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ON ZAMFIRESCU'S FIXED POINT THEOREM

VASILE BERINDE and MĂDĂLINA BERINDE

Zamfirescu's fixed point theorem, established for a class of quasi-nonexpansive operators, is one of the most interesting extensions of Banach's well known contraction principle. It is the main aim of this paper to survey some recent developments in metrical fixed point theory that are connected to, extend or generalize Zamfirescu's fixed point theorem.

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1. INTRODUCTION

An operator T which satisfies the contractive conditions of Theorem Z in Section 2 will be called a *Zamfirescu operator*. The class of Zamfirescu operators is one of the most studied classes of quasi-nonexpansive type operators. In this class of mappings, all important fixed point iteration procedures, i.e., Picard [22], Mann [15] and Ishikawa [17] iterations, are known to converge to the unique fixed point of T.

Harder and Hicks [11] introduced a concept of stability for fixed point iteration procedures and proved that, in a complete metric space setting, the Picard iteration is stable with respect to the class of Zamfirescu operators. The same authors proved that, in a linear normed space, certain Mann iterations are stable with respect to any Zamfirescu operator.

Rhoades [20, Theorem 31] completed the previous results, showing that the Ishikawa iteration converges to the fixed point of and is stable with respect to a Zamfirescu operator.

Some of the convergence results in [16] and [17] were very recently extended from uniformly convex to arbitrary Banach spaces, by simultaneously weakening the assumptions on the sequence $\{\alpha_n\}$ which is involved in the definition of Mann and Ishikawa fixed point iterations, see [5], [7].

Starting from the fact that Picard iteration, Mann iteration, Ishikawa iteration and, in particular, Krasnoselskij iteration can be used to approximate fixed points of Zamfirescu operators, the first author has shown in [4] that Picard iteration converges faster than Mann iteration for this class of mappings, as suggested by some previous empirical studies reported in [1]. Very recently, Babu and Vara Prashad [23] established that Mann iteration is faster than Ishikawa iteration in the same class of Zamfirescu operators.

Recently the first author [8] used a contractive condition that is satisfied in particular by any Zamfirescu operator, in order to obtain further generalizations of Zamfirescu's fixed point theorem.

Using the main idea of proving Zamfirescu's fixed point theorem in [1], we also introduced in [6] a more general contractive condition that also includes Zamfirescu's conditions, amongst other important contractive type conditions, and obtained very general constructive fixed point theorems. These results were then extended to weak φ-contractions in [2]. A comparison of our weak contractive condition to other important contractive conditions in metrical fixed point theory was presented in [3].

Motivated by such a rich literature concerning Zamfirescu operators, the present paper aims to survey some of the most important recent results on that topic.

The next section presents some basic results in metrical fixed point theory, including the statement of Zamfirescu's fixed point theorem, while the subsequent sections are devoted to some extensions of it.

2. PRELIMINARIES

Banach's classical contraction principle is one of the most useful results in fixed point theory, see for example [21]. In a metric space setting it can be briefly stated as follows.

THEOREM B. *Let (X,d) be a complete metric space and $T: X \to X$ an a-contraction, i.e., a map satisfying*

$$d(Tx,Ty) \le a\, d(x,y) \tag{2.1}$$

for all $x, y \in X$, where $0 \le a < 1$ is constant. Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \dots, \tag{2.2}$$

converges to p, for any $x_0 \in X$.

Theorem B has many applications in solving nonlinear equations, but suffers from one drawback – the contractive condition (2.1) forces T to be continuous on X. Kannan [13] obtained in 1968 a fixed point theorem which extends Theorem B to mappings that need not be continuous, by considering instead of (2.1) the condition: there exists $b \in [0, \frac{1}{2})$ such that

$$(2.3) \qquad d(Tx, Ty) \le b\big[d(x, Tx) + d(y, Ty)\big]$$

for all $x, y \in X$.

Following Kannan's theorem, a lot of papers were devoted to obtaining fixed point theorems for various classes of contractive type conditions which do not require the continuity of T, see for example [21] and references therein. One of them, actually a sort of dual of Kannan's fixed point theorem, due to Chatterjea [9], is based on a condition similar to (2.3): there exists $c \in [0, \frac{1}{2})$ such that

$$(2.4) \qquad d(Tx, Ty) \le c\big[d(x, Ty) + d(y, Tx)\big]$$

for all $x, y \in X$

It is known, see Rhoades [18], that (2.1) and (2.3), (2.1) and (2.4), (2.3) and (2.4), respectively, are independent contractive conditions.

Zamfirescu [22] obtained in 1972 a very interesting fixed point theorem, by combining in an ingenious way the three independent conditions (2.1), (2.3) and (2.4).

THEOREM Z. *Let (X, d) be a complete metric space and $T : X \to X$ a map for which there exist the real numbers a, b and c satisfying $0 \le a < 1$, $0 \le b$, $c < \frac{1}{2}$ such that for each pair x, y in X, at least one of the following is true:*

(z_1) $d(Tx, Ty) \le a\, d(x, y)$;

(z_2) $d(Tx, Ty) \le b\big[d(x, Tx) + d(y, Ty)\big]$;

(z_3) $d(Tx, Ty) \le c\big[d(x, Ty) + d(y, Tx)\big]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots$$

converges to p, for any $x_0 \in X$.

Remarks. 1) One of the most general contraction conditions for which the (unique) fixed point can be approximated by means of Picard iteration has been obtained by Ciric [10] in 1974: there exists $0 \le h < 1$ such that

$$(2.5) \quad d(Tx, Ty) \le h \cdot \max\big\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big\}$$

for all $x, y \in X$.

2) A mapping satisfying (2.5) is commonly called *quasi contraction*.

It is obvious that each of conditions (2.1), (2.3), (2.4) and (z_1)–(z_3) implies (2.5). It is also known, see [18], that any quasi contraction with $0 \le h < 1/2$ is a Zamfirescu operator.

Actually, as shown by Rhoades [18], Zamfirescu's contractive conditions are equivalent to the following Ciric type contractive condition: there exists $0 \le h < 1$ such that

$$d(Tx,Ty) \le h \cdot \max\{d(x,y), [d(x,Tx)+d(y,Ty)]/2,$$
$$[d(x,Ty)+d(y,Tx)]/2\}$$

for all $x, y \in X$.

For the sake of completeness, we conclude this section with the definitions of Picard, Krasnoselskij, Mann and Ishikawa iteration procedures.

Let E be a normed linear space and $T : E \to E$ a given operator. Let $x_0 \in E$ be arbitrary and $\{\alpha_n\} \subset [0,1]$ a sequence or real numbers. The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n = 0,1,2,\ldots \tag{2.6}$$

is called the *Mann iteration* or *Mann iterative procedure* (cf. [15]).

The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, & n = 0,1,2,\ldots, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, & n = 0,1,2,\ldots, \end{cases} \tag{2.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0,1]$, and $x_0 \in E$ arbitrary, is called the *Ishikawa iteration* or *Ishikawa iterative procedure* (cf. [12]).

Remark. For $\alpha_n = \lambda$ (constant), iteration (2.6) reduces to the so called *Krasnoselskij iteration*, cf. [14], while for $\alpha_n \equiv 1$ we obtain the *Picard iteration* (2.2) or method of successive approximations, as it is commonly known, cf. [1]. Obviously, for $\beta_n \equiv 0$, Ishikawa iteration (2.7) reduces to Mann iteration (2.6).

3. PICARD ITERATION

We start this section by presenting a proof of Zamfirescu's fixed point theorem (Theorem Z), adapted after [1].

We first fix $x, y \in X$. At least one of (z_1), (z_2) or (z_3) is true. If (z_2) holds, then

$$d(Tx,Ty) \le \beta[d(x,Tx)+d(y,Ty)] \le$$
$$\le \beta\{d(x,Tx) + [d(y,x)+d(x,Tx)+d(Tx,Ty)]\}.$$

So

$$(1-\beta)\, d(Tx,Ty) \le 2\beta\, d(x,Tx) + \beta\, d(x,y),$$

which yields

$$d(Tx,Ty) \le \frac{2\beta}{1-\beta}\, d(x,Tx) + \frac{\beta}{1-\beta}\, d(x,y). \tag{3.1}$$

If (z_3) holds, then we similarly get

$$d(Tx,Ty) \le \frac{2\gamma}{1-\gamma}\, d(x,Tx) + \frac{\gamma}{1-\gamma}\, d(x,y). \tag{3.2}$$

Therefore, denoting

$$\delta = \max\left\{ \alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma} \right\},$$

we have $0 \le \delta < 1$. Then, for all $x, y \in X$, the inequality

$$d(Tx,Ty) \le 2\delta \cdot d(x,Tx) + \delta \cdot d(x,y) \tag{3.3}$$

holds. In a similar manner we obtain

$$d(Tx,Ty) \le 2\delta \cdot d(x,Ty) + \delta \cdot d(x,y) \tag{3.4}$$

for all $x, y \in X$. It follows from (3.3) that $\operatorname{card} F_T \le 1$. We will show that T has a unique fixed point. Let $x_0 \in X$ be arbitrary and $\{x_n\}_{n=0}^{\infty}$, with

$$x_n = T^n x_0, \quad n = 0, 1, 2, \dots$$

be the Picard iteration associated with T.

If $x := x_n$, $y := x_{n-1}$ are two successive approximations, then by (3.4) we have

$$d(x_{n+1}, x_n) \le \delta \cdot d(x_n, x_{n-1}). \tag{3.5}$$

Hence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence, thus a convergent sequence, too. Let $x^* \in X$ be its limit. In particular, we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

By the triangle rule and (3.4) we have

$$d(x^*, Tx^*) \le d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \le$$

$$\le d(x^*, x_{n+1}) + \delta\, d(x^*, x_n) + 2\,\delta d(x_n, Tx_n),$$

which, by letting $n \to \infty$, yields

$$d(x^*, Tx^*) = 0 \Longleftrightarrow x^* = Tx^*$$

and therefore

$$F_T = \{x^*\} \quad \text{and} \quad x_n \to x^* (n \to \infty)$$

for each $x_0 \in X$. □

Remark. It is a simple task to obtain by (3.5) the *a priori* and *a posteriori* error estimates for the Picard iteration associated with a Zamfirescu operator, that is,

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} \cdot d(x_0, x_1), \quad n = 0, 1, 2, \ldots, \tag{3.6}$$

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} \cdot d(x_n, x_{n-1}), \quad n = 0, 1, 2, \ldots . \tag{3.7}$$

4. MANN AND ISHIKAWA ITERATION PROCEDURES

Let E be a Banach space and $T : E \to E$ an operator for which there exist $0 \leq \delta < 1$ and $L \geq 0$ such that

$$\|Tx - Ty\| \leq \delta \cdot \|x - y\| + L \cdot \|x - Tx\| \tag{4.1}$$

for all $x, y \in E$.

It is known that the operators satisfying (4.1) need not have a fixed point but, if $F(T) = \{x \in E : Tx = x\} \neq \varnothing$, then $F(T)$ is a singleton.

Condition (4.1) is in fact condition (3.4), fulfilled by any Zamfirescu operator.

All arguments in proving Theorem Z do work if we consider, instead of Zamfirescu operators, all operators that have at least one fixed point and satisfy (4.1).

THEOREM 1 ([8]). *Let E be a normed linear space, K a closed convex subset of E, and $T : K \to K$ an operator with $F(T) \neq \varnothing$ satisfying* (4.1).

Let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration defined by (2.7) *and $x_0 \in K$, arbitrary, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ with $\{\alpha_n\}$ satisfying*

$$\sum_{n=0}^{\infty} \alpha_n = \infty. \tag{i}$$

Then $\{x_n\}$ converges strongly to the unique fixed point of T.

Assuming that the normed linear space E in Theorem 1 is a Banach space and T a Zamfirescu operator, we get the main result in [5], while by taking $\beta_n = 0$ in Theorem 1 and assuming that T is a Zamfirescu operator, we obtain the main result in [7]:

COROLLARY 1. *Let E be an arbitrary Banach space, K a closed convex subset of E, and $T : K \to K$ a Zamfirescu operator. Let $\{x_n\}_{n=0}^{\infty}$ be the Mann iteration defined by* (2.6) *and $x_0 \in K$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0, 1]$ with $\{\alpha_n\}$ satisfying* (i). *Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Remarks. 1) Since both Kannan's and Chatterjea's contractive conditions do imply the Zamfirescu conditions, from Theorem 1 we obtain a corresponding convergence theorems for the Ishikawa iteration in these classes of operators.

2) Other important results, amongst them those due to B.E. Rhoades [16, 17], are special cases of Theorem 1.

We conclude this section by stating two results, the first one taken from our paper [4], regarding the comparison of Picard and Mann iterations in the class of Zamfirescu operators, and the second one from Babu and Vara Prashad [23], which both basically show that, to efficiently approximate fixed points of Zamfirescu operators, one should always use Picard iteration.

THEOREM 2. *Let E be an arbitrary Banach space, K a closed convex subset of E, and $T : K \to K$ a Zamfirescu operator. Let $\{y_n\}_{n=0}^{\infty}$ be the Mann iteration defined by* (2.6) *and $y_0 \in K$, with $\{\alpha_n\} \subset [0,1]$ satisfying* (i). *Then $\{y_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T and, moreover, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by* (2.3) *and $x_0 \in K$ converges faster than the Mann iteration.*

The previous result was very recently extended by Babu and Vara Prashad [23] for the Mann and Ishikawa iterative procedures. Their result is given by

THEOREM 3 ([23]). *Let E be an arbitrary Banach space, K a closed convex subset of E, and $T : K \to K$ a Zamfirescu operator. Let $\{y_n\}_{n=0}^{\infty}$ be the Mann iteration defined by* (2.6) *and $y_0 \in K$, and let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration defined by* (2.7) *and $x_0 \in K$, with $\{\alpha_n\} \subset [0,1]$ satisfying* (i). *Then $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ converge strongly to the unique fixed point of T and, moreover, the Mann iteration converges faster than the Ishikawa iteration to the fixed point of T.*

The concept of rate of convergence used in Theorems 2 and 3 is that considered in [1] and [4]. Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two fixed point iteration procedures that converge to the same fixed point p and satisfy, respectively,

$$\|u_n - p\| \leq a_n, \quad n = 0, 1, 2, \dots \tag{4.2}$$

and

$$\|v_n - p\| \leq b_n, \quad n = 0, 1, 2, \dots, \tag{4.3}$$

where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers (converging to zero). If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$, then we say that $\{u_n\}_{n=0}^{\infty}$ given by (4.2) *converges faster to* p than $\{v_n\}_{n=0}^{\infty}$ given by (4.3).

5. WEAK CONTRACTIONS

Let (X, d) be a metric space. A map $T : X \to X$ is called *weak contraction* [6], if there exist a constant $\delta \in [0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx) \tag{5.1}$$

for all $x, y \in X$. By the symmetry of the distance, any weak contraction must satisfy the dual condition

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + L \cdot d(x, Ty) \tag{5.2}$$

for all $x, y \in X$, obtained from (5.1) by formally replacing $d(Tx, Ty)$ and $d(x, y)$ by $d(Ty, Tx)$ and $d(y, x)$, respectively, and then interchanging x and y.

It is easy to see that any Zamfirescu mapping is a weak contraction. As we already mentioned, any quasi contraction with $0 \leq h < 1/2$ is a weak contraction, too.

We state here the main results in [3], i.e., Theorem 4 (an existence theorem) and Theorem 5 (an existence and uniqueness theorem). Their main merit is that they extend both Theorem B and Theorem Z to the larger class of weak contractions, in the spirit of Theorem B, that is, in such a way that they offer a method for approximating the fixed point, for which both a priori and a posteriori estimates are available.

THEOREM 4. *Let (X, d) be a complete metric space and $T : X \to X$ a weak contraction. Then*

1) $F(T) = \{x \in X : Tx = x\} \neq \varnothing$;

2) *for any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by* (2.2) *converges to some $x^* \in F(T)$;*

3) *estimates* (3.6) *and* (3.7) *hold, where δ is the constant appearing in* (5.1).

As we have shown in [6], a weak contraction need not have a unique fixed point. The simplest example to illustrate this is $T : [0, 1] \to [0, 1]$, $Tx = x$, $x \in [0, 1]$.

It is however possible to force the uniqueness of the fixed point of a weak contraction, by imposing an additional contractive condition, quite similar to (5.1), as shown by the next result.

THEOREM 5. *Let (X, d) be a complete metric space and $T : X \to X$ a weak contraction for which there exist $\theta \in (0, 1)$ and some $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \theta \cdot d(x, y) + L_1 \cdot d(x, Tx) \tag{5.3}$$

for all $x, y \in X$. Then

1) *T has a unique fixed point, i.e., $F(T) = \{x^*\}$;*

2) *the Picard iteration $\{x_n\}_{n=0}^{\infty}$ converges to x^*, for any $x_0 \in X$;*

3) *the a priori and a posteriori error estimates* (3.6) *and* (3.7) *hold;*
4) *the rate of convergence of the Picard iteration is given by*

$$d(x_n, x^*) \le \theta\, d(x_{n-1}, x^*), \quad n = 1, 2, \ldots$$

It is easy to see that Theorem Z is a special case of Theorem 5, since any Zamfirescu operator does satisfy conditions (5.1) and (5.3), which are actually conditions (3.4) and (3.3), respectively.

Starting from the fact that φ-contractions are natural generalizations of strict contractions, see [21], we furthermore extended in [2] the results given in [6], from weak contractions to the more general class of weak φ-contractions.

6. CONCLUSIONS

The class of Zamfirescu operators is a very important subclass of quasi-nonexpansive mappings ($T : K \to K$ is quasi-nonexpansive if T has at least one fixed point and, for any fixed point p, the inequality $\|Tx - p\| \le \|x - p\|$ holds for any $x \in K$). This is due to the fact that the class of quasi-nonexpansive operators is independent of the class of nonexpansive operators, also very important in metrical fixed point theory. Indeed, if we consider $X = \mathbb{R}$ and $T : X \to X$, $Tx = 0$, if $x \in (-\infty, 2]$ and $Tx = -\frac{1}{2}$, if $x > 2$, then (i) T is not continuous; (ii) T fulfills (2.3) (with $a = \frac{1}{5}$) and hence it is a Zamfirescu mapping; (iii) T is not nonexpansive, since $\left|T2 - T\frac{9}{2}\right| = \frac{1}{2} > \frac{1}{4} = \left|2 - \frac{9}{4}\right|$.

In this paper we were only interested in presenting the most recent developments around Zamfirescu's fixed point theorem, in order to illustrate its main features. Other important results, as for example those regarding the stability of Picard and Mann iterations with respect to a Zamfirescu operator [11], were not included in this paper. For more details and results, and also for a comprehensive bibliography, see our recent monograph [1].

Starting from the properties of a Zamfirescu operator we obtained one of the most general classes of contractive mappings for which the fixed points can by approximated by means of Picard iteration, i.e., the class of weak contractive mappings.

Indeed, the great majority of the metrical contractive conditions known in literature (see [18]) which involve in the right-hand side the displacements

$$d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)$$

with nonnegative coefficients

$$a(x,y), b(x,y), c(x,y), d(x,y), e(x,y),$$

respectively, are commonly based on a restrictive assumption of the kind

$$0 \le a(x,y) + b(x,y) + c(x,y) + d(x,y) + e(x,y) < 1,$$

while our condition (5.1) does not require $\delta + L$ be less than 1, thus providing a large class of contractive type mappings.

Let us also mention that in Rhoades' classification of the main contractive conditions [18], in a list that comprises 25 contractive conditions, which are basically ordered with respect to their generality, Banach's contraction condition is numbered (1), Kannan condition is numbered (4), while Zamfirescu's contractive condition is numbered (19) and Ciric's contractive condition is numbered (24).

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North University of Baia Mare
Department of Mathematics and Computer Science
Str. Victoriei 76
430122 Baia Mare, Romania
vberinde@ubm.ro
vasile_berinde@yahoo.com

and

"Babes-Bolyai" University of Cluj-Napoca
Department of Statistics, Analysis, Forecast
and Mathematics, Faculty of Economics
Str. T. Mihali 58-60
400591 Cluj-Napoca, Romania
mberinde@math.ubbcluj.ro
madalina_berinde@yahoo.com