, T^n x_0) \to 0 \ \ as \ \ n \to \infty$. We discuss two cases.

Case 1. If $y_n \in \left[0, \frac{1}{2}\right)$, then $Ty_n = \frac{2}{3}y_n$ and by

$$\lim_{n\to\infty} d(y_{n+1}, Ty_n) = \lim_{n\to\infty} d\left(y_{n+1}, \frac{2}{3}y_n\right) = 0,$$

we obtain that $\lim_{n\to\infty} y_n = 0$.

Indeed, by $\left|y_{n+1} - \frac{2}{3}y_n\right| \to 0$, as $n \to \infty$, we have $y_{n+1} - \frac{2}{3}y_n = \alpha_n$, with $\alpha_n \to 0$, as $n \to \infty$. Then, $y_{n+1} = \frac{2}{3}y_n + \alpha_n$, so $y_{n+1} \leq \frac{2}{3}y_n + \alpha_n$, and applying Lemma 1.1, we get $\lim_{n\to\infty} y_n = 0$.

There exists $x_0 \in \left[0, \frac{1}{2}\right)$, such that

$$\lim_{n\to\infty} d(y_n, x_n) = \lim_{n\to\infty} d\left(\left(\frac{2}{3}\right)^n y_0, \left(\frac{2}{3}\right)^n x_0\right) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n d\left(y_0, x_0\right) = 0.$$

Case 2. If $y_n \in \left[\frac{1}{2}, 1\right]$, then $Ty_n = \frac{2}{3}y_n + \frac{1}{3}$.

So, from $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = \lim_{n\to\infty} d(y_{n+1}, \frac{2}{3}y_n + \frac{1}{3}) = 0$ it results that $\lim_{n\to\infty} y_n = 1$ and therefore, there exists $x_0 \in \left[\frac{1}{2}, 1\right]$, with $\lim_{n\to\infty} x_n = 1$, such that $\lim_{n\to\infty} d(y_n, x_n) =$

$$= \lim_{n\to\infty} d\left(\left(\frac{2}{3}\right)^n y_0 + 1 - \left(\frac{2}{3}\right)^n, \left(\frac{2}{3}\right)^n x_0 + 1 - \left(\frac{2}{3}\right)^n\right) = 0,$$

so, the Picard iteration is $T$-stable in sense of Rus.

**Example 3.37.** *(**Timiş**, [157])*

Let $T : [0, 1] \to [0, 1]$ be given by

$$Tx = \begin{cases} x^2, & x \in \left[0, \frac{1}{4}\right) \\ 0, & x \in \left[\frac{1}{4}, 1\right], \end{cases}$$

where $[0, 1]$ is endowed with the usual metric. $T$ has a fixed point at 0.

Păcurar [**116**] showed that $T$ is an almost contraction, i.e., there exists the constants $\delta = \frac{1}{2} \in [0, 1)$ and $L = \frac{1}{3} \geq 0$, such that, for any $x, y \in [0, 1]$, we have that

$$d(Tx, Ty) \leq \delta d(x, y) + L d(y, Tx).$$

Note that in this case $\delta + L = \frac{5}{6} < 11$.

In the following, we show that Picard iteration is $T$-stable in sense of Harder and it is also $T$-stable in sense of Rus.

Let $x_0 \in X$ and assume that Picard iteration procedure $\{x_n\}_{n=1}^{\infty}$, given by $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, converges to a fixed point $p$ of $T$.

Let $x_0 \in \left[0, \frac{1}{4}\right)$, so $x_n = (x_0)^{2n}$, with $\lim_{n\to\infty} x_n = 0 = p$.

Now, for an arbitary $\{y_n\}_{n=0}^{\infty}$, we discuss two cases.

Case 1. If $y_n \in \left[\frac{1}{4}, 1\right]$, then $Ty_n = 0$ and from $\lim_{n\to\infty} d\left(y_{n+1}, Ty_n\right) = 0$, it results that $\lim_{n\to\infty} y_n = 0$ and this is a contradiction, as long as $y_n \in \left[\frac{1}{4}, 1\right]$.

Case 2. If $y_n \in \left[0, \frac{1}{4}\right)$, then $Ty_n = y_n^2$ and from $\lim_{n\to\infty} d\left(y_{n+1}, Ty_n\right) =$ $= \lim_{n\to\infty} d\left(y_{n+1}, y_n^2\right) = 0$, we obtain that $\lim_{n\to\infty} y_n = 0$.

Indeed, from $|y_{n+1} - y_n^2| \to 0$, as $n \to \infty$, we have that

$$y_{n+1} = y_n^2 + \alpha_n, \quad (*)$$

with $\alpha_n \to 0$, as $n \to \infty$. Denote $\lim_{n\to\infty} y_n := l$ and by taking to the limit in $(*)$, we get $l = l^2$, so $l = 0$, or $l = 1$.

Because $y_n \in \left[0, \frac{1}{4}\right)$, we have $l = 0$, so $\lim_{n\to\infty} y_n = 0$.

Then, $\lim_{n\to\infty} d\left(y_n, p\right) = 0$, so the Picard iteration is $T$-stable in sense of Harder.

According to Proposition 1.2, if Picard iteration is $T$-stable in the sense of Harder, it is also stable in the sense of Rus.

**Example 3.38.** ***(Timiş*, [157]*)***

Let $T : [0, 1] \to [0, 1]$ be given by

$$Tx = \begin{cases} \frac{2}{3}, & x \in [0, 1) \\ \\ 0, & x = 1, \end{cases}$$

where $[0, 1]$ is endowed with the usual metric.

$T$ has one fixed point at $\frac{2}{3}$, $Fix(T) = \left\{\frac{2}{3}\right\}$.

Păcurar [**116**] showed that $T$ is an almost contraction, i.e., there exists the constants $\delta = \frac{2}{3} \in [0, 1)$ and $L \geq \delta \geq 0$, such that, for any $x, y \in [0, 1]$, we have that

$$d(Tx, Ty) \leq \delta d(x, y) + L d(y, Tx).$$

Note that in this case $\delta + L \geq \frac{4}{3} > 1$.

In the following, we show that Picard iteration is $T$-stable in sense of Harder and hence it is also $T$-stable in sense of Rus.

Let $x_0 \in X$ and assume that Picard iteration procedure $\{x_n\}_{n=1}^{\infty}$, given by $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, converges to a fixed point $p$ of $T$.

For any $x_0 \in [0,1]$, $x_n = \frac{2}{3}$, so $\lim_{n\to\infty} x_n = \frac{2}{3} = p$.

Now, for an arbitary $\{y_n\}_{n=0}^{\infty}$, we discuss two cases.

Case 1. If $y_n = 1$, then $Ty_n = 0$ and then $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 1 \neq 0$ and it is a contradiction.

Case 2. If $y_n \in [0,1)$, then $Ty_n = \frac{2}{3}$ and from $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$, it results that $\lim_{n\to\infty} y_n = \frac{2}{3}$.

Then, $\lim_{n\to\infty} d(y_n, p) = 0$, so the Picard iteration is $T$-stable in sense of Harder.

According to Proposition 1.2, if Picard iteration is $T$-stable in the sense of Harder, it is also stable in the sense of Rus.

**Example 3.39.** ***(Timiş,*** *[157])*

Let $T : [0,1] \to [0,1]$ be given by

$$Tx = \begin{cases} 0, & x \in \left[0, \frac{1}{2}\right] \\ \frac{x}{2}, & x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

where $[0,1]$ is endowed with the usual metric.

$T$ has one fixed point at $\frac{1}{2}$, $Fix(T) = \left\{\frac{1}{2}\right\}$.

Păcurar [116] showed that $T$ is an almost contraction, i.e., there exists two constants $\delta_u = \frac{1}{2} \in [0,1)$ and $L_u = 1 \geq 0$, such that, for any $x, y \in [0,1]$, we have that

$$d(Tx, Ty) \leq \delta_u d(x,y) + L_u d(x, Tx).$$

Note that in this case $\delta + L = \frac{3}{2} > 1$.

In the following, we show that Picard iteration is $T$-stable in sense of Harder and it is also $T$-stable in sense of Rus.

Let $x_0 \in X$ and assume that Picard iteration procedure $\{x_n\}_{n=1}^{\infty}$, given by $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, converges to a fixed point $p$ of $T$.

For any $x_0 \in [0,1]$, we have that $\lim_{n\to\infty} x_n = 0 = p$.

Now, for an arbitary $\{y_n\}_{n=0}^{\infty}$, we discuss two cases.

Case 1. If $y_n \in \left(\frac{1}{2}, 1\right]$, then $Ty_n = \frac{y_n}{2}$ and by $\lim_{n\to\infty} d\left(y_{n+1}, Ty_n\right) = 0$, it results that $\lim_{n\to\infty} y_n = 0$ and it is a contradiction, as long as $y_n \in \left(\frac{1}{2}, 1\right]$.

Case 2. If $y_n \in \left[0, \frac{1}{2}\right]$, then $Ty_n = 0$ and by $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$, we obtain that $\lim_{n\to\infty} y_n = 0$.

Hence, $\lim_{n\to\infty} d\left(y_n, p\right) = 0$, so the Picard iteration is $T$-stable in sense of Harder.

According to Proposition 1.2, if Picard iteration is $T$-stable in the sense of Harder, it is also stable in the sense of Rus.

**Example 3.40.** ***(Timiş**, [157]**)***

Let $T : [0, 1] \to [0, 1]$ be given by

$$Tx = \begin{cases} \frac{1}{2}, & x \in \left[0, \dfrac{1}{2}\right] \\ 0, & x \in \left(\dfrac{1}{2}, 1\right] \end{cases},$$

where $[0, 1]$ is endowed with the usual metric. $T$ is continuous at each point of $[0, 1]$ except at $\frac{1}{2}$.

$T$ has an unique fixed point at $\frac{1}{2}$, $Fix(T) = \left\{\frac{1}{2}\right\}$.

We already showed in Example 6.8 that for each $x, y \in [0, 1]$, with $x \neq y$, $T$ satisfies the condition

$$d(Tx, Ty) < \max \left\{d(x, Tx), d(y, Ty)\right\},$$

and also we showed that the associated Picard iteration is not $T$-stable in the sense of Harder, by using a divergent sequence $\{y_n\}_{n=0}^{\infty} = \frac{1}{2}, \frac{1}{4}, \frac{1}{2} + \frac{1}{4^2}, \frac{1}{4^3}, \frac{1}{2} + \frac{1}{4^4}, \frac{1}{4^5}, \cdots$.

In the following, we prove that it is stable in the sense of Rus.

By Definition 1.25 of Rus, for any $y_n \in [0, 1]$, we have that $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$ and it implies that there exists $x_0 \in X$, such that $\lim_{n\to\infty} d(y_n, T^n x_0) = 0$.

From $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$, it results that $y_n \in \left[0, \frac{1}{2}\right]$ and hence, $Ty_n = \frac{1}{2}$ and $\lim_{n\to\infty} y_n = \frac{1}{2}$.

Now, for any $x_0 \in [0, 1]$, we have $x_n = \frac{1}{2}$, $n \geq 2$, and so $\lim_{n\to\infty} x_n = \frac{1}{2}$.

Hence,

$$\lim_{n\to\infty} d(y_n, T^n x_0) = \lim_{n\to\infty} d(y_n, x_n) = 0,$$

so, Picard iteration is $T$-stable in the sense of Rus.

**Example 3.41.** ***(Timiş,* [157]*)***

Let $T : [0,1] \to [0,1]$ be given by

$$Tx = \begin{cases} 0, & x \in \left[0, \frac{1}{2}\right] \\ \frac{1}{2}, & x \in \left(\frac{1}{2}, 1\right] \end{cases},$$

where $[0,1]$ is endowed with the usual metric. $T$ is continuous at every point of $[0,1]$ except at $\frac{1}{2}$.

$T$ has an unique fixed point at 0, $Fix(T) = \{0\}$.

We already showed in Example 6.9 that for each $x, y \in [0,1]$, with $x \neq y$, $T$ satisfies the condition

$$d(Tx, Ty) < \max\{d(x, Ty), d(y, Tx)\},$$

and also showed that the associated Picard iteration is not $T$-stable in the sense of Harder, using $\{y_n\}_{n=0}^{\infty}$, with $y_n = \frac{n+2}{2n}, \quad n \geq 1$.

In the following, we prove that it is stable in the sense of Rus.

According to Definition 1.25 of Rus, for any $y_n \in [0,1]$, we have to prove that $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$ implies that there exists $x_0 \in X$, such that

$$\lim_{n\to\infty} d(y_n, T^n x_0) = 0.$$

We discuss two cases.

Case 1. If $y_n \in \left[0, \frac{1}{2}\right]$, then $Ty_n = 0$, and hence from $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$, it results that $\lim_{n\to\infty} y_n = 0$.

Case 2. If $y_n \in \left(\frac{1}{2}, 1\right]$, then $Ty_n = \frac{1}{2}$, and hence from $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$, it results that $\lim_{n\to\infty} y_n = \frac{1}{2}$.

Now, definitely, there exists $x_0 \in [0,1]$, such that

$$\lim_{n\to\infty} d(y_n, T^n x_0) = \lim_{n\to\infty} d(y_n, x_n) = 0,$$

so, Picard iteration is $T$-stable in the sense of Rus.

**Example 3.42.** ***(Timiş,* [157]*)***

Let $T : \mathbb{R} \to \left\{0, \frac{1}{4}, \frac{1}{2}\right\}$ be defined by

$$Tx = \begin{cases} \frac{1}{2}, & x < 0 \\ \frac{1}{4}, & x \in \left[0, \dfrac{1}{2}\right] \\ 0, & x > \frac{1}{2} \end{cases},$$

where $\mathbb{R}$ is endowed with the usual metric. $T$ is continuous at every point in $\mathbb{R}$ except at 0 and $\frac{1}{2}$.

The only fixed point of $T$ is $\frac{1}{4}$, $Fix(T)$.

We already showed in Example 6.10 that for each $x, y \in \mathbb{R}$, with $x \neq y$, $T$ satisfies the condition

$$d(Tx, Ty) < \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\},$$

and also showed that the associated Picard iteration is not $T$-stable in the sense of Harder by using the sequence $\{y_n\}_{n=0}^{\infty}$ of real numbers $y_n = \frac{1}{2} + \frac{1}{n}$, for each positive odd integer and $y_n = -\frac{1}{n}$, for each positive even integer.

In the following, we prove that it is stable in the sense of Rus.

According to Definition 1.25 of Rus, for any $y_n \in \mathbb{R}$, we have that $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$ and it implies that there exists $x_0 \in \mathbb{R}$, such that $\lim_{n\to\infty} d(y_n, T^n x_0) = 0$. We discuss three cases.

Case 1. If $y_n < 0$, then $Ty_n = \frac{1}{2}$, so, from $d(y_{n+1}, Ty_n) = d(y_{n+1}, \frac{1}{2}) \to 0$, as $n \to \infty$, it results that $\lim_{n\to\infty} y_n = \frac{1}{2}$, and this is a contradiction, as long as $y_n < 0$.

Case 2. If $y_n > \frac{1}{2}$, then $Ty_n = 0$, so, from $d(y_{n+1}, Ty_n) = d(y_{n+1}, 0) \to 0$, as $n \to \infty$, it results that $\lim_{n\to\infty} y_n = 0$, and this is another contradiction, as long as $y_n > \frac{1}{2}$.

Case 3. If $y_n \in \left[0, \dfrac{1}{2}\right]$, then $Ty_n = \frac{1}{4}$, so, from $d(y_{n+1}, Ty_n) = d(y_{n+1}, \frac{1}{4}) \to 0$, as $n \to \infty$, it results that $\lim_{n\to\infty} y_n = \frac{1}{4}$.

Now, definitely, there exists $x_0 \in \mathbb{R}$, such that $\lim_{n\to\infty} x_n = \frac{1}{4}$ and $\lim_{n\to\infty} d(y_n, T^n x_0) = \lim_{n\to\infty} d(y_n, x_n) = 0$, so, Picard iteration is $T$-stable in the sense of Rus.

# 4. New stability concepts of fixed point iteration for common fixed points and contractive type mappings

By adapting Definition 1.24 of limit shadowing property of Eirola, Nevanlinna and Pilyugin [57] to common fixed points, we introduce the following:

**Definition 4.26.** *(**Timiş**, [156]) Let $(X, d)$ be a metric space and $S, T : X \to X$ be two mappings such that $T(X) \subseteq S(X)$. Let $u$ be a common fixed point of $S$ and $T$, that is, $Tu = Su = u$.*

*For any $x_0 \in X$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$ be generated by the Jungck type iterative procedure*

$$Sx_{n+1} = Tx_n, \;\; n = 0, 1, 2, ..., \tag{4.29}$$

*and assume that it converges to $u$.*

*Then, we say that the mappings $T$ and $S$ have the limit shadowing property with respect to Jungck type iteration procedure, if*

$$Sy_n \in X, \quad n \in \mathbb{N}, \quad d(Sy_{n+1}, Ty_n) \to 0 \;\; as \;\; n \to \infty$$

*imply that there exists $x_0 \in X$, such that*

$$d(Sy_n, T^n x_0) \to 0 \;\; as \;\; n \to \infty.$$

**Remark 4.30.** *If $S = I$, the identity map on $X$, then, by Definition 4.26, we get Definition 1.24 of the limit shadowing property introduced by Eirola, Nevanlinna and Pilyugin [57].*

The notion of stability introduced by Rus [136] in Definition 1.25 will be transposed to common fixed points, as follows:

**Definition 4.27.** *(**Timiş**, [156]) Let $(X, d)$ be a metric space and $S, T : X \to X$ be two mappings such that $T(X) \subseteq S(X)$. Let $u$ be a common fixed point of $S$ and $T$, that is, $Tu = Su = u$.*

*For any $x_0 \in X$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$ be generated by the Jungck type iterative procedure $Sx_{n+1} = Tx_n$, $n = 0, 1, 2, ...,$ and assume that it converges to $u$.*

*Then, the Jungck type iteration procedure is stable with respect to the mappings $T$ and $S$ if it is convergent with respect to $T$ and $S$ and the mappings $T$ and $S$ have the limit shadowing property with respect to this iterative procedure.*

In the following, we study the relationship between the stability concept introduced by Singh and Prasad [**152**] in Definition 2.15 which is given for a pair of mappings $(S, T)$ with a coincidence point and our new stability concept introduced by Definition 4.27.

**Proposition 4.3.** *(**Timiş**, [**156**])*

*Let $(X, d)$ be a metric space and let $S$, $T : X \to X$, where $T(X) \subseteq S(X)$ and the mappings $S$ and $T$ have a common fixed point, that is, $Su = Tu = u$.*

*For any $x_0 \in X$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$ be defined by (4.29) and assume that it converges to $u \in X$.*

*Suppose that the Jungck type iteration procedure is stable in the sense of Singh and Prasad [**152**], by Definition 2.15.*

*Then, the Jungck type iteration procedure is also stable in the sense of Definition 4.27..*

Proof. Let $\{Sy_n\}_{n=0}^{\infty}$ be an arbitrary sequence in $X$ and set

$$\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} d\left(Sy_{n+1}, Ty_n\right) = 0.$$

According to Definition 2.15, fixed point iteration procedure is $(S, T)$-stable if and only if

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} Sy_n = u.$$

Now, according to Definition 4.27, we take $Sy_n \in X$, with $d(Sy_{n+1}, Ty_n) \to 0$, as $n \to \infty$.

So, there exists $x_0 = u \in X$, such that $T^n x_0 = T^n u = u$, and hence, $\lim_{n\to\infty} d\left(Sy_n, T^n x_0\right) = \lim_{n\to\infty} d\left(Sy_n, u\right) = 0$, and we get the conclusion. □

**Remark 4.31.** *If If $S = I$, the identity map on $X$, Proposition 4.3 reduces to Proposition 1.2.*

In the following, we give some stability results for the iteration procedure defined by (4.29), with respect to two mappings which satisfy various contractive conditions.

**Theorem 4.20.** *(**Timiş**, [**156**]) Let $(X, d)$ be a complete metric space and $S, T : X \to X$ be two mappings, satisfying*

$$d(Tx, Ty) \le ad(Sx, Sy), \tag{4.30}$$

*for each $x, y \in X$ and some constant $a \in [0, 1)$.*

*$S$ and $T$ have an unique common fixed point $u$, with $Tu = Su = u$, if*

*i)* $T(X) \subseteq S(X)$*;*

*ii)* $S$ *is continuous;*

*iii)* $S$ *and* $T$ *commute.*

*For any* $x_0 \in X$*, let the sequence* $\{Sx_n\}_{n=0}^{\infty}$ *be generated by (4.29) which is assumed to converge to* $u$*.*

*Then, the Jungck type iteration procedure is stable with respect to the mappings* $T$ *and* $S$*, in the sense of Definition 4.27;*

Proof. By Definition 4.27, iteration procedure defined by (4.29) iteration is stable with respect to $S$ and $T$ if it is convergent with respect to $S$ and $T$, and mappings $S$ and $T$ have the limit shadowing property with respect to this iterative procedure.

Let $\{Sy_n\}_{n=0}^{\infty}$ an arbitrary sequence in $X$ and if

$$Sy_n \in X, \quad n \in \mathbb{N}, \quad d(Sy_{n+1}, Ty_n) \to 0 \quad as \quad n \to \infty,$$

imply that there exists $x_0 \in X$, such that

$$d(Sy_n, T^n x_0) \to 0 \quad as \quad n \to \infty.$$

Assume that $\lim_{n\to\infty} d(Sy_{n+1}, Ty_n) = 0$.

Therefore, there exists $x_0 = u \in X$, such that $T^n x_0 = T^n u = u$. Then,

$$d(Sy_{n+1}, u) \le d(Sy_{n+1}, Ty_n) + d(Ty_n, Tx_n) + d(Tx_n, u).$$

From the contraction condition,

$$d(Ty_n, Tx_n) \le ad(Sy_n, Sx_n), \quad a \in [0, 1),$$

so it yields that

$$d(Sy_{n+1}, u) \le ad(Sy_n, u) + \epsilon_n,$$

where

$$\epsilon_n := d(Sy_{n+1}, Ty_n) + d(Tx_n, u) \to 0,$$

so, applying Lemma 1.1, we obtain that $\lim_{n\to\infty} d(Sy_n, u) = 0$.

Therefore, $S$ and $T$ have the limit shadowing property with respect to iteration procedure defined by (4.29) and because it is convergent with respect to $S$ and $T$, we get the conclusion. □

**Remark 4.32.** *If* $S = I$*, the identity map on* $X$*, the stability result in the case of Jungck type iteration procedure in the sense of Rus, i.e., Theorem 4.20, reduces to the stability result of Picard iteration procedure, i.e., Theorem 2.18.*

**Theorem 4.21.** *(**Timiş**, [156]) Let $(X,d)$ be a metric space and $S,T:X\to X$ be two mappings. Suppose there exists $h\in[0,1)$ such that, for every $x,y\in X$,*

$$d(Tx,Ty)\le h\max\{d(Sx,Ty),d(Sy,Tx)\}. \tag{4.31}$$

*$S$ and $T$ have an unique common fixed point $u$, with $Tu=Su=u$, if*

*i) $T(X)\subseteq S(X)$;*

*ii) $S$ is continuous;*

*iii) $S$ and $T$ commute.*

*For any $x_0\in X$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$ be generated by (4.31) which is assumed to converge to $u$.*

*Then, iteration procedure defined by (4.31) is stable with respect to the mappings $T$ and $S$, in the sense of Definition 4.27.*

Proof. By Definition 4.27, iteration procedure defined by (4.29) is stable with respect to $S$ and $T$ if it is convergent with respect to $S$ and $T$, and mappings $S$ and $T$ have the limit shadowing property with respect to this iterative procedure.

Let $\{Sy_n\}_{n=0}^{\infty}$ to be an arbitrary sequence in $X$ and if

$$Sy_n\in X,\quad n\in\mathbb{N},\quad d(Sy_{n+1},Ty_n)\to 0\quad as\quad n\to\infty,$$

imply that there exists $x_0\in X$, such that

$$d(Sy_n,T^nx_0)\to 0\quad as\quad n\to\infty.$$

Therefore, suppose that $\exists x_0=u\in X$ and then, $T^nx_0=T^nu=u$.

Assume that $\lim_{n\to\infty}d(Sy_{n+1},Ty_n)=0$. Then

$$d(Sy_{n+1},u)\le d(Sy_{n+1},Ty_n)+d(Ty_n,u).$$

From the contraction condition,

$$d(Ty_n,Tu)\le h\max\{d(Sy_n,u),d(Ty_n,u)\},\quad h\in[0,1),$$

so it yields that

$$d(Sy_{n+1},u)\le h\max\{d(Sy_n,u),d(Ty_n,u)\}+\epsilon_n,$$

where

$$\epsilon_n:=d(Sy_{n+1},Ty_n)\to 0.$$

We discuss two cases. First, let $\max\{d(Sy_n,u),d(Ty_n,u)\}=d(Sy_n,u)$.

We obtain

$$d(Sy_{n+1},u)\le hd(Sy_n,u)+\epsilon_n,$$

and applying Lemma 1.1, we get $\lim_{n\to\infty}d(Sy_n,u)=0$.

In the second case, let $\max\{d(Sy_n,u), d(Ty_n,u)\} = d(Ty_n,u)$. Then, we have

$$d(Sy_{n+1},u) \leq d(Ty_n,u) + \epsilon_n \leq hd(Ty_n,u) + \epsilon_n,$$

which is a contradiction, since $h \in [0,1)$.

Therefore, $S$ and $T$ have the limit shadowing property with respect to iteration procedure defined by (4.29 and because it is convergent with respect to $S$ and $T$, we get the conclusion. □

## 5. New stability of Picard iteration for mappings defined by implicit relations

We recommence the set of all continuous real functions $\mathbb{F}$, introduced by Popa [120], [121] and used in Chapter 3, Section 2, i.e., $F : \mathbb{R}_+^6 \to R_+$ for which we consider the following conditions:

(1) (a) $F$ is non-increasing in the fifth variable and $F(u,v,v,u,u+v,0) \leq 0$ for $u,v \geq 0 \Longrightarrow \exists h \in [0,1)$ such that $u \geq hv$;
  (b) $F$ is non-increasing in the fourth variable and $F(u,v,0,u+v,u,v) \leq 0$ for $u,v \geq 0 \Longrightarrow \exists h \in [0,1)$ such that $u \geq hv$;
  (c) $F$ is non-increasing in the third variable and $F(u,v,u+v,0,v,u) \leq 0$ for $u,v \geq 0 \Longrightarrow \exists h \in [0,1)$ such that $u \geq hv$;

(2) $F(u,u,0,0,u,u) > 0$, for all $u > 0$.

For the complete metric space $(X,d)$ and $T : X \to X$ a self mapping for which there exists $F \in \mathbb{F}$ such that for all $x,y \in X$,

$$F\left(d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right) \leq 0, \tag{5.32}$$

Berinde [30] proved that if $F$ satisfies (1a) and (2), then

- $T$ has an unique fixed point $x^*$ in $X$;
- The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$, $n = 0,1,2,...$ converges to $x^*$, for any $x_0 \in X$.

In the following, using the above assumptions, we study the stability of Picard iteration in the sense of Definition 1.25.

**Theorem 5.22.** *(**Timiş**, [155]) Let $(X,d)$ be a complete metric space and $T : X \to X$ a self mapping for which there exists $F \in \mathbb{F}$ such that for all $x,y \in X$, $F$*

*satisfies (5.32), i.e.*

$$F\left(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\right)\leq 0.$$

*If $F$ satisfies (1a), (1b) and (2), then the Picard iteration is $T$-stable in the sense of Definition 1.25.*

PROOF. By Definition 1.25, Picard iteration is stable with respect to $T$ if it is convergent with respect to $T$ and $T$ has the limit shadowing property with respect to this iterative procedure.

Let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration associated to $T$ and defined by $x_{n+1}=Tx_n$, $n=0,1,2,...$, converging to the fixed point $x^*$ of $T$, which exists and is unique, according to Theorem 3.3 of Berinde [**30**], since $F$ satisfies (1a) and (2).

In order to prove that $T$ has the limit shadowing property with respect to Picard iteration, by Definition 1.24, let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in $X$ and if

$$d(y_{n+1},Ty_n)\to 0,\quad as\quad n\to\infty,$$

we have that there exists $x_0\in X$, such that

$$d(y_n,T^n x_0)\to 0,\quad as\quad n\to\infty.$$

Therefore, suppose that $\exists x_0\in X$ and from the definition of Picard iterative procedure,

$$x_1=Tx_0,\quad x_2=Tx_1=T^2x_0,\quad \dots\ ,\quad x_n=T^nx_0.$$

Assume that $\lim_{n\to\infty} d(y_{n+1},Ty_n)=0$. Then

$$d(y_{n+1},x_n)\leq d(y_{n+1},Ty_n)+d(Ty_n,x_n). \tag{5.33}$$

If we take $x:=x_n$ and $y:=y_n$ in (5.32), then we obtain $F(u,v,0,w,u,v)\leq 0$, where $u:=d(Ty_n,x_n)$, $v:=d(y_n,Tx_n)$, $w:=d(y_n,Ty_n)$. By the triangle inequality, $d(y_n,Ty_n)\leq d(y_n,Tx_n)+d(x_n,Ty_n)$, that is, $w\leq u+v$. Now, according to (1b), since $F$ is non-increasing in the fourth variable, we have that

$$F(u,v,0,u+v,u,v)\leq F(u,v,0,w,u,v)\leq 0$$

and by the same assumption (1b), there exists $h\in[0,1)$ such that $u\leq hv$, that is, $d(x_n,Ty_n)\leq hd(y_n,Tx_n)$, which, by (5.33) yields

$$d(y_{n+1},x_n)\leq hd(y_n,x_n)+d(y_{n+1},Ty_n)$$

and applying Lemma 1.1, we get the conclusion. □

CHAPTER 5

# Stability of tripled fixed point iteration procedures

In this chapter we introduce the concept of stability for tripled fixed point iterative procedures and also establish some stability results for mixed monotone mappings and monotone mappings, satisfying various contractive conditions. An illustrative example is also given.

The author's original contributions in this chapter are: Definition 2.30, Theorem 2.23, Corollary 2.11, Theorem 2.24, Theorem 2.25, Lemma 3.4, Definition 3.33, Theorem 3.26, Corollary 3.12, Theorem 3.27, Theorem 3.28, Example 4.43 and the contractive conditions (2.35)-(2.40), (3.46)-(3.51).

Most of them were published in [**166**] (Timiş, I., *Stability of tripled fixed point iteration procedures for monotone mappings*, Ann. Univ. Ferrara (2012) DOI 10.1007/s11565-012-0171-7).

## 1. Tripled fixed point iterative procedures

Banach-Caccioppoli-Picard Principle has been generalized by enriching the metric space structure with a partial order. The first result of this kind for monotone mappings in ordered metric spaces was obtained by Ran and Reurings [**126**].

Following the same approach, Bhaskar and Laksmikantham [**33**] obtained some coupled fixed point results for mixed-monotone operators of Picard type, obtaining results involving the existence, the existence and the uniqueness of the coincidence points for mixed-monotone operators $T : X^2 \to X$ in the presence of a contraction type condition, in a partially ordered metric space.

This concept of coupled fixed points in partially ordered metric spaces and cone metric spaces have been studied by several authors, including Abbas, Ali Khan and Radenovic [**2**], Berinde [**22**], [**23**], [**25**], Choudhury and Kundu [**48**], Ciric and Lakshmikantham [**53**], Karapinar [**82**], Lakshmikantham and Ciric [**85**], Olatinwo [**96**], Sabetghadam, Masiha and Sanatpour [**144**].

Recently, Berinde and Borcut [32], [38] obtained extensions to the concept of tripled fixed points and tripled coincidence fixed points and also obtained tripled fixed points theorems and tripled coincidence fixed points theorems for contractive type mappings in partially ordered metric spaces.

The research on tripled fixed point was continued by Abbas, Aydi and Karapinar [3], Aydi and Karapinar [10], Aydi, Karapinar and Vetro [11], Amini-Harandi [9], Borcut [35], [36], [37], Charoensawan [44], Rao and Kishore [126].

By adapting the concept of stability from fixed point iterative procedures, Olatinwo [102] studied the stability of the coupled fixed point iterative procedures using some contractive conditions for which the existence of a unique coupled fixed point has been established by Sabetghadam, Masiha and Sanatpour [144].

In the following, we introduce the concept of stability for tripled fixed point iterative procedures and establish stability results for mixed monotone mappings and monotone mappings, satisfying various contractive conditions by extension from coupled fixed points to tripled fixed points of contractive conditions employed by Olatinwo [102].

## 2. Stability of tripled fixed point iteration procedures for monotone mappings

Let $(X, \leq)$ be a partially ordered set and $d$ be a metric on $X$ such that $(X, d)$ is a complete metric space. Borcut [37] endowed the product space $X^3$ with the following partial order:

$$(x, y, z),\ (u, v, w) \in X^3,\ (u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u,\ y \leq v,\ z \geq w.$$

**Definition 2.28.** [37] *Let $(X, \leq)$ be a partially ordered set and $T : X^3 \to X$ a mapping. We say that $T$ has the monotone property if $T(x, y, z)$ is monotone nondecreasing in $x$, $y$ and $z$, that is, for any $x, y, z \in X$,*

$$x_1, x_2 \in X,\ x_1 \leq x_2 \Rightarrow T(x_1, y, z) \leq T(x_2, y, z),$$

$$y_1, y_2 \in X,\ y_1 \leq y_2 \Rightarrow T(x, y_1, , z) \leq T(x, y_2, z),$$

$$z_1, z_2 \in X,\ z_1 \leq z_2 \Rightarrow T(x, y, z_1) \leq T(x, y, z_2).$$

**Definition 2.29.** [37] *An element $(x, y, z) \in X^3$ is called tripled fixed point of $T : X^3 \to X$, if $T(x, y, z) = x$, $T(y, x, z) = y$, $T(z, y, x) = z$.*

A mapping $T : X^3 \to X$ is said to be a $(k, \mu, \rho)$-contraction, if and only if there exists the constants $k \geq 0$, $\mu \geq 0$, $\rho \geq 0$, $k + \mu + \rho < 1$, such that $\forall x, y, z, u, v, w \in X$,

$$d(T(x, y, z), T(u, v, w)) \leq kd(x, u) + \mu d(y, v) + \rho d(z, w). \tag{2.34}$$

In relation to (2.34), we introduce some new contractive conditions.

Let $(X, d)$ be a metric space. For a mapping $T : X^3 \to X$, suppose there exists $a_1, a_2, a_3, b_1, b_2, b_3 \geq 0$, with $a_1 + a_2 + a_3 < 1$, $b_1 + b_2 + b_3 < 1$, such that $\forall x, y, z, u, v, w \in X$,

$$(i)\ d\left(T(x, y, z), T(u, v, w)\right) \leq a_1 d\left(T(x, y, z), x\right) + b_1 d\left(T(u, v, w), u\right); \tag{2.35}$$

$$d\left(T(y, x, z), T(v, u, w)\right) \leq a_2 d\left(T(y, x, z), y\right) + b_2 d\left(T(v, u, w), v\right); \tag{2.36}$$

$$d\left(T(w, y, x), T(z, v, u)\right) \leq a_3 d\left(T(z, y, x), z\right) + b_3 d\left(T(w, v, u), w\right); \tag{2.37}$$

$$(ii)\ d\left(T(x, y, z), T(u, v, w)\right) \leq a_1 d\left(T(x, y, z), u\right) + b_1 d\left(T(u, v, w), x\right); \tag{2.38}$$

$$d\left(T(y, x, z), T(v, u, w)\right) \leq a_2 d\left(T(y, x, z), v\right) + b_2 d\left(T(v, u, w), y\right); \tag{2.39}$$

$$d\left(T(w, y, x), T(z, v, u)\right) \leq a_3 d\left(T(z, y, x), w\right) + b_3 d\left(T(w, v, u), z\right). \tag{2.40}$$

Let $(X, d)$ be a metric space and $T : X^3 \to X$ a mapping. For $(x_0, y_0, z_0) \in X^3$, the sequence $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ defined by

$$x_{n+1} = T(x_n, y_n, z_n),\ \ y_{n+1} = T(y_n, x_n, z_n),\ \ z_{n+1} = T(z_n, y_n, x_n), \tag{2.41}$$

with $n = 0, 1, 2, \ldots$, is said to be a *tripled fixed point iterative procedure.*

We give the following definition of stability with respect tho $T$, in metric spaces, relative to tripled fixed points iterative procedures:

**Definition 2.30.** *(**Timiş**, [166]) Let $(X, d)$ be a complete metric space and a maping $T : X^3 \to X$, with*

$$Fix_t(T) = \Big\{(x^*, y^*, z^*) \in X^3\ \mid T(x^*, y^*, z^*) = x^*,\ T(y^*, x^*, z^*) = y^*,$$
$$T(z^*, y^*, x^*) = z^*\},$$ *the set of tripled fixed points of $T$.*

*Let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the sequence generated by the iterative procedure defined by (2.41), where $(x_0, y_0, z_0) \in X^3$ is the initial value, which converges to a tripled fixed point $(x^*, y^*, z^*)$ of $T$.*

*Let $\{(u_n, v_n, w_n)\}_{n=0}^{\infty} \subset X^3$ an arbitrary sequence and set*

$$\epsilon_n = d\left(u_{n+1}, T\left(u_n, v_n, w_n\right)\right), \quad \delta_n = d\left(v_{n+1}, T\left(v_n, u_n, w_n\right)\right),$$

$$\gamma_n = d\left(w_{n+1}, T\left(w_n, v_n, u_n\right)\right), \quad n = 0, 1, 2, \ldots.$$

*Then, the tripled fixed point iterative procedure defined by (2.41) is $T$-stable or stable with respect to $T$, if and only if*

$$\lim_{n\to\infty}(\epsilon_n, \delta_n, \gamma_n) = 0_{R^3} \text{ implies that } \lim_{n\to\infty}(u_n, v_n, w_n) = (x^*, y^*, z^*).$$

**Theorem 2.23.** ***(Timiş***, [**166**]*) Let $(X, \leq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space.*

*Let $T : X^3 \to X$ be a continuous mapping having the monotone property on $X$ and satisfying (2.34).*

*If there exists $x_0$, $y_0$, $z_0 \in X$ such that*

$$x_0 \leq T(x_0, y_0, z_0), \quad y_0 \leq T(y_0, x_0, z_0) \text{ and } z_0 \leq T(z_0, y_0, x_0),$$

*then there exists $x^*, y^*, z^* \in X$ such that*

$$x^* = T(x^*, y^*, z^*), \quad y^* = T(y^*, x^*, z^*) \text{ and } z^* = T(z^*, y^*, x^*).$$

*Assume that for every $(x, y, z)$, $(x_1, y_1, z_1) \in X^3$, there exists $(u, v, w) \in X^3$ which is comparable to $(x, y, z)$ and $(x_1, y_1, z_1)$.*

*For $(x_0, y_0, z_0) \in X^3$, let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the tripled fixed point iterative procedure defined by (2.41).*

*Then, the tripled fixed point iterative procedure is $T$-stable.*

Proof. From the suppositions of the hypothesis, Borcut [**37**] proved the existence and uniqueness of the tripled fixed point and now, using these results, we can study the stability of the tripled fixed point iterative procedures.

Let $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{z_n\}_{n=0}^{\infty} \subset X^3$, $\epsilon_n = d\left(u_{n+1}, T\left(u_n, v_n, w_n\right)\right)$, $\delta_n = d\left(v_{n+1}, T\left(v_n, u_n, w_n\right)\right)$ and $\gamma_n = d\left(w_{n+1}, T\left(w_n, v_n, u_n\right)\right)$. Assume also that $\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \gamma_n = 0$ in order to establish that $\lim_{n\to\infty} u_n = x^*$, $\lim_{n\to\infty} v_n = y^*$ and $\lim_{n\to\infty} w_n = z^*$.

Therefore, using (2.34), we obtain

$$\begin{aligned} d(u_{n+1}, x^*) &\leq d(u_{n+1}, T\left(u_n, v_n, w_n\right)) + d(T\left(u_n, v_n, w_n\right), x^*) = \\ &= d(T\left(u_n, v_n, w_n\right), T(x^*, y^*, z^*)) + \epsilon_n \leq \\ &\leq kd(u_n, x^*) + \mu d(v_n, y^*) + \rho d(w_n, z^*) + \epsilon_n. \end{aligned} \tag{2.42}$$

$$\begin{aligned} d(v_{n+1}, y^*) &\leq d(v_{n+1}, T\left(v_n, u_n, w_n\right)) + d(T\left(v_n, u_n, w_n\right), y^*) = \\ &= d(T\left(v_n, u_n, w_n\right), T(y^*, x^*, z^*)) + \delta_n \leq \\ &\leq kd(v_n, y^*) + \mu d(u_n, x^*) + \rho d(w_n, z^*) + \delta_n. \end{aligned} \tag{2.43}$$

$$d(w_{n+1}, z^*) \leq d(w_{n+1}, T(w_n, v_n, u_n)) + d(T(w_n, v_n, u_n), z^*) =$$

$$= d(T(w_n, v_n, u_n), T(z^*, y^*, x^*)) + \gamma_n \leq$$

$$\leq kd(w_n, z^*) + \mu d(v_n, y^*) + \rho d(u_n, x^*) + \gamma_n. \tag{2.44}$$

From (2.42), (2.43) and (2.44), we obtain

$$d(u_{n+1}, x^*) + d(v_{n+1}, y^*) + d(w_{n+1}, z^*) \leq$$

$$\leq (k + \mu + \rho)(d(u_n, x^*) + d(v_n, y^*) + d(w_n, z^*)) + (\epsilon_n + \delta_n + \gamma_n).$$

Hence, applying Lemma 1.1, for $a_n := d(u_{n+1}, x^*) + d(v_{n+1}, y^*) + d(w_{n+1}, z^*)$ and $h := k + \mu + \rho \in [0, 1)$, we get the conclusion. $\square$

**Remark 2.33.** *Theorem 2.23 completes the existence theorem of tripled fixed points of Borcut [37] with the stability result for the tripled fixed point iterative procedures, using monotone operators.*

**Corollary 2.11.** *(**Timiş**, [166]) Let $(X, \leq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space.*

*Let $T : X^3 \to X$ be a continuous mapping having the monotone property on $X$.*

*Assume that there exists $\kappa \in [0, 1)$, such that for each $x, y, z, u, v, w \in X$, $T$ satisfies the following contraction condition:*

$$d(T(x, y, z), T(u, v, w)) \leq \frac{\kappa}{3} \{d(x, u) + d(y, v) + d(z, w)\}.$$

*If there exists $x_0$, $y_0$, $z_0 \in X$ such that*

$$x_0 \leq T(x_0, y_0, z_0), \quad y_0 \geq T(y_0, x_0, y_0) \quad \textit{and} \quad z_0 \leq T(z_0, y_0, x_0),$$

*then there exists $x^*, y^*, z^* \in X$ such that*

$$x^* = T(x^*, y^*, z^*), \quad y^* = T(y^*, x^*, z^*) \quad \textit{and} \quad z^* = T(z^*, y^*, x^*).$$

*Assume that for every $(x, y, z)$, $(x_1, y_1, z_1) \in X^3$, there exists $(u, v, w) \in X^3$ which is comparable to $(x, y, z)$ and $(x_1, y_1, z_1)$.*

*For $(x_0, y_0, z_0) \in X^3$, let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the tripled fixed point iterative procedure defined by (2.41).*

*Then, the tripled fixed point iterative procedure is $T$-stable.*

Proof. We apply Theorem 2.23, for $k = \mu = \rho := \frac{\kappa}{3}$. $\square$

**Remark 2.34.** *Corollary 2.11 completes the existence theorem of tripled fixed points of Borcut [37] with the stability result for the tripled fixed point iterative procedures, using monotone operators.*

**Theorem 2.24.** *(**Timiş**, [166]) Let $(X, \leq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space.*

*Let $T : X^3 \to X$ be a continuous mapping having the monotone property on $X$ and satisfying (2.35), (2.36) and (2.37).*

*If there exists $x_0$, $y_0$, $z_0 \in X$ such that*

$$x_0 \leq T(x_0, y_0, z_0), \quad y_0 \geq T(y_0, x_0, y_0) \quad \text{and} \quad z_0 \leq T(z_0, y_0, x_0),$$

*then there exists $x^*, y^*, z^* \in X$ such that*

$$x^* = T(x^*, y^*, z^*), \quad y^* = T(y^*, x^*, z^*) \quad \text{and} \quad z^* = T(z^*, y^*, x^*).$$

*Assume that for every $(x, y, z)$, $(x_1, y_1, z_1) \in X^3$, there exists $(u, v, w) \in X^3$ which is comparable to $(x, y, z)$ and $(x_1, y_1, z_1)$.*

*For $(x_0, y_0, z_0) \in X^3$, let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the tripled fixed point iterative procedure defined by (2.41).*

*Then, the tripled fixed point iterative procedure is $T$-stable.*

Proof. Let $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{z_n\}_{n=0}^{\infty} \subset X^3$, $\epsilon_n = d\left(u_{n+1}, T\left(u_n, v_n, w_n\right)\right)$, $\delta_n = d\left(v_{n+1}, T\left(v_n, u_n, w_n\right)\right)$ and $\gamma_n = d\left(w_{n+1}, T\left(w_n, v_n, u_n\right)\right)$. Assume also that $\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \gamma_n = 0$ in order to establish that $\lim_{n\to\infty} u_n = x^*$, $\lim_{n\to\infty} v_n = y^*$ and $\lim_{n\to\infty} w_n = z^*$.

Therefore, using the contraction condition (2.35), we obtain

$$d(u_{n+1}, x^*) \leq d(u_{n+1}, T\left(u_n, v_n, w_n\right)) + d(T\left(u_n, v_n, w_n\right), x^*) =$$

$$= d(T\left(u_n, v_n, w_n\right), T(x^*, y^*, z^*)) + \epsilon_n \leq$$

$$\leq a_1 d\left(T(x^*, y^*, z^*), x^*\right) + b_1 d\left(T(u_n, v_n, w_n), u_n\right) + \epsilon_n \leq$$

$$\leq a_1 d(x^*, x^*) + b_1 d\left(T(u_n, v_n, w_n), u_{n+1}\right) + b_1 d(u_{n+1}, x^*) + b_1 d(x^*, u_n) + \epsilon_n =$$

$$= a_1 d(x^*, x^*) + b_1 d(u_{n+1}, x^*) + b_1 d(x^*, u_n) + (b_1 + 1)\epsilon_n.$$

Hence, $(1-b_1)d(u_{n+1}, x^*) \leq b_1 d(x^*, u_n) + \epsilon'_n$, where $\epsilon'_n := (b_1+1)\epsilon_n + a_1 d(x^*, x^*)$. Passing it to the limit and applying Lemma 1.1 for $\frac{b_1}{1-b_1} \in [0, 1)$, we obtain that $\lim_{n\to\infty} u_n = x^*$.

Now, using the contraction condition (2.36), we obtain

$$d(v_{n+1}, y^*) \leq d(v_{n+1}, T\left(v_n, u_n, w_n\right)) + d(T\left(v_n, u_n, w_n\right), y^*) =$$

$$= d(T(v_n, u_n, w_n), T(y^*, x^*, z^*)) + \delta_n \leq$$

$$\leq a_2 d(T(y^*, x^*, z^*), y^*) + b_2 d(T(v_n, u_n, w_n), v_n) + \delta_n \leq$$

$$\leq a_2 d(y^*, y^*) + b_2 d(T(v_n, u_n, w_n), v_{n+1}) + b_2 d(v_{n+1}, y^*) + b_2 d(y^*, v_n) + \delta_n =$$

$$= a_2 d(y^*, y^*) + b_2 d(v_{n+1}, y^*) + b_2 d(y^*, v_n) + (b_2 + 1)\delta_n.$$

So, $(1 - b_2)d(v_{n+1}, y^*) \leq b_2 d(y^*, v_n) + \delta'_n$, where $\delta'_n := (b_2 + 1)\delta_n + a_2 d(y^*, y^*)$. Passing it to the limit and applying Lemma 1.1 for $\frac{b_2}{1-b_2} \in [0, 1)$, we obtain that $\lim_{n\to\infty} v_n = y^*$.

Similarly, using the contraction condition (2.37), we obtain

$$d(w_{n+1}, z^*) \leq d(w_{n+1}, T(z_n, v_n, u_n)) + d(T(z_n, v_n, u_n), z^*) =$$

$$= d(T(w_n, v_n, u_n), T(z^*, y^*, x^*)) + \gamma_n \leq$$

$$\leq a_3 d(T(z^*, y^*, x^*), z^*) + b_3 d(T(w_n, v_n, u_n), w_n) + \gamma_n \leq$$

$$\leq a_3 d(z^*, z^*) + b_3 d(T(w_n, v_n, u_n), w_{n+1}) + b_3 d(w_{n+1}, z^*) + b_3 d(z^*, w_n) + \gamma_n =$$

$$= a_3 d(z^*, z^*) + b_3 d(w_{n+1}, z^*) + b_3 d(z^*, w_n) + (b_3 + 1)\gamma_n.$$

Therefore, $(1 - b_3)d(w_{n+1}, z^*) \leq b_3 d(z^*, w_n) + \gamma'_n$, where $\gamma'_n := (b_3 + 1)\gamma_n + a_3 d(z^*, z^*)$. Passing it to the limit and applying Lemma 1.1 for $\frac{b_3}{1-b_3} \in [0, 1)$, we obtain that $\lim_{n\to\infty} w_n = z^*$ and then, we get the conclusion. □

**Theorem 2.25.** *(**Timiş**, [166]) Let $(X, \leq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space.*

*Let $T : X^3 \to X$ be a continuous mapping having the monotone property on $X$ and satisfying (2.38), (2.39) and (2.40).*

*If there exists $x_0$, $y_0$, $z_0 \in X$ such that*

$$x_0 \leq T(x_0, y_0, z_0), \quad y_0 \geq T(y_0, x_0, y_0) \quad \textit{and} \quad z_0 \leq T(z_0, y_0, x_0),$$

*then there exists $x^*, y^*, z^* \in X$ such that*

$$x^* = T(x^*, y^*, z^*), \quad y^* = T(y^*, x^*, z^*) \quad \textit{and} \quad z^* = T(z^*, y^*, x^*).$$

*Assume that for every $(x, y, z)$, $(x_1, y_1, z_1) \in X^3$, there exists $(u, v, w) \in X^3$ which is comparable to $(x, y, z)$ and $(x_1, y_1, z_1)$.*

*For $(x_0, y_0, z_0) \in X^3$, let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the tripled fixed point iterative procedure defined by (2.41).*

*Then, the tripled fixed point iterative procedure is $T$-stable.*

PROOF. Let $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{z_n\}_{n=0}^{\infty} \subset X^3$, $\epsilon_n = d\left(u_{n+1}, T\left(u_n, v_n, w_n\right)\right)$, $\delta_n = d\left(v_{n+1}, T\left(v_n, u_n, w_n\right)\right)$ and $\gamma_n = d\left(w_{n+1}, T\left(w_n, v_n, u_n\right)\right)$. Assume also that $\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \gamma_n = 0$ in order to establish that $\lim_{n\to\infty} u_n = x^*$, $\lim_{n\to\infty} v_n = y^*$ and $\lim_{n\to\infty} w_n = z^*$.

Therefore, using the contraction condition (2.38), we obtain

$$\begin{aligned}
d(u_{n+1}, x^*) &\leq d(u_{n+1}, T\left(u_n, v_n, w_n\right)) + d(T\left(u_n, v_n, w_n\right), x^*) = \\
&= d(T\left(u_n, v_n, w_n\right), T(x^*, y^*, z^*)) + \epsilon_n \leq \\
&\leq a_1 d\left(T(x^*, y^*, z^*), u_n\right) + b_1 d\left(T(u_n, v_n, w_n), x^*\right) + \epsilon_n \leq \\
&\leq a_1 d(u_n, x^*) + b_1 d\left(T(u_n, v_n, w_n), u_n\right) + b_1 d(u_n, x^*) + \epsilon_n = \\
&= (a_1 + b_1) d(u_n, x^*) + \epsilon_n + b_1 \epsilon_{n-1}.
\end{aligned}$$

Hence, passing it to the limit and applying Lemma 1.1 for $h := a_1 + b_1 \in [0, 1)$ and for $\epsilon_n' := \epsilon_n + b_1 \epsilon_{n-1} \to 0$, as $n \to \infty$, we obtain that $\lim_{n\to\infty} u_n = x^*$.

Now, using the contraction condition (2.39), we obtain

$$\begin{aligned}
d(v_{n+1}, y^*) &\leq d(v_{n+1}, T\left(v_n, u_n, w_n\right)) + d(T\left(v_n, u_n, w_n\right), y^*) = \\
&= d(T\left(v_n, u_n, w_n\right), T(y^*, x^*, z^*)) + \delta_n \leq \\
&\leq a_2 d\left(T(y^*, x^*, z^*), v_n\right) + b_2 d\left(T(v_n, u_n, w_n), y^*\right) + \delta_n \leq \\
&\leq a_2 d(v_n, y^*) + b_2 d\left(T(v_n, u_n, w_n), v_n\right) + b_2 d(v_n, y^*) + \delta_n = \\
&= (a_2 + b_2) d(v_n, y^*) + \delta_n + b_2 \delta_{n-1}.
\end{aligned}$$

So, passing it to the limit and applying Lemma 1.1 for $h := a_2 + b_2 \in [0, 1)$ and for $\delta_n' := \delta_n + b_2 \delta_{n-1} \to 0$, as $n \to \infty$, we get $\lim_{n\to\infty} v_n = y^*$.

Similarly, using the contraction condition (2.40), we obtain

$$\begin{aligned}
d(w_{n+1}, z^*) &\leq d(w_{n+1}, T\left(z_n, v_n, u_n\right)) + d(T\left(z_n, v_n, u_n\right), z^*) = \\
&= d(T\left(w_n, v_n, u_n\right), T(z^*, y^*, x^*)) + \gamma_n \leq \\
&\leq a_3 d\left(T(z^*, y^*, x^*), w_n\right) + b_3 d\left(T(w_n, v_n, u_n), z^*\right) + \gamma_n \leq \\
&\leq a_3 d(w_n, z^*) + b_3 d\left(T(w_n, v_n, u_n), w_n\right) + b_3 d(w_n, z^*) + \gamma_n = \\
&= a_3 d(w_n, z^*) + b_3 d(w_n, z^*) + b_3 d(T(w_n, v_n, u_n), w_n) + \gamma_n = \\
&= (a_3 + b_3) d(w_n, z^*) + \gamma_n + b_3 \gamma_{n-1}.
\end{aligned}$$

Hence, passing it to the limit and applying Lemma 1.1 for $h := a_3 + b_3 \in [0, 1)$ and for $\gamma_n' := \gamma_n + b_3 \gamma_{n-1} \to 0$, as $n \to \infty$, we obtain that $\lim_{n\to\infty} w_n = z^*$ and then, we get the conclusion. $\square$

## 3. Stability of tripled fixed point iteration procedures for mixed monotone mappings

Let $(X, \leq)$ be a partially ordered set and $d$ be a metric on $X$ such that $(X, d)$ is a complete metric space. Berinde and Borcut [32] endowed the product space $X^3$ with the following partial order:

$$(x, y, z),\ (u, v, w) \in X^3,\ (u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u,\ y \leq v,\ z \geq w.$$

**Definition 3.31.** [32] *Let $(X, \leq)$ be a partially ordered set and $T : X^3 \to X$ a mapping. We say that $T$ has the mixed monotone property if $T(x, y, z)$ is monotone nondecreasing in $x$, monotone nonincreasing in $y$ and monotone nondecreasing in $z$, that is, for any $x, y, z \in X$,*

$$x_1, x_2 \in X,\ x_1 \leq x_2 \Rightarrow T(x_1, y, z) \leq T(x_2, y, z),$$

$$y_1, y_2 \in X,\ y_1 \leq y_2 \Rightarrow T(x, y_1, , z) \geq T(x, y_2, z),$$

$$z_1, z_2 \in X,\ z_1 \leq z_2 \Rightarrow T(x, y, z_1) \leq T(x, y, z_2).$$

**Definition 3.32.** [32] *An element $(x, y, z) \in X^3$ is called tripled fixed point of $T : X^3 \to X$, if*

$$T(x, y, z) = x, \quad T(y, x, y) = y, \quad T(z, y, x) = z.$$

**Remark 3.35.** *The concept of tripled fixed point from this context is different from the concept used in above section.*

A mapping $T : X^3 \to X$ is said to be a $(k, \mu, \rho)$-contraction, if and only if there exists three constants $k \geq 0,\ \mu \geq 0,\ \rho \geq 0,\ k + \mu + \rho < 1$, such that $\forall x, y, z, u, v, w \in X$,

$$d(T(x, y, z), T(u, v, w)) \leq kd(x, u) + \mu d(y, v) + \rho d(z, w). \tag{3.45}$$

In relation to (3.45), we introduce some new contractive conditions:

Let $(X, d)$ be a metric space. For a mapping $T : X^3 \to X$, there exists $a_1, a_2, a_3, b_1, b_2, b_3 \geq 0$, with $a_1 + a_2 + a_3 < 1$, $b_1 + b_2 + b_3 < 1$, such that $\forall x, y, z, u, v, w \in X$, we introduce the following definitions of contractive conditions:

$$(i)\ d\left(T(x, y, z), T(u, v, w)\right) \leq a_1 d\left(T(x, y, z), x\right) + b_1 d\left(T(u, v, w), u\right); \tag{3.46}$$

$$d\left(T(y, x, y), T(v, u, v)\right) \leq a_2 d\left(T(y, x, y), y\right) + b_2 d\left(T(v, u, v), v\right); \tag{3.47}$$

$$d\left(T(w,y,x),T(z,v,u)\right)\leq a_3 d\left(T(z,y,x),z\right)+b_3 d\left(T(w,v,u),w\right); \tag{3.48}$$

$$(ii)\ d\left(T(x,y,z),T(u,v,w)\right)\leq a_1 d\left(T(x,y,z),u\right)+b_1 d\left(T(u,v,w),x\right); \tag{3.49}$$

$$d\left(T(y,x,y),T(v,u,v)\right)\leq a_2 d\left(T(y,x,y),v\right)+b_2 d\left(T(v,u,v),y\right); \tag{3.50}$$

$$d\left(T(w,y,x),T(z,v,u)\right)\leq a_3 d\left(T(z,y,x),w\right)+b_3 d\left(T(w,v,u),z\right). \tag{3.51}$$

In the case of two matrices $A, B \in M_{(m,n)}(\mathbb{R})$, we say that $A \leq B$, if $a_{ij} \leq b_{ij}$, for all $i = \overline{1,m}$, $j = \overline{1,n}$.

In order to prove our main stability result, we give the next result which extends Lemma 1.1 to vector sequences, where inequalities between vectors means inequality on its elements:

**Lemma 3.4.** *(**Timiş**, [165]) Let $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$ be sequences of nonnegative real numbers and consider a matrix $A \in M_{3,3}(\mathbb{R})$ with nonnegative elements, so that*

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \\ w_{n+1} \end{pmatrix} \leq A \cdot \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} + \begin{pmatrix} \epsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix},\ n \geq 0, \tag{3.52}$$

*with*

*(i)* $\lim_{n\to\infty} A^n = O_3$*;*

*(ii)* $\sum_{k=0}^{\infty} \epsilon_k < \infty$, $\sum_{k=0}^{\infty} \delta_k < \infty$ *and* $\sum_{k=0}^{\infty} \gamma_k < \infty$.

*Then,* $\lim_{n\to\infty} \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Proof. For $A = 0 \in M_{(3,3)}$, the conclusion is obvious.

We rewrite (3.52) with $n := k$:

$$\begin{pmatrix} u_{k+1} \\ v_{k+1} \\ w_{k+1} \end{pmatrix} \leq A \cdot \begin{pmatrix} u_k \\ v_k \\ w_k \end{pmatrix} + \begin{pmatrix} \epsilon_k \\ \delta_k \\ \gamma_k \end{pmatrix},\ k \geq 0,$$

and sum the inequalities obtained for $k = 0, 1, 2, ..., n$.

After doing all cancellations, we obtain

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \\ w_{n+1} \end{pmatrix} \leq A^{n+1} \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} + \sum_{k=0}^{n} A^k \begin{pmatrix} \epsilon_{n-k} \\ \delta_{n-k} \\ \gamma_{n-k} \end{pmatrix}. \tag{3.53}$$

By (ii), it follows that the sequences of partial sums $\{E_n\}_{n=0}^{\infty}$, $\{\Delta_n\}_{n=0}^{\infty}$ and $\{\Gamma_n\}_{n=0}^{\infty}$, given respectively by $E_n = \epsilon_0 + \epsilon_1 + ... + \epsilon_n$, $\Delta_n = \delta_0 + \delta_1 + ... + \delta_n$ and $\Gamma_n = \gamma_0 + \gamma_1 + ... + \gamma_n$, for $n \geq 0$, converge respectively to some $E \geq 0$, $\Delta \geq 0$ and $\Gamma \geq 0$ and hence, they are bounded.

Let $M > 0$ be such that $\begin{pmatrix} E_n \\ \Delta_n \\ \Gamma_n \end{pmatrix} \leq M \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\forall n \geq 0$.

By (i), we have that $\forall e > 0$, there exists $N = N(e)$ such that

$$A^n \leq \frac{e}{2M} \cdot I_3, \quad \forall\ n \geq N,\ M > 0.$$

We can write $\sum_{k=0}^{n} A^k \begin{pmatrix} \epsilon_{n-k} \\ \delta_{n-k} \\ \gamma_{n-k} \end{pmatrix} = A^n \begin{pmatrix} \epsilon_0 \\ \delta_0 \\ \gamma_0 \end{pmatrix} + ... + A^N \begin{pmatrix} \epsilon_{n-N} \\ \delta_{n-N} \\ \gamma_{n-N} \end{pmatrix} +$
$+A^{N-1} \begin{pmatrix} \epsilon_{n-N+1} \\ \delta_{n-N+1} \\ \gamma_{n-N+1} \end{pmatrix} + ... + I_3 \begin{pmatrix} \epsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix}$.

But $A^n \begin{pmatrix} \epsilon_0 \\ \delta_0 \\ \gamma_0 \end{pmatrix} + ... + A^N \begin{pmatrix} \epsilon_{n-N} \\ \delta_{n-N} \\ \gamma_{n-N} \end{pmatrix} \leq \frac{e}{2M} \cdot I_3 \left[ \begin{pmatrix} \epsilon_0 \\ \delta_0 \\ \gamma_0 \end{pmatrix} + ... + \begin{pmatrix} \epsilon_{n-N} \\ \delta_{n-N} \\ \gamma_{n-N} \end{pmatrix} \right] =$
$\frac{e}{2M} \cdot I_3 \cdot \begin{pmatrix} E_{n-N} \\ \Delta_{n-N} \\ \Gamma_{n-N} \end{pmatrix} \leq \frac{e}{2M} \cdot I_3 \cdot M \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{e}{2} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, for all $n \geq N$.

On the other hand, if we denote $A' = \max\left\{I_3, A, ..., A^{N-1}\right\}$, we obtain

$$A^{N-1} \begin{pmatrix} \epsilon_{n-N+1} \\ \delta_{n-N+1} \\ \gamma_{n-N+1} \end{pmatrix} + ... + I_3 \begin{pmatrix} \epsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix} \leq A' \left[ \begin{pmatrix} \epsilon_{n-N+1} \\ \delta_{n-N+1} \\ \gamma_{n-N+1} \end{pmatrix} + ... + \begin{pmatrix} \epsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix} \right] =$$

$$= A' \begin{pmatrix} E_n - E_{n-N} \\ \Delta_n - \Delta_{n-N} \\ \Gamma_n - \Gamma_{n-N} \end{pmatrix}.$$

As $N$ is fixed, then

$$\lim_{n\to\infty} E_n = \lim_{n\to\infty} E_{n-N} = E, \quad \lim_{n\to\infty} \Delta_n = \lim_{n\to\infty} \Delta_{n-N} = \Delta,$$

and $\lim_{n\to\infty}\Gamma_n = \lim_{n\to\infty}\Gamma_{n-N} = \Gamma$, which shows that there exists a positive integer $k$ such that

$$A'\begin{pmatrix} E_n - E_{n-N} \\ \Delta_n - \Delta_{n-N} \\ \Gamma_n - \Gamma_{n-N} \end{pmatrix} < \frac{e}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \forall n \geq k.$$

Now, for $m = \max\{k, N\}$, we get

$$A^n\begin{pmatrix} \epsilon_0 \\ \delta_0 \\ \gamma_0 \end{pmatrix} + ... + I_3\begin{pmatrix} \epsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix} < e\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \forall n \geq m,$$

and therefore, $\lim_{n\to\infty}\sum_{k=0}^{n} A^k\begin{pmatrix} \epsilon_{n-k} \\ \delta_{n-k} \\ \gamma_{n-k} \end{pmatrix} = 0.$

Now, by letting the limit in (3.53), as $\lim_{n\to\infty} A^n = 0$, we get

$$\lim_{n\to\infty}\begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

as required. □

Let $(X, d)$ be a metric space and $T : X^3 \to X$ a mapping. For $(x_0, y_0, z_0) \in X^3$, the sequence $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ defined by

$$x_{n+1} = T(x_n, y_n, z_n),\ y_{n+1} = T(y_n, x_n, y_n),\ z_{n+1} = T(z_n, y_n, x_n), \tag{3.54}$$

with $n = 0, 1, 2, ...$, is said to be a *tripled fixed point iterative procedure.*

We give the following definition of stability with respect tho $T$, in metric spaces, relative to tripled fixed points iterative procedures:

**Definition 3.33.** *(**Timiş**, [165]) Let $(X, d)$ be a complete metric space and a mapping $T : X^3 \to X$, with*

$$Fix_t(T) = \Big\{(x^*, y^*, z^*) \in X^3 \ \mid T(x^*, y^*, z^*) = x^*,\ T(y^*, x^*, y^*) = y^*,$$

*$T(z^*, y^*, x^*) = z^*\}$, the set of tripled fixed points of $T$.*

*Let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the sequence generated by the iterative procedure defined by (3.54), where $(x_0, y_0, z_0) \in X^3$ is the initial value, which is supposed to converge to a tripled fixed point $(x^*, y^*, z^*)$ of $T$.*

*Let $\{(u_n, v_n, w_n)\}_{n=0}^{\infty} \subset X^3$ an arbitrary sequence and set*

$$\epsilon_n = d\left(u_{n+1}, T\left(u_n, v_n, w_n\right)\right), \quad \delta_n = d\left(v_{n+1}, T\left(v_n, u_n, v_n\right)\right),$$

$$\gamma_n = d\left(w_{n+1}, T\left(w_n, v_n, u_n\right)\right), \quad n = 0, 1, 2, \ldots.$$

*Then, the tripled fixed point iterative procedure defined by (3.54) is $T$-stable or stable with respect to $T$, if and only if*

$$\lim_{n\to\infty}(\epsilon_n, \delta_n, \gamma_n) = 0_{\mathbb{R}^3} \text{ implies that } \lim_{n\to\infty}(u_n, v_n, w_n) = (x^*, y^*, z^*).$$

**Theorem 3.26.** *(**Timiş**, [165]) Let $(X, \leq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space.*

*Let $T : X^3 \to X$ be a continuous mapping having the mixed monotone property on $X$ and satisfying (3.45).*

*If there exists $x_0$, $y_0$, $z_0 \in X$ such that*

$$x_0 \leq T(x_0, y_0, z_0), \quad y_0 \geq T(y_0, x_0, y_0) \text{ and } z_0 \leq T(z_0, y_0, x_0),$$

*then there exists $x^*, y^*, z^* \in X$ such that*

$$x^* = T(x^*, y^*, z^*), \quad y^* = T(y^*, x^*, y^*) \text{ and } z^* = T(z^*, y^*, x^*).$$

*Assume that for every $(x, y, z)$, $(x_1, y_1, z_1) \in X^3$, there exists $(u, v, w) \in X^3$ which is comparable to $(x, y, z)$ and $(x_1, y_1, z_1)$.*

*For $(x_0, y_0, z_0) \in X^3$, let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the tripled fixed point iterative procedure defined by (3.54).*

*Then, the tripled fixed point iterative procedure is $T$-stable or stable with respect to $T$.*

Proof. From the suppositions of the hypothesis, Berinde and Borcut[32] proved the existence and uniqueness of the tripled fixed point and now, using these results, we can study the stability of the tripled fixed point iterative procedures.

Let $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{z_n\}_{n=0}^{\infty} \subset X^3$, and set

$$\epsilon_n = d\left(u_{n+1}, T\left(u_n, v_n, w_n\right)\right), \ \delta_n = d\left(v_{n+1}, T\left(v_n, u_n, v_n\right)\right),$$

$$\gamma_n = d\left(w_{n+1}, T\left(w_n, v_n, u_n\right)\right).$$

Assume also that $\lim_{n\to\infty}\epsilon_n = \lim_{n\to\infty}\delta_n = \lim_{n\to\infty}\gamma_n = 0$ in order to establish that $\lim_{n\to\infty}u_n = x^*$, $\lim_{n\to\infty}v_n = y^*$ and $\lim_{n\to\infty}w_n = z^*$.

Therefore, using the $(k, \mu, \rho)$-contraction condition (3.45), we obtain

$$d(u_{n+1}, x^*) \leq d(u_{n+1}, T\left(u_n, v_n, w_n\right)) + d(T\left(u_n, v_n, w_n\right), x^*) =$$

$$= d(T\left(u_n, v_n, w_n\right), T(x^*, y^*, z^*)) + \epsilon_n \leq$$

$$\leq kd(u_n, x^*) + \mu d(v_n, y^*) + \rho d(w_n, z^*) + \epsilon_n. \tag{3.55}$$

$$d(v_{n+1}, y^*) \leq d(v_{n+1}, T(v_n, u_n, v_n)) + d(T(v_n, u_n, v_n), y^*) =$$

$$= d(T(v_n, u_n, v_n), T(y^*, x^*, y^*)) + \delta_n \leq$$

$$\leq kd(v_n, y^*) + \mu d(u_n, x^*) + \rho d(v_n, y^*) + \delta_n. \tag{3.56}$$

$$d(w_{n+1}, z^*) \leq d(w_{n+1}, T(w_n, v_n, u_n)) + d(T(w_n, v_n, u_n), z^*) =$$

$$= d(T(w_n, v_n, u_n), T(z^*, y^*, x^*)) + \gamma_n \leq$$

$$\leq kd(w_n, z^*) + \mu d(v_n, y^*) + \rho d(u_n, x^*) + \gamma_n. \tag{3.57}$$

From (3.55), (3.56) and (3.57), we obtain

$$\begin{pmatrix} d(u_{n+1}, x^*) \\ d(v_{n+1}, y^*) \\ d(w_{n+1}, z^*) \end{pmatrix} \leq \begin{pmatrix} k & \mu & \rho \\ \mu & k+\rho & 0 \\ \rho & \mu & k \end{pmatrix} \cdot \begin{pmatrix} d(u_n, x^*) \\ d(v_n, y^*) \\ d(w_n, z^*) \end{pmatrix} + \begin{pmatrix} \epsilon_n \\ \delta_n \\ \gamma_n \end{pmatrix}.$$

We denote $A := \begin{pmatrix} k & \mu & \rho \\ \mu & k+\rho & 0 \\ \rho & \mu & k \end{pmatrix}$, where $0 \leq k + \mu + \rho < 1$, as in (3.45).

In order to apply Lemma 3.4, we need that $A^n \to 0$, as $n \to \infty$.

Simplifying the writing, $A := \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & b_1 & h_1 \end{pmatrix}$, where

$$a_1 + b_1 + c_1 = d_1 + e_1 + f_1 = g_1 + b_1 + h_1 = k + \mu + \rho < 1.$$

$$\text{Then, } A^2 = \begin{pmatrix} k & \mu & \rho \\ \mu & k+\rho & 0 \\ \rho & \mu & k \end{pmatrix} \cdot \begin{pmatrix} k & \mu & \rho \\ \mu & k+\rho & 0 \\ \rho & \mu & k \end{pmatrix} =$$

$$= \begin{pmatrix} k^2+\mu^2+\rho^2 & 2k\mu+2\mu\rho & 2k\rho \\ 2k\mu+\rho\mu & k^2+\mu^2+\rho^2+2k\rho & \mu\rho \\ 2k\rho+\mu^2 & 2k\mu+2\rho\mu & k^2+\rho^2 \end{pmatrix} := \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & b_2 & h_2 \end{pmatrix},$$

where

$$a_2 + b_2 + c_2 = d_2 + e_2 + f_2 = g_2 + b_2 + h_2 = (k + \mu + \rho)^2 < k + \mu + \rho < 1.$$

Now, we prove by induction that

$$A^n = \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & b_n & h_n \end{pmatrix},$$

where

$$a_n + b_n + c_n = d_n + e_n + f_n = g_n + b_n + h_n = (k+\mu+\rho)^n < k+\mu+\rho < 1. \tag{3.58}$$

If we assume that (3.58) is true for $n$, then since

$$A^{n+1} = \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & b_n & h_n \end{pmatrix} \cdot \begin{pmatrix} k & \mu & \rho \\ \mu & k+\rho & 0 \\ \rho & \mu & k \end{pmatrix} =$$

$$= \begin{pmatrix} ka_n + \mu b_n + \rho c_n & \mu a_n + kb_n + \rho b_n + \mu c_n & \rho a_n + kc_n \\ kd_n + \mu e_n + \rho f_n & \mu d_n + ke_n + \rho e_n + \mu f_n & \rho d_n + kf_n \\ kg_n + \mu b_n + \rho h_n & \mu g_n + kb_n + \rho b_n + \mu h_n & \rho g_n + kh_n \end{pmatrix}.$$

We have

$$a_{n+1} + b_{n+1} + c_{n+1} = ka_n + \mu b_n + \rho c_n + \mu a_n + kb_n + \rho b_n + \mu c_n + \rho a_n + kc_n =$$
$$= (k+\mu+\rho)a_n + (k+\mu+\rho)b_n + (k+\mu+\rho)c_n = (k+\mu+\rho)(a_n+b_n+c_n) =$$
$$= (k+\mu+\rho)(k+\mu+\rho)^n = (k+\mu+\rho)^{n+1} < k+\mu+\rho < 1.$$

Similarly, we obtain

$$d_{n+1} + e_{n+1} + f_{n+1} = g_{n+1} + b_{n+1} + h_{n+1} = (k+\mu+\rho)^{n+1} < k+\mu+\rho < 1.$$

Therefore, $\lim_{n\to\infty} A^n = O_3$ and now, having satisfied the conditions of the hypothesis of Lemma 3.4, we can apply it and we get

$$\lim_{n\to\infty} \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = \begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix},$$

so the tripled fixed point iteration procedure defined by (3.54) is $T$-stable. □

**Remark 3.36.** *Theorem 3.26 completes the existence theorem of tripled fixed points of Berinde and Borcut [**32**] with the stability result for the tripled fixed point iterative procedures, using mixed-monotone operators.*

**Corollary 3.12.** *(**Timiş**, [**165**]) Let $(X, \leq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space.*

*Let $T : X^3 \to X$ be a continuous mapping having the mixed monotone property on $X$.*

*There exists $\kappa \in [0, 1)$, such that for each $x, y, z, u, v, w \in X$, $T$ satisfies the following contraction condition:*

$$d(T(x,y,z), T(u,v,w)) \leq \frac{\kappa}{3}\left\{d(x,u) + d(y,v) + d(z,w)\right\}. \tag{3.59}$$

*If there exists $x_0$, $y_0$, $z_0 \in X$ such that*

$$x_0 \leq T(x_0, y_0, z_0), \quad y_0 \geq T(y_0, x_0, y_0) \quad \text{and} \quad z_0 \leq T(z_0, y_0, x_0),$$

*then there exists $x^*, y^*, z^* \in X$ such that*

$$x^* = T(x^*, y^*, z^*), \quad y^* = T(y^*, x^*, y^*) \quad \text{and} \quad z^* = T(z^*, y^*, x^*).$$

*Assume that for every $(x, y, z)$, $(x_1, y_1, z_1) \in X^3$, there exists $(u, v, w) \in X^3$ which is comparable to $(x, y, z)$ and $(x_1, y_1, z_1)$.*

*For $(x_0, y_0, z_0) \in X^3$, let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the tripled fixed point iterative procedure defined by (3.54).*

*Then, the tripled fixed point iterative procedure is $T$-stable or stable with respect to $T$.*

Proof. We apply Theorem 3.26, for $k = \mu = \rho := \frac{\kappa}{3}$. □

**Remark 3.37.** *Corollary 3.12 completes the existence theorem of tripled fixed points of Berinde and Borcut [**32**] with the stability result for the tripled fixed point iterative procedures, using mixed-monotone operators.*

**Theorem 3.27.** *(**Timiş**, [**165**]) Let $(X, \leq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space.*

*Let $T : X^3 \to X$ be a continuous mapping having the mixed monotone property on $X$ and satisfying (3.46), (3.47) and (3.48).*

*If there exists $x_0$, $y_0$, $z_0 \in X$ such that*

$$x_0 \leq T(x_0, y_0, z_0), \quad y_0 \geq T(y_0, x_0, y_0) \quad \text{and} \quad z_0 \leq T(z_0, y_0, x_0),$$

*then there exists $x^*, y^*, z^* \in X$ such that*

$$x^* = T(x^*, y^*, z^*), \quad y^* = T(y^*, x^*, y^*) \quad \text{and} \quad z^* = T(z^*, y^*, x^*).$$

*Assume that for every $(x, y, z)$, $(x_1, y_1, z_1) \in X^3$, there exists $(u, v, w) \in X^3$ which is comparable to $(x, y, z)$ and $(x_1, y_1, z_1)$.*

*For $(x_0, y_0, z_0) \in X^3$, let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the tripled fixed point iterative procedure defined by (3.54).*

*Then, the tripled fixed point iterative procedure is $T$-stable or stable with respect to $T$.*

Proof. Let $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{z_n\}_{n=0}^{\infty} \subset X^3$, $\epsilon_n = d\left(u_{n+1}, T\left(u_n, v_n, w_n\right)\right)$, $\delta_n = d\left(v_{n+1}, T\left(v_n, u_n, v_n\right)\right)$ and $\gamma_n = d\left(w_{n+1}, T\left(w_n, v_n, u_n\right)\right)$. Assume also that

$\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \gamma_n = 0$ in order to establish that $\lim_{n\to\infty} u_n = x^*$, $\lim_{n\to\infty} v_n = y^*$ and $\lim_{n\to\infty} w_n = z^*$.

Therefore, using the contraction condition (3.46), we obtain

$$d(u_{n+1}, x^*) \leq d(u_{n+1}, T\left(u_n, v_n, w_n\right)) + d(T\left(u_n, v_n, w_n\right), x^*) =$$

$$= d(T\left(u_n, v_n, w_n\right), T(x^*, y^*, z^*)) + \epsilon_n \leq$$

$$\leq a_1 d\left(T(x^*, y^*, z^*), x^*\right) + b_1 d\left(T(u_n, v_n, w_n), u_n\right) + \epsilon_n \leq$$

$$\leq a_1 d(x^*, x^*) + b_1 d\left(T(u_n, v_n, w_n), u_{n+1}\right) + b_1 d(u_{n+1}, x^*) + b_1 d(x^*, u_n) + \epsilon_n =$$

$$= a_1 d(x^*, x^*) + b_1 d(u_{n+1}, x^*) + b_1 d(x^*, u_n) + (b_1 + 1)\epsilon_n.$$

Hence, $(1-b_1)d(u_{n+1}, x^*) \leq b_1 d(x^*, u_n) + \epsilon'_n$, where $\epsilon'_n := (b_1+1)\epsilon_n + a_1 d(x^*, x^*)$. Passing it to the limit and applying Lemma 1.1 for $\frac{b_1}{1-b_1} \in [0, 1)$, we obtain that $\lim_{n\to\infty} u_n = x^*$.

Now, using the contraction condition (3.47), we obtain

$$d(v_{n+1}, y^*) \leq d(v_{n+1}, T\left(v_n, u_n, v_n\right)) + d(T\left(v_n, u_n, v_n\right), y^*) =$$

$$= d(T\left(v_n, u_n, v_n\right), T(y^*, x^*, y^*)) + \delta_n \leq$$

$$\leq a_2 d\left(T(y^*, x^*, y^*), y^*\right) + b_2 d\left(T(v_n, u_n, v_n), v_n\right) + \delta_n \leq$$

$$\leq a_2 d(y^*, y^*) + b_2 d\left(T(v_n, u_n, v_n), v_{n+1}\right) + b_2 d(v_{n+1}, y^*) + b_2 d(y^*, v_n) + \delta_n =$$

$$= a_2 d(y^*, y^*) + b_2 d(v_{n+1}, y^*) + b_2 d(y^*, v_n) + (b_2 + 1)\delta_n.$$

So, $(1 - b_2)d(v_{n+1}, y^*) \leq b_2 d(y^*, v_n) + \delta'_n$, where $\delta'_n := (b_2 + 1)\delta_n + a_2 d(y^*, y^*)$. Passing it to the limit and applying Lemma 1.1 for $\frac{b_2}{1-b_2} \in [0, 1)$, we obtain that $\lim_{n\to\infty} v_n = y^*$.

Similarly, using the contraction condition (3.48), we obtain

$$d(w_{n+1}, z^*) \leq d(w_{n+1}, T\left(z_n, v_n, u_n\right)) + d(T\left(z_n, v_n, u_n\right), z^*) =$$

$$= d(T\left(w_n, v_n, u_n\right), T(z^*, y^*, x^*)) + \gamma_n \leq$$

$$\leq a_3 d\left(T(z^*, y^*, x^*), z^*\right) + b_3 d\left(T(w_n, v_n, u_n), w_n\right) + \gamma_n \leq$$

$$\leq a_3 d(z^*, z^*) + b_3 d\left(T(w_n, v_n, u_n), w_{n+1}\right) + b_3 d(w_{n+1}, z^*) + b_3 d(z^*, w_n) + \gamma_n =$$

$$= a_3 d(z^*, z^*) + b_3 d(w_{n+1}, z^*) + b_3 d(z^*, w_n) + (b_3 + 1)\gamma_n.$$

Therefore, $(1 - b_3)d(w_{n+1}, z^*) \leq b_3 d(z^*, w_n) + \gamma'_n$, where $\gamma'_n := (b_3 + 1)\gamma_n + a_3 d(z^*, z^*)$. Passing it to the limit and applying Lemma 1.1 for $\frac{b_3}{1-b_3} \in [0, 1)$, we obtain that $\lim_{n\to\infty} w_n = z^*$ and then, we get the conclusion. □

**Theorem 3.28.** *(**Timiş**, [165]) Let $(X, \leq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space.*

*Let $T : X^3 \to X$ be a continuous mapping having the mixed monotone property on $X$ and satisfying (3.49), (3.50) and (3.51).*

*If there exists $x_0$, $y_0$, $z_0 \in X$ such that*

$$x_0 \leq T(x_0, y_0, z_0), \quad y_0 \geq T(y_0, x_0, y_0) \quad \text{and} \quad z_0 \leq T(z_0, y_0, x_0),$$

*then there exists $x^*, y^*, z^* \in X$ such that*

$$x^* = T(x^*, y^*, z^*), \quad y^* = T(y^*, x^*, y^*) \quad \text{and} \quad z^* = T(z^*, y^*, x^*).$$

*Assume that for every $(x, y, z)$, $(x_1, y_1, z_1) \in X^3$, there exists $(u, v, w) \in X^3$ which is comparable to $(x, y, z)$ and $(x_1, y_1, z_1)$.*

*For $(x_0, y_0, z_0) \in X^3$, let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset X^3$ be the tripled fixed point iterative procedure defined by (3.54).*

*Then, the tripled fixed point iterative procedure is $T$-stable or stable with respect to $T$.*

Proof. Let $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{z_n\}_{n=0}^{\infty} \subset X^3$, $\epsilon_n = d\left(u_{n+1}, T\left(u_n, v_n, w_n\right)\right)$, $\delta_n = d\left(v_{n+1}, T\left(v_n, u_n, v_n\right)\right)$ and $\gamma_n = d\left(w_{n+1}, T\left(w_n, v_n, u_n\right)\right)$. Assume also that $\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \gamma_n = 0$ in order to establish that $\lim_{n\to\infty} u_n = x^*$, $\lim_{n\to\infty} v_n = y^*$ and $\lim_{n\to\infty} w_n = z^*$.

Therefore, using the contraction condition (3.49), we obtain

$$\begin{aligned}
d(u_{n+1}, x^*) &\leq d(u_{n+1}, T\left(u_n, v_n, w_n\right)) + d(T\left(u_n, v_n, w_n\right), x^*) = \\
&= d(T\left(u_n, v_n, w_n\right), T(x^*, y^*, z^*)) + \epsilon_n \leq \\
&\leq a_1 d\left(T(x^*, y^*, z^*), u_n\right) + b_1 d\left(T(u_n, v_n, w_n), x^*\right) + \epsilon_n \leq \\
&\leq a_1 d(u_n, x^*) + b_1 d\left(T(u_n, v_n, w_n), u_n\right) + b_1 d(u_n, x^*) + \epsilon_n = \\
&= (a_1 + b_1) d(u_n, x^*) + \epsilon_n + b_1 \epsilon_{n-1}.
\end{aligned}$$

Hence, passing it to the limit and applying Lemma 1.1 for $h := a_1 + b_1 \in [0, 1)$ and for $\epsilon_n' := \epsilon_n + b_1 \epsilon_{n-1} \to 0$, as $n \to \infty$, we obtain that $\lim_{n\to\infty} u_n = x^*$.

Now, using the contraction condition (3.50), we obtain

$$\begin{aligned}
d(v_{n+1}, y^*) &\leq d(v_{n+1}, T\left(v_n, u_n, v_n\right)) + d(T\left(v_n, u_n, v_n\right), y^*) = \\
&= d(T\left(v_n, u_n, v_n\right), T(y^*, x^*, y^*)) + \delta_n \leq \\
&\leq a_2 d\left(T(y^*, x^*, y^*), v_n\right) + b_2 d\left(T(v_n, u_n, v_n), y^*\right) + \delta_n \leq \\
&\leq a_2 d(v_n, y^*) + b_2 d\left(T(v_n, u_n, v_n), v_n\right) + b_2 d(v_n, y^*) + \delta_n =
\end{aligned}$$

$$= (a_2 + b_2)d(v_n, y^*) + \delta_n + b_2\delta_{n-1}.$$

So, passing it to the limit and applying Lemma 1.1 for $h := a_2 + b_2 \in [0, 1)$ and for $\delta'_n := \delta_n + b_2\delta_{n-1} \to 0$, as $n \to \infty$, we get $\lim_{n\to\infty} v_n = y^*$.

Similarly, using the contraction condition (3.51), we obtain

$$d(w_{n+1}, z^*) \leq d(w_{n+1}, T(z_n, v_n, u_n)) + d(T(z_n, v_n, u_n), z^*) =$$

$$= d(T(w_n, v_n, u_n), T(z^*, y^*, x^*)) + \gamma_n \leq$$

$$\leq a_3 d(T(z^*, y^*, x^*), w_n) + b_3 d(T(w_n, v_n, u_n), z^*) + \gamma_n \leq$$

$$\leq a_3 d(w_n, z^*) + b_3 d(T(w_n, v_n, u_n), w_n) + b_3 d(w_n, z^*) + \gamma_n =$$

$$= a_3 d(w_n, z^*) + b_3 d(w_n, z^*) + b_3 d(T(w_n, v_n, u_n), w_n) + \gamma_n =$$

$$= (a_3 + b_3)d(w_n, z^*) + \gamma_n + b_3\gamma_{n-1}.$$

Hence, passing it to the limit and applying Lemma 1.1 for $h := a_3 + b_3 \in [0, 1)$ and for $\gamma'_n := \gamma_n + b_3\gamma_{n-1} \to 0$, as $n \to \infty$, we obtain that $\lim_{n\to\infty} w_n = z^*$ and then, we get the conclusion. □

## 4. Illustrative example

**Example 4.43.** *(**Timiş**, [**165**])*

Let $(X, d)$ be a complete metric space, where $X = \mathbb{R}$, $d(x, y) = |x - y|$ and a continuous and mixed monotone mapping $T : \mathbb{R}^3 \to \mathbb{R}$, with $T(x, y, z) = \frac{2x-2y+2z+1}{12}$.

Berinde and Borcut [**32**] proved the existence and the uniqueness of the tripled fixed point of $T$, respectively $(x^*, y^*, z^*) = \left(\frac{1}{10}, \frac{1}{10}, \frac{1}{10}\right)$, using $(x_0, y_0, z_0) = \left(\frac{1}{20}, \frac{1}{5}, \frac{1}{20}\right)$.

For $k = \frac{1}{2}$, $T$ satisfies the contraction condition (3.59), i.e.,

$$d(T(x, y, z), T(u, v, w)) \leq \frac{\kappa}{3}\left[d(x, u) + d(y, v) + d(z, w)\right],$$

for each $x, y, z, u, v, w \in X$, with $x \geq u$, $y \leq v$ and $z \geq w$.

We apply Corollary 3.12 in order to prove the stability of the tripled fixed point iteration procedure.

Let $\{(x_n, y_n, z_n)\}_{n=0}^{\infty} \subset \mathbb{R}^3$ be the sequence generated by the iterative procedure defined by (2.41), where $(x_0, y_0, z_0) = \left(\frac{1}{20}, \frac{1}{5}, \frac{1}{20}\right) \in \mathbb{R}^3$ is the initial value, which converges to a tripled fixed point $(x^*, y^*, z^*) = (\frac{1}{10}, \frac{1}{10}, \frac{1}{10})$ of $T$.

Let $\{(u_n, v_n, w_n)\}_{n=0}^{\infty} \subset \mathbb{R}^3$ an arbitrary sequence and set

$$\epsilon_n = d\left(u_{n+1}, T\left(u_n, v_n, w_n\right)\right), \quad \delta_n = d\left(v_{n+1}, T\left(v_n, u_n, v_n\right)\right),$$

$$\gamma_n = d\left(w_{n+1}, T\left(w_n, v_n, u_n\right)\right), \quad n = 0, 1, 2, \ldots.$$

Assume that $\lim_{n\to\infty}(\epsilon_n, \delta_n, \gamma_n) = 0_{\mathbb{R}^3}$. Then,

$$\epsilon_n = d\left(u_{n+1}, T\left(u_n, v_n, w_n\right)\right) = \left|u_{n+1} - \frac{2u_n - 2v_n + 2w_n + 1}{12}\right|,$$

$$\delta_n = d\left(v_{n+1}, T\left(v_n, u_n, v_n\right)\right) = \left|v_{n+1} - \frac{2v_n - 2u_n + 2v_n + 1}{12}\right|,$$

$$\gamma_n = d\left(w_{n+1}, T\left(w_n, v_n, u_n\right)\right) = \left|w_{n+1} - \frac{2w_n - 2v_n + 2u_n + 1}{12}\right|,$$

and passing to the limit for $n \to \infty$, we obtain that

$$\lim_{n\to\infty}(u_n, v_n, w_n) = \left(\frac{1}{10}, \frac{1}{10}, \frac{1}{10}\right),$$

which is the unique tripled fixed point of $T$.

Hence, the tripled fixed point iterative procedure defined by (2.41) is $T$-stable.

CHAPTER 6

# Conclusions

Fixed point theory has an important role in the nonlinear analysis domain, with an expansive evolution in the last decades and with many concrete results.

Following the basic result from the metrical fixed point theory, respectively the Contraction Principle of Picard-Banach-Caccioppoli [14], an important part of the scientific literature appeared, with applications to functional equations, differential equations, integral equations etc.

In order to solve a nonlinear equation, we appeal to approximating fixed points of a corresponding contractive type mappings. From the existing methods for approximating fixed points, we studied the Picard iteration and the Jungck type iteration procedure.

Establishing the stability of these methods is very important in practical applications, because a fixed point iteration which is numerically stable will produce small modifications on the approximate value of the fixed point during the computational process.

The concept of stability of a fixed point iteration procedure has been systematically studied by Harder [60], Harder and Hicks [61], [62], and since then, many other stability results for several fixed point iteration procedures and for various classes of nonlinear operators were obtained.

In this paper, we treat the problem of stability of fixed point, common fixed point, coincidence point and tripled fixed point iteration procedures, for certain class of mappings.

In the chapter named **Stability of fixed point, common fixed point and coincidence point iterative procedures for mappings satisfying an explicit contractive condition**, we present the concept of stability of fixed point iteration procedures and we survey the most significant contributions to this area.

Berinde [**27**] introduced a natural concept of stability, called *weak stability*, and we transposed this notion to the case of two mappings $S$ and $T$ with a coincidence point, named $(S,T)$-*weak stability*.

We established weak stability results for common fixed points iteration procedures, on the metric space $(X,d)$, with $Y \subset X$ and $S,T : Y \to X$ two nonself mappings with a coincidence point, satisfying the following contraction condition:

(i) $d(Tx,Ty) \leq qd(Sx,Sy)$, for all $x,y \in Y$ and $q \in (0,1)$;
(ii) $d(Tx,Ty) \leq qd(Sx,Sy)+Ld(Sx,Tx)$, for all $x,y \in Y$, $q \in (0,1)$ and $L \geq 0$.

Because some fixed point iteration procedures are not weakly stable and because the stability can be obtained in the meaning of a new concept, we developed a weaker notion, named $w^2$-*stability*.

Therefore, we gave some stability results on a complete metric space $(X,d)$ and using a mapping $T : X \to X$ satisfying the following contractive conditions:

(1) $d(Tx,Ty) < \max\{d(x,Tx), d(y,Ty)\}$;
(2) $d(Tx,Ty) < \max\{d(x,Tx), d(y,Ty), d(x,y)\}$;
(3) $d(Tx,Ty) < \max\{d(x,Tx), d(y,Ty), d(x,y), d(x,Ty), d(y,Tx)\}$;
(4) $d(Tx,Ty) < \max\left\{d(x,Tx), d(y,Ty), d(x,y), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$;

for all $x,y \in X$ and $x \neq y$.

Moreover, we gave stability results on a complete metric space $(X,d)$, using two mappings $S,T : X \to X$ with a coincidence point and satisfying the following contractive conditions:

(1) $d(Tx,Ty) < \max\{d(Sx,Tx), d(Sy,Ty)\}$;
(2) $d(Tx,Ty) < \max\{d(Sx,Ty), d(Sy,Tx)\}$;
(3) $d(Tx,Ty) < \max\{d(Sx,Tx), d(Sy,Ty), d(Sx,Sy)\}$;
(4) $d(Tx,Ty) < \max\{d(Sx,Tx), d(Sy,Ty), d(Sx,Sy), d(Sx,Ty), d(Sy,Tx)\}$;
(5) $d(Tx,Ty) < \max\left\{d(Sx,Tx), d(Sy,Ty), d(Sx,Sy), \frac{d(Sx,Ty)+d(Sy,Tx)}{2}\right\}$;

for all $x,y \in X$ and $x \neq y$.

We also presented some examples of weak stable, $w^2$-stable but nor stable iterations with respect to $T$ and with respect to $(S, T)$.

Our research can be extended by using other iterative methods, e.g. Ishikawa, Mann, or another contractive conditions.

Following the development initiated by Popa [**119**], [**120**], [**121**], several classical fixed point theorems and common fixed point theorems have been recently unified by considering general contractive conditions expressed by an implicit relation.

On the chapter entitled **Stability of fixed point, common fixed point and coincidence point iterative procedures for contractive mappings defined by implicit relations**, we continued to study the stability of Picard iterative procedure and also of Jungck iterative procedure for common fixed points and coincidence points, for contractive mappings satisfying various implicit relations, with different number of parameters.

Using the set of all continuous real functions $\mathbb{F}$ introduced by Popa [**119**], $F : \mathbb{R}_+^5 \to \mathbb{R}$, with the following conditions:

(1) $F$ is continuous in each coordinate variable,

(2) there exists $h \in [0, 1)$ such that, for all $u, v, w \geq 0$ satisfying
- $F(u, v, u, v, w) \leq 0$ or
- $F(u, v, v, u, w) \leq 0$,

we have that $u \leq h \max\{v, w\}$,

we established a general stability result for the Picard iteration procedure, on the complete metric space $(X, d)$, for a mapping $T : X \to X$, with $Fix(X) \neq \emptyset$, for which there exists $F \in \mathbb{F}$, such that for all $x, y \in X$,

$$F\left(d(Tx, Ty), d(x, y), d(x, Ty), d(y, Tx), \frac{d(x, Tx) + d(y, Ty)}{2}\right) \leq 0.$$

We also gave a stability result for the common fixed point iteration procedure of Jungck-type using weakly compatible mappings satisfying (E.A) property and defined by an implicit contraction condition on the complete metric space $(X, d)$, $S, T : X \to X$, for which there exists $F \in \mathbb{F}$, such that for all $x, y \in X$,

$$F\left(d(Tx, Ty), d(Sx, Sy), d(Sx, Ty), d(Sy, Tx), \frac{d(Sx, Tx) + d(Sy, Ty)}{2}\right) \leq 0.$$

On the other hand, using the set of all continuous real functions $\mathbb{F}$ introduced by Popa [**120**], [**121**], $F : \mathbb{R}_+^6 \to R_+$, with the following conditions:

(1) (a) $F$ is non-increasing in the fifth variable and $F(u, v, v, u, u + v, 0) \le 0$ for $u, v \ge 0 \Longrightarrow \exists h \in [0, 1)$ such that $u \ge hv$;
    (b) $F$ is non-increasing in the fourth variable and $F(u, v, 0, u+v, u, v) \le 0$ for $u, v \ge 0 \Longrightarrow \exists h \in [0, 1)$ such that $u \ge hv$;
    (c) $F$ is non-increasing in the third variable and $F(u, v, u+v, 0, v, u) \le 0$ for $u, v \ge 0 \Longrightarrow \exists h \in [0, 1)$ such that $u \ge hv$;

(2) $F(u, u, 0, 0, u, u) > 0$, for all $u > 0$,

we also established a stability result for common fixed point iterative procedures, on the complete metric space $(X, d)$, for two mappings $S, T : X \to X$, with $Fix(X) \neq \emptyset$, for which there exists $F \in \mathbb{F}$, such that for all $x, y \in X$,

$$F\left(d(Tx, Ty), d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\right) \le 0.$$

Our research can be extended by using another iterative methods, e.g. Ishikawa, Mann, another contractive conditions, or by modifying the number of the parameters.

The idea of the chapter **A new point of view on the stability of fixed point iterative procedures** was due to Professor I. A. Rus [**136**], who unified the notions of stability in difference equations, dynamical systems, differential equations, operator theory and numerical analysis by new ones.

By considering these new notions, we gave some stability result for Picard iteration procedure for mappings which satisfy certain contractive conditions.

We studied the relationship between the two stability definitions, the one of Harder [**60**] and the other one due to Rus [**136**].

We gave stability results on the metric space $(X, d)$, for self mappings $T : X \to X$ satisfying the following contraction conditions:

(1) $d\left(Tx, Ty\right) \le \delta_u d(x, y) + L_u d(x, Tx)$, $\delta_u \in [0, 1)$, $L_u \ge 0$;
(2) $d\left(Tx, Ty\right) \le \delta d(x, y) + L \min\left\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right\}$, $\delta \in (0, 1)$, $L \ge 0$;

for all $x, y \in X$.

We also presented some examples of mappings satisfying certain contractive conditions for which the associated Picard iteration was not stable in the sense of Harder but it was actually stable in the sense of Rus.

On the other hand, we transposed the notion of stability introduced by Rus [**136**] to common fixed points and we studied the relationship between the stability concept introduced by Singh and Prasad [**152**] given for a pair of mappings $(S,T)$ with a coincidence point and our new stability concept.

We gave some stability results for the Jungck-type iteration procedure, with respect to two mappings which satisfy the following contractive conditions:

(1) $d(Tx,Ty) \le ad(Sx,Sy)$, $a \in [0,1)$;

(2) $d(Tx,Ty) \le h \max\{d(Sx,Ty), d(Sy,Tx)\}$, $h \in [0,1)$;

for each $x, y \in X$.

Our research can be extended by using another iterative methods, e.g. Ishikawa, Mann, or another contractive conditions.

An open problem is the study of the stability in the sense of Rus for general nonexpansive mappings as well as for general almost contractions, that do not satisfy a certain uniqueness condition.

In the chapter **Stability of tripled fixed point iteration procedures**, we introduced the concept of stability for tripled fixed point iterative procedures and also established stability results for mixed monotone mappings and monotone mappings, satisfying various contractive conditions by extension from coupled fixed points to tripled fixed points of contractive conditions employed by Olatinwo [**102**].

We established stability results for the tripled fixed point iteration procedure, on the metric space $(X,d)$, for mappings $T : X^3 \to X$, in the case of the monotone property and also in the case of the mixed monotone property of $T$, satisfying the following contraction conditions:

(1) for $k \ge 0$, $\mu \ge 0$, $\rho \ge 0$, $k+\mu+\rho < 1$,

$$d(T(x,y,z),T(u,v,w)) \le kd(x,u) + \mu d(y,v) + \rho d(z,w);$$

(2) for $a_1, a_2, a_3, b_1, b_2, b_3 \ge 0$, $a_1+a_2+a_3 < 1$, $b_1+b_2+b_3 < 1$,

$$d\left(T(x,y,z),T(u,v,w)\right) \le a_1 d\left(T(x,y,z),x\right) + b_1 d\left(T(u,v,w),u\right);$$

$$d\left(T(y,x,z),T(v,u,w)\right) \le a_2 d\left(T(y,x,z),y\right) + b_2 d\left(T(v,u,w),v\right);$$

$$d\left(T(w,y,x),T(z,v,u)\right)\leq a_3 d\left(T(z,y,x),z\right)+b_3 d\left(T(w,v,u),w\right);$$

(3) for $a_1,a_2,a_3,b_1,b_2,b_3\geq 0,\ a_1+a_2+a_3<1,\ b_1+b_2+b_3<1,$

$$d\left(T(x,y,z),T(u,v,w)\right)\leq a_1 d\left(T(x,y,z),u\right)+b_1 d\left(T(u,v,w),x\right);$$

$$d\left(T(y,x,z),T(v,u,w)\right)\leq a_2 d\left(T(y,x,z),v\right)+b_2 d\left(T(v,u,w),y\right);$$

$$d\left(T(w,y,x),T(z,v,u)\right)\leq a_3 d\left(T(z,y,x),w\right)+b_3 d\left(T(w,v,u),z\right);$$

$\forall x,y,z,u,v,w\in X$.

Moreover, we have illustrated these results with an example.

Our research can be extended by using some other contractive conditions, for mappings satisfying various properties.

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[155] **Timiş, I.**, *New stability of Picard iteration for mappings defined by implicit relations* (in preparation)

[156] **Timiş, I.**, *New stability results of Picard iteration for common fixed points and contractive type mappings* (in preparation)

[157] **Timiş, I.**, *New stability results of Picard iteration for contractive type mappings* (submitted)

[158] **Timiş, I.**, *On the weak stability of fixed point iterative methods* (in preparation)

[159] **Timiş, I.**, *On the weak stability of Picard iteration for some contractive type mappings*, An. Univ. Craiova Ser. Mat. Inform. **37** (2) (2010), 106-114

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## Addend: Published and Communicated Research Papers

This thesis is developed on the basis of the following published and communicated papers:

### I. List of published research papers:

1. Timiş, I., *On the weak stability of Picard iteration for some contractive type mappings*, An. Univ. Craiova Ser. Mat. Inform. 37 (2) (2010), 106-114
2. Timiş, I., *On the weak stability of Picard iteration for some contractive type mappings and coincidence theorems*, International Journal of Computer Applications 37 (4) (2012), 9-13
3. Timiş, I., *Stability of Jungck-type iterative procedure for some contractive type mappings via implicit relations*, Miskolc Math. Notes 13 (2) (2012), 555-567
4. Timiş, I., *Stability of the Picard iterative procedure for mappings which satisfy implicit relations*, Comm. Appl. Nonlinear Anal. 19 (2012), no. 4, 37-44
5. Timiş, I., *Stability of tripled fixed point iteration procedures for monotone mappings*, Ann. Univ. Ferrara (2012) DOI 10.1007/s11565-012-0171-7
6. Timiş, I. and Berinde, V., *Weak stability of iterative procedures for some coincidence theorems*, Creative Math. Inform. 19 (2010), 85-95

### II. List of communicated research papers:

1. Timiş, I., *New stability results of Picard iteration for common fixed points and contractive type mappings*, presented at SYNASC 2012, Timişoara, 26-29 Sept. 2012
2. Timiş, I., *On the weak stability of fixed point iterative methods*, presented at ICAM7, Baia Mare, 1-4 Sept. 2010

3. Timiş, I., *Stability of Jungck-type iterative procedure for common fixed points and contractive mappings via implicit relations*, presented at ICAM8, Baia Mare, 27-30 Oct. 2011

**III. List of submitted research papers:**

1. Timiş, I., *New stability results of Picard iteration for contractive type mappings*
2. Timiş, I., *Stability of Jungck-type iterative procedure for common fixed points and contractive mappings satisfying an implicit relation*
3. Timiş, I., *Stability of tripled fixed point iteration procedures for mixed monotone mappings*